# Crossing Number is Hard for Cubic Graphs

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January 16, 2004

**Abstract**

It was proved by [Garey, Johnson] that computing the crossing number of a graph is an $NP$-hard problem. Their reduction, however, used parallel edges and vertices of very high degrees. We prove here that it is $NP$-hard to determine the crossing number of a simple cubic graph. In particular, this implies that the minor-monotone version of crossing number is also $NP$-hard, which has been open till now.

**Keywords:** crossing number, cubic graph, NP-completeness.
2000 Math Subjects Classification: **05C10, 05C62, 68R10**

*ITI is supported by the Ministry of Education of the Czech Republic as project LN00A056.

# 1 Background on Crossing Number

We assume that the reader is familiar with basic terms of graph theory. In this paper we consider finite simple graphs, unless we specifically speak about multigraphs. A graph is cubic if it has all vertices of degree 3.

In a *(proper) drawing* of a graph $G$ in the plane the vertices of $G$ are points and the edges are simple curves joining their endvertices. Moreover, it is required that no edge passes through a vertex (except at its ends), and that no three edges intersect in a common point which is not a vertex. An *edge crossing* is an intersection point of two edges-curves in the drawing which is not a vertex. The *crossing number* $\mathrm{cr}(G)$ of a graph $G$ is the minimum number of edge crossings in a proper drawing of $G$ in the plane (thus, a graph is planar if and only if its crossing number is 0). We remark that there are other possible definitions of crossing number which are supposed, but not(!) known, to be equivalent to each other.

The algorithmic problem CROSSINGNUMBER is given as follows:

*Input:* A multigraph $G$ and an integer $k$.

*Question:* Is it true that $\mathrm{cr}(G) \leq k$?

It has been proved in a classical paper by Garey and Johnson [3] that CROSSINGNUMBER is an $NP$-complete problem for $k$ on the input.

Since then, a new significant complexity result about graph crossing number has appeared only recently — a paper by Grohe [4] presenting a quadratic-time ("FPT") algorithm for CROSSINGNUMBER($k$) with constant $k$. There is also a long-standing open question, originally asked by Seese: What is the complexity of CROSSINGNUMBER for graphs of fixed tree-width? (Here we leave aside other results dealing with various restricted versions of the crossing number problem appearing in connection with VLSI design or with graph drawing, such as the "layered crossing" number etc.)

Before the above mentioned "FPT" algorithm of Grohe for crossing number has appeared; Fellows [1] observed that there are finitely many excluded minors for the cubic graphs of crossing number at most $k$, which implied a (non-constructive) algorithm for CROSSINGNUMBER($k$) with constant $k$ over cubic graphs. That observation might still suggest that CROSSINGNUMBER was easier to solve over cubic graphs than in general. However, that is not so, as we show in this paper.

A *minor* $F$ of a graph $G$ is a graph obtained from a subgraph of $G$ by contractions of edges. Let us further define the *minor-monotone crossing number* $\mathrm{mcr}(G)$ as the smallest crossing number $\mathrm{cr}(H)$ over all graphs $H$ having $G$ as a minor. The traditional crossing number does not behave well with respect to taking minors; one may find graphs $G$ such that $\mathrm{cr}(G) = 1$ but $\mathrm{cr}(G')$ is arbitrarily large for a minor $G'$ of $G$. On the other hand, $\mathrm{mcr}(G') \leq \mathrm{mcr}(G)$ for a minor $G'$ of $G$ by definition. Our main result immediately extends to a proof that also the minor-monotone crossing number is $NP$-hard to compute, which has been an open question till now.

# 2 Crossing Number and OLA

We first define another classical $NP$-complete combinatorial problem [2] called <u>OptimalLinearArrangement</u>, which is given as follows:

*Input:* An $n$-vertex graph $G$, and an integer $a$.

*Question:* Is there a bijection $\alpha : V(G) \to \{1, \dots, n\}$ (a *linear arrangement* of vertices) such that the following holds

$$\sum_{uv\in E(G)} |\alpha(u) - \alpha(v)| \ \leq \ a\,? \tag{1}$$

The sum on the left of (1) is called the *weight* of $\alpha$.

The above mentioned paper [3] actually reduces CrossingNumber from OptimalLinearArrangement. We, however, consider that reduction "unrealistic" in the following sense: The reduction in [3] creates many large classes of parallel edges, and it uses vertices of very high degrees. (There seems to be no easy modification avoiding those.) So we consider it natural to ask what can be said about the crossing number problem on simple graphs with small vertex degrees.

It might be tempting to construct a "nicer" polynomial reduction for CrossingNumber from another $NP$-complete problem called Planar-SAT (a version of the satisfiability problem with a planar incidence graph). There have been, to our knowledge, a few attempts in this directions, so far unsuccessful. We consider this phenomenon remarkable since Planar-SAT seems to be much closer to the crossing-number problem than the Linear Arrangement is.

Still, we have found another construction reducing CrossingNumber from OptimalLinearArrangement, which produces cubic graphs. The basic idea of our construction is similar to [3], but the restriction to degree-3 vertices brings many more difficulties to the proofs. The construction establishes our main result which reads:

**Theorem 2.1** *The problem* CrossingNumber *is NP-complete for* 3-*connected (simple) cubic graphs.*

Let us, moreover, define so called MM-CrossingNumber problem (from "Minor-Monotone") as follows:

*Input:* A multigraph $G$ and an integer $k$.

*Question:* Is it true that $\mathrm{mcr}(G) \le k$?

We immediately conclude the following new result.

**Corollary 2.2** *The problem* MM-CrossingNumber *is NP-complete.*

**Observation** Let a cubic graph $G$ be a minor of a multigraph $H$. Then some subdivision of $G$ is contained as a subgraph in $H$. Hence $\mathrm{cr}(G) \le \mathrm{cr}(H)$.

Thus $\mathrm{cr}(G) = \mathrm{mcr}(G)$ for cubic graphs, and the corollary follows directly from Theorem 2.1.

# 3 Our Cubic Construction

Let us call a *cubic grid* the graph illustrated in Figure 1 (looking like a "brick wall"). We say that the cubic-grid *height* equals the number of the "horizontal" paths, and the *length* equals the number of edges on the "top-most" horizontal path. (The positions are referred to as in Figure 1.) Formally, the cubic grid of even height $h$ and length $\ell$, denoted by $\mathcal{C}'_{h,\ell}$, is defined

$$V(\mathcal{C}'_{h,\ell}) = \{v_{i,j} : i = 1, 2, \dots, h;\ j = 0, 1, \dots, \ell\} \cup$$

$$\cup \{w_{i,j} : i = 2, 3, \dots, h-1;\ j = 1, 2, \dots, \ell\},$$

$$E(\mathcal{C}'_{h,\ell}) = \{v_{2i-1,j} v_{2i,j} : i = 1, 2, \dots, h/2;\ j = 0, 1, \dots, \ell\} \cup$$

$$\cup \{w_{2i,j} w_{2i+1,j} : i = 2, 3, \dots, h/2 - 1;\ j = 1, 2, \dots, \ell\} \cup$$

$$\cup\{v_{i,j-1}w_{i,j}, w_{i,j}v_{i,j} : i = 2, 3, \ldots, h-1;\ j = 1, 2, \ldots, \ell\}\cup$$
$$\cup\{v_{i,j-1}v_{i,j} : i = 1, h;\ j = 1, 2, \ldots, \ell\}\,.$$

Suppose we now identify the "left-most" vertices in the grid $\mathcal{C}'_{h,\ell}$ with the "right-most" ones, formally $v_{i,0} = v_{i,\ell}$ for $i = 1, 2, \ldots, h$, and simplify the resulting graph. Then we obtain the *cyclic cubic grid* $\mathcal{C}_{h,\ell}$ (which is, indeed, a cubic graph).

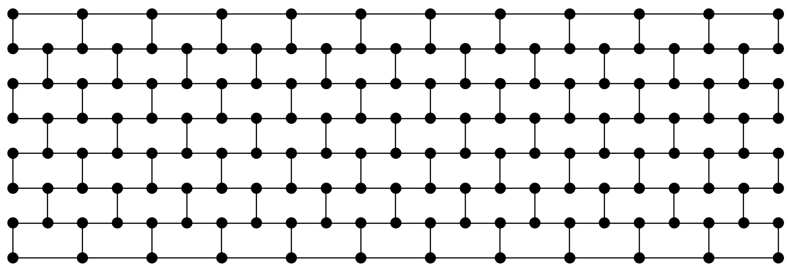

Figure 1: An illustration of a cubic grid (a fragment of length 11 and height 8).

Let us have a cubic grid $\mathcal{C}'_{h,\ell}$ or $\mathcal{C}_{h,\ell}$ as above. We say that an edge $f$ is *attached to the grid at low position* $j$ if the edge $v_{1,j-1}v_{1,j}$ is subdivided with a vertex $x_f$, where $x_f$ is an endvertex of $f$ as well. We say that $f$ is *attached at high position* $j$ if an analogous construction is done for the edge $v_{h,j-1}v_{h,j}$. Notice that the new vertex $x_f$ introduced when attaching an edge $f$ has degree 3, and that the degrees of other vertices are unchanged. Similarly, a vertex $x$ is *attached to the grid at position* $j$ if two new edges $f, f'$ with a common endvertex $x$ are attached via their other endvertices at low and high positions $j$, respectively, to our cubic grid. This is illustrated on a detailed picture in Figure 2.

In a cyclic cubic grid $\mathcal{C}_{h,\ell}$, the cycles $M^i$ on vertices $v_{i,0}w_{i,1}v_{i,1}w_{i,2}$-$\ldots v_{i,\ell-1}w_{i,\ell}$ for $i = 2, 3, \ldots, h-1$, and on vertices $v_{i,0}v_{i,1}\ldots v_{i,\ell-1}$ for $i = 1, h$, are called the *main cycles* of the grid $\mathcal{C}_{h,\ell}$. $M^1$ and $M^h$ are also referred to as the *outer* main cycles. We use the same names, main cycles, for the subdivisions of the cycles $M^i$ in graphs created from the grid $\mathcal{C}_{h,\ell}$ by attaching edges.

Assume now that we are given a graph $G$ on $n$ vertices. In order to prove Theorem 2.1, we are going to construct a cubic graph $H_G$ depending on $G$. (Although our graph $H_G$ is huge, it has polynomial size in $G$.) We show then how one can compute the weight of an optimal linear arrangement for

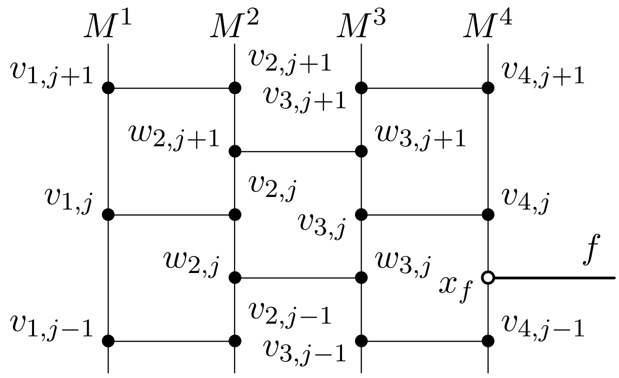


Figure 2: A detail of the cyclic cubic grid $\mathcal{C}_{4,\ell}$, with an edge $f$ attached at high position $j$.

$G$ from the crossing number $\mathrm{cr}(H_G)$, and vice versa. Our construction uses several size parameters defined next:

$$
\begin{aligned}
n &= |V(G)|, \;\; m = |E(G)|, \\
t &= 2mn, \qquad\qquad (2)\\
r &= t^2 = 4m^2n^2, \\
s &= m^3 r = 4m^5n^2, \\
q &= (m^3+n+1)r = 4m^5n^2 + 4m^2n^3 + 4m^2n^2, \\
z &= 2((s+rn)nt + r) = 16m^6n^4 + 16m^3n^5 + 8m^2n^2.
\end{aligned}
$$

Without loss of generality we may assume that the graph $G$ is sufficiently large, say

$$m > n > 100\,. \qquad (3)$$

We start with two copies $B_1, B_2$ of the cyclic cubic grid $\mathcal{C}_{z,q}$, called here the *boulders* (for their huge size that keeps the rest of our graph "in place"). Then we make $n$ disjoint copies $R_1, \ldots, R_n$ of the cyclic cubic grid $\mathcal{C}_{t,q}$, called here the *rings*. An intermediate step in the construction – our graph $H_{m,n}$, is obtained by the following operations:

- Start with the disjoint union $B_1 \cup B_2 \cup R_1 \cup \ldots \cup R_n$ of the two boulders and the $n$ rings.

- For every pair of integers $0 \le i < m^3$ and $0 \le j < r$, take a new edge $\epsilon^{\mathrm{s}}_{i+jm^3}$, and attach $\epsilon^{\mathrm{s}}_{i+jm^3}$ at low positions $i + j(m^3 + n + 1) < q$ to the boulder $B_1$ via one end, and to $B_2$ via the other end. These $s$ new edges $\epsilon^{\mathrm{s}}_0, \dots, \epsilon^{\mathrm{s}}_{s-1}$ are called the *free spokes* in $H_{m,n}$.

- For every pair of integers $1 \le i \le n$ and $0 \le j < r$, set $p = i - 1 + m^3 + j(m^3 + n + 1) < q$, and take two new vertices $\nu^{\mathrm{r1}}_{i,j}$ and $\nu^{\mathrm{r3}}_{i,j}$ connected by an edge $\epsilon^{\mathrm{p2}}_{i,j}$. Then attach a new edge $\epsilon^{\mathrm{p1}}_{i,j}$ (new edge $\epsilon^{\mathrm{p3}}_{i,j}$) with one end $\nu^{\mathrm{r1}}_{i,j}$ (one end $\nu^{\mathrm{r3}}_{i,j}$) to the boulder $B_1$ ($B_2$) at low position $p$ via the other end. Finally, attach a new edge $\epsilon^{\mathrm{r1}}_{i,j}$ (new edge $\epsilon^{\mathrm{r3}}_{i,j}$) to the ring $R_i$ at low (high) position $p$ via the other end. The path formed by three edges $\epsilon^{\mathrm{p1}}_{i,j}, \epsilon^{\mathrm{p2}}_{i,j}, \epsilon^{\mathrm{p3}}_{i,j}$ is called the *$j$-th ring spoke* of $R_i$ in $H_{m,n}$.

We remark that the above construction attaches only one edge at the same position of each of the boulders and rings, and so the operations are well-defined. (Figure 3.) This remark applies also to further constructions on the graph $H_G$.

To simplify our notation, the above names of the boulders $B_1, B_2$ and the rings $R_i$ are inherited to the subdivisions of those boulders and rings created in the construction of $H_{m,n}$. The same simplified notation is used further for the graph $H_G$, too.

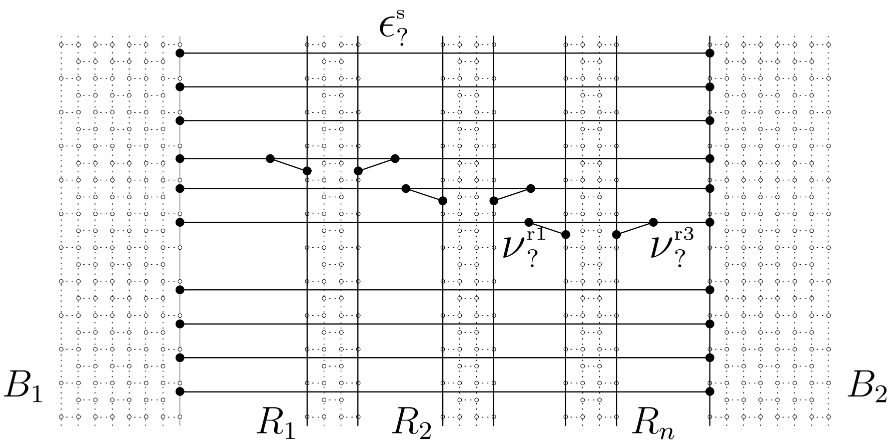


Figure 3: How to attach free and ring spokes in the graph $H_{m,n}$.

So far, the constructed graph $H_{m,n}$ does not depend on a particular structure of $G$, but only on its size and our choice of the parameters (2). In the next lemma we show that drawings of $H_{m,n}$ are very "flexible", and hence suitable for modeling the linear arrangement problem of $G$. (Actually, all the drawings described in the next lemma are optimal, as we see in Section 5.)

**Lemma 3.1** *For any permutation $\pi$ of the set $\{1, 2, \ldots, n\}$; there is a drawing of the graph $H_{m,n}$ with $(s + rn)nt$ crossings, such that the subdrawings of the rings are pairwise disjoint, and that any free spoke in the drawing intersects all the rings in order $R_{\pi(1)}, \ldots, R_{\pi(n)}$ from $B_1$ to $B_2$.*

*Proof.* We start with the unique planar embedding of the boulders and the free and ring spokes of $H_{m,n}$. Then we draw each ring $R_i$ of $H_{m,n}$ so that $R_i$ separates the boulders from each other in the drawing, and that the rings are nested into each other in the required order $R_{\pi(1)}, \ldots, R_{\pi(n)}$. So each of the $s$ free spokes, and each of the $rn$ ring spokes, has $t$ crossings with each ring (one with every main cycle), summing to a total of $(s+rn)nt$ crossings. We finally attach, in a suitable drawing, each of the ring spokes to its ring by the edges $\epsilon^{\mathrm{r1}}_{i,j}$ and $\epsilon^{\mathrm{r3}}_{i,j}$ with no additional crossings. See Figure 3 for an illustration. ∎

Finally, the graph $H_G$ needed for our polynomial reduction from $G$ is constructed as follows:

- Start with the graph $H_{m,n}$, for $n = |V(G)|$ and $m = |E(G)|$. Number the vertices $V(G) = \{1, 2, \ldots, n\}$.
- For every ordered pair $0 < i, j \leq n$ such that $\{i, j\} \in E(G)$, set $p = (i - 1 + jn - n)4m^2(m^3 + n + 1) + m^3 + n < q$. In the graph $H_{m,n}$, attach new vertices $\chi_{ij}, \chi'_{ij}$ to the rings $R_i, R_j$, respectively, at positions $p$, and add a new edge $\{\chi_{ij}, \chi'_{ij}\}$. The subgraph $X_{i,j}$ induced on the five new edges incident with $\chi_{ij}, \chi'_{ij}$ is called a *handle* of the edge $ij$ in $H_G$. (Figure 4.)

That is, the rings in $H_G$ model the vertices of $G$, and the handles model the edges of $G$. As we show later, an optimal drawing of $H_G$ uniquely determines an ordering of the rings of $H_{m,n}$, and hence the weight of an optimal linear arrangement of $G$ corresponds to the number of crossings between the rings and the handles in an optimal drawing of the graph $H_G$.

We conclude this section with an upper bound on the crossing number of our constructed graph.

**Lemma 3.2** *Let us, for a given graph $G$, construct the graph $H_G$ as described above. If $G$ has a linear arrangement of weight $A$, then the crossing number of $H_G$ is*

$$\mathrm{cr}(H_G) \leq (s + rn)nt + 2(A + m)t - 4m\,,$$

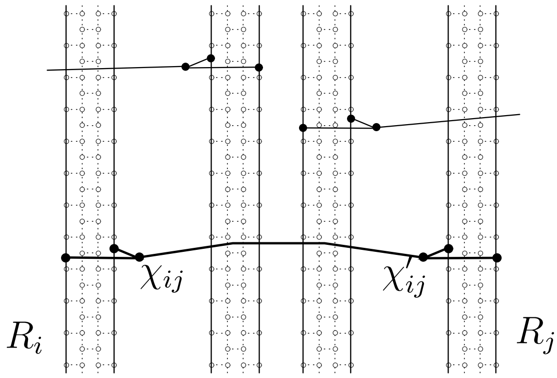


Figure 4: How to attach handles of the edges of $G$ in the graph $H_G$.

*where the weight of a linear arrangement is defined by* (1) *on page 3, and* $m, n, r, s, t$ *are given by* (2) *on page 6.*

*Proof.* Let $\alpha$ be the linear arrangement of $G$ of weight $A$. We draw the graph $H_{m,n} \subset H_G$ by Lemma 3.1 with $(s+rn)nt$ edge crossings, such that the rings are ordered as $R_{\alpha^{-1}(1)}, \dots, R_{\alpha^{-1}(n)}$ from $B_1$ to $B_2$. Then we draw the handles in $H_G$ for all edges of $G$ in the natural (shortest) way, as illustrated in Figure 4.

Now for $0 < i, j \le n$ such that $\{i, j\} \in E(G)$, the handle of $ij$ in $H_G$ has $t-1$ crossings with the main cycles of the ring $R_i$ and $t-1$ crossings with those of $R_j$. Moreover, the handle has $t \cdot |\alpha(i) - \alpha(j)| - t$ crossings with the rings "between $R_i$ and $R_j$". Keeping in mind that each edge of $G$ actually makes two handles of $ij$ and of $ji$, we sum the crossings of the handles:

$$\sum\nolimits_{ij \in E(G)} 2\left(t \cdot |\alpha(i) - \alpha(j)| - t + 2t - 2\right) =$$

$$= 2t \sum\nolimits_{ij \in E(G)} |\alpha(i) - \alpha(j)| + 2mt - 4m = 2At + 2mt - 4m\,.$$

Altogether, the described drawing of $H_G$ has $(s + rn)nt + 2(A + m)t - 4m$ crossings. ∎

Now, using obvious $A \le m(n-1)$, it is easy to conclude:

**Corollary 3.3** *For any $G$ conforming to* (3), $\mathrm{cr}(H_G) < z/2 = (s+rn)nt+r$.

# 4 Assorted Topological Lemmas

We need to be a bit more formal in this section. A *curve* $\gamma$ is a continuous function mapping the interval $[0,1]$ to a topological space. A curve $\gamma$ is a *closed* curve if $\gamma(0)=\gamma(1)$. A closed curve $\gamma$ is *contractible* in a topological space if $\gamma$ can be continuously deformed to a single point there. We call a *cylinder* the topological space obtained from the unit square by identifying one pair of opposite edges in the same direction. (A cylinder has two disjoint closed curves as the boundary.)

We are going to deal with collections of curves having somehow special structure. A set $\Gamma$ of curves is called *nice* if all of the following are true:

- No three curves in $\Gamma$ have a common intersection.
- If $x$ is a self-intersection point of a curve $\gamma \in \Gamma$, i.e. $x=\gamma(a)=\gamma(b)$ for distinct $a,b \in [0,1]$, then no other curve in $\Gamma$ passes through $x$.
- If $x$ in an intersection point of $\gamma,\gamma' \in \Gamma$, then in a sufficiently small neighbourhood $U$ of $x$, the curves $\gamma,\gamma'$ are otherwise disjoint, and they intersect the boundary of $U$ in a cyclic order of $\gamma,\gamma',\gamma,\gamma'$ (they "properly cross").

A subset of a nice set of curves is nice as well by the definition. Naturally, we call a *crossing* of curves the intersection point of two curves in a nice set. This obviously corresponds with the notion of an edge crossing in a topological graph.

**Lemma 4.1** *Let $k,\ell$ be positive integers, let $p=k(\ell+1)$, and let $\Pi$ be a cylinder with two closed boundary curves $\pi_1,\pi_2$. Let $X_1,\dots,X_p$ be distinct points on $\pi_1$ in this cyclic order, and let $Y_1,\dots,Y_p$ be distinct points on $\pi_2$ in the corresponding cyclic order. Suppose that $\mathcal{S}=\{\sigma_i : i=1,\dots,p\}$ is a nice set of $p$ curves on $\Pi$ such that each $\sigma_i$ has ends $X_i$ and $Y_i$, and that $\tau$ is a contractible closed curve on $\Pi$ disjoint from $\pi_1,\pi_2$. Moreover, assume that $\tau$ intersects each one of the curves in $\mathcal{S}_0=\{\sigma_{i(\ell+1)} : i=1,\dots,k\}\subset\mathcal{S}$, and that $(\mathcal{S}\setminus\mathcal{S}_0)\cup\{\tau\}$ also forms a nice set of curves. Then at least one of the two cases happens:*

- *$\tau$ crosses twice at least $\frac{3}{5}k\ell$ of the curves in $\mathcal{S}\setminus\mathcal{S}_0$, or*
- *there are at lest $(\frac{2}{25}k^2-\frac{1}{5}k)\ell$ crossings of curves in $\mathcal{S}$.*

*Proof.* First notice that since $\tau$ is a contractible closed curve, it divides $\Pi$ into two connected regions, one of them containing both $\pi_1, \pi_2$. So if a curve $\sigma \in \mathcal{S} \setminus \mathcal{S}_0$ intersects $\tau$ (and, recall $\{\sigma, \tau\}$ is nice), then $\sigma$ has (at lest) two crossings with $\tau$ by Jordan's curve theorem. Hence, let us assume that more than $\frac{2}{5}k\ell$ of the curves in $\mathcal{S} \setminus \mathcal{S}_0$ are disjoint from $\tau$, and denote their subset by $\mathcal{S}_1 \subseteq \mathcal{S} \setminus \mathcal{S}_0$.

*Claim 4.1.1.* For any two $\varphi, \varphi' \in \mathcal{S}_1$ and two $\theta, \theta' \in \mathcal{S}_0$ such that the ends of $\varphi, \varphi'$ on $\pi_1$ separate the ends of $\theta, \theta'$ there, one of $\theta$ or $\theta'$ has at least two crossings with $\varphi \cup \varphi'$.

To see that the claim holds true; realize that one of the connected components of the topological space $\Pi \setminus (\varphi \cup \varphi')$ contains the whole curve $\tau$ by our choice of $\varphi, \varphi'$, and no such component may contain an end of $\theta$ and an end of $\theta'$ at the same time. See Figure 5.

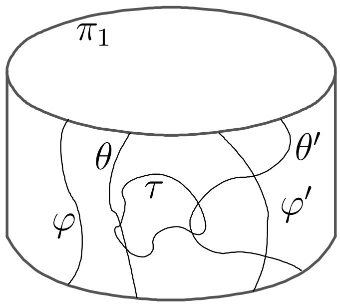


Figure 5: An illustration to Claim 4.1.1.

For the rest of the proof of Lemma 4.1 we focus on the collection of curves $\mathcal{S}_0 \cup \mathcal{S}_1$, and consider their cyclic ordering determined by their ends on $\pi_1$. (Also the same cyclic ordering as determined by their ends on $\pi_2$.) By the assumptions, every $\ell + 1$ consecutive curves of $\mathcal{S}_0 \cup \mathcal{S}_1$ must contain at least one curve from $\mathcal{S}_0$. So we find $a, b$ such that $\sigma_{a(\ell+1)}, \sigma_{b(\ell+1)} \in \mathcal{S}_0$ divide the cyclic ordering of $\mathcal{S}_1$ into two parts of size at least $\frac{1}{2}(|\mathcal{S}_1| - \ell)$ each. Hence we may apply Claim 4.1.1 to $\theta = \sigma_{a(\ell+1)}, \theta' = \sigma_{b(\ell+1)}$ and to $\frac{1}{2}(|\mathcal{S}_1| - \ell)$ choices of disjoint pairs from $\mathcal{S}_1$, accounting for at least $|\mathcal{S}_1| - \ell$ crossings in $\mathcal{S}$.

More generally, with indices modulo $p$ the pair $\sigma_{(a+i)(\ell+1)}, \sigma_{(b+i)(\ell+1)} \in \mathcal{S}_0$ divides the cyclic ordering of $\mathcal{S}_1$ into two parts of size at least $\frac{1}{2}(|\mathcal{S}_1| - (i+1)\ell)$ each, for $i = 0, 1, \ldots, \frac{2}{5}k - 2$. By applying the previous idea for each pair $\theta = \sigma_{(a+i)(\ell+1)}$, $\theta' = \sigma_{(b+i)(\ell+1)}$, we find at least this number of distinct crossings of curves in $\mathcal{S}$:

$$\sum_{i=0}^{2k/5-2} (|\mathcal{S}_1| - (i+1)\ell) \geq \sum_{i=0}^{2k/5-2} \left(\frac{2}{5}k\ell - (i+1)\ell\right) =$$

$$= \sum_{i=1}^{2k/5-1} i\ell = \binom{2k/5}{2}\ell = \left(\frac{2}{25}k^2 - \frac{1}{5}k\right)\ell \qquad \blacksquare$$

**Lemma 4.2** *Let $n, t$ be positive integers. Suppose that, for each $i = 1, 2, \ldots, n$, we have a set $\mathcal{R}_i$ of $t$ closed curves. If the union $\mathcal{R}_1 \cup \mathcal{R}_2 \cup \ldots \cup \mathcal{R}_n$ has less than $t^2$ intersecting pairs of curves, then there are pairwise disjoint representatives $\varrho_i \in \mathcal{R}_i$, $i = 1, 2, \ldots, n$.*

*Proof.* We select the curve $\varrho_1 \in \mathcal{R}_1$ which is intersected by the least number of other curves. Then we replace $\mathcal{R}_1$ by the collection $\mathcal{R}'_1$ of $t$ copies of $\varrho_1$, but we do not count the pairs from $\mathcal{R}'_1$ as intersecting. It follows from the way we have chosen $\varrho_1$, that the number of intersecting pairs in $\mathcal{R}'_1 \cup \mathcal{R}_2 \cup \ldots \cup \mathcal{R}_n$ is not larger than in $\mathcal{R}_1 \cup \mathcal{R}_2 \cup \ldots \cup \mathcal{R}_n$.

In the next step we analogously select $\varrho_2 \in \mathcal{R}_2$ with the least number of intersections, and replace $\mathcal{R}_2$ by $\mathcal{R}'_2$ consisting of $t$ copies of $\varrho_2$ (not considered pairwise intersecting); and so on..., up to $\varrho_n$. We claim that $\varrho_1, \varrho_2, \ldots, \varrho_n$ are the desired pairwise disjoint representatives. Indeed, if $\varrho_i$ intersected $\varrho_j$, then there would be $t^2$ intersecting pairs from $\mathcal{R}'_i$ and $\mathcal{R}'_j$, and hence also the number of intersecting pairs in the original collection $\mathcal{R}_1 \cup \mathcal{R}_2 \cup \ldots \cup \mathcal{R}_n$ would had been at least $t^2$, a contradiction. $\blacksquare$

For the next two lemmas, we define a set $\mathcal{X}(c, d)$ of pairwise disjoint cycles in a cyclic cubic grid $D = \mathcal{C}_{t,\ell}$ as follows: We shall use the notation from the definition of a cubic grid (page 4, Figure 2). Let $C_j$ denote the cycle of the cubic grid $D$ on vertices $v_{1,c+2j}, v_{2,c+2j}, w_{2,c+2j}, w_{3,c+2j}, v_{3,c+2j}, \ldots, w_{h-1,c+2j}, v_{h-1,c+2j}, v_{h,c+2j}, v_{h,c+2j+1}, v_{h-1,c+2j+1}, \ldots, v_{1,c+2j+1}$. (Such a cycle is also depicted in Figure 6.) Then $\mathcal{X}(c, d) = \{C_j : 0 \leq j < \frac{1}{2}(d-c)\}$.

**Lemma 4.3** *Let $k, \ell, t$ be integers, and let $(p_1, p_2, \ldots, p_k)$ be an increasing sequence of integers such that $p_1 > 4kt$, $p_k < \ell$, and $p_{j+1} - p_i \geq 4kt$ for $j = 1, 2, \ldots, k$. Assume that the graph $F$ is constructed from the cyclic cubic grid $D = \mathcal{C}_{t,\ell}$ by attaching a vertex $z_j$ at position $p_j$ for each $j = 1, 2, \ldots, k$. Then* $\mathrm{cr}(F) = k(t-2)$ .

*Proof.* There is an obvious drawing of $F$ with exactly $k(t-2)$ edge crossings — when the edges incident with each $z_j$ cross all the main cycles of $F$ except the outer ones. Conversely, we prove that every proper drawing of $F$ must have at least $k(t-2)$ crossings. Let us fix $a \in \{0, 1, \dots, k-1\}$. Assuming $p_0 = 0$, we denote by $\mathcal{X} = \mathcal{X}(p_a, p_{a+1})$ a collection of disjoint cycles in $D \subset F$. Since one edge crossing may involve edges of at most two of the cycles in $\mathcal{X}$, and since $\mathrm{cr}(F) < kt \leq \frac{1}{2}|\mathcal{X}|$, we conclude:

*Claim 4.3.1.* Any optimal drawing of $F$ must have a cycle $C \in \mathcal{X}(p_a, p_{a+1})$ with no crossed edge.

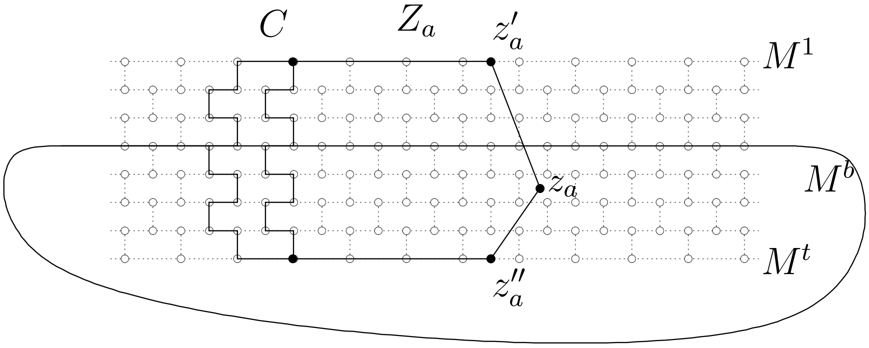


Figure 6: An illustration to Claim 4.3.2.

Denote the $i$-th main cycle in the cyclic cubic grid of $F$ by $M^i$, for $i = 1, \dots, t$. Denote the neighbours of $z_a$ subdividing the outer main cycles $M^1, M^t$ in $F$ by $z'_a, z''_a$, respectively. We define a path $Z_a$ in $F$ consisting of the path $z'_a z_a z''_a$, and of the subpaths of $M^1, M^t$ connecting $z'_a, z''_a$, respectively, to the cycle $C$ from Claim 4.3.1; as depicted in Figure 6. Let us further fix $b \in \{2, 3 \dots, t-1\}$. By Claim 4.3.1 the cycle $C$ is drawn as a simple closed curve with no edge crossing, and so a drawing of another main cycle $M^b$ (which intersects $C$ in two edges) separates the ends of $Z_a$ on $C$. We conclude:

*Claim 4.3.2.* The path $Z_a$ must cross the main cycle $M^b$, for all pairs $a \in \{0, 1, \dots, k-1\}$ and $b \in \{2, 3 \dots, t-1\}$.

Now realize that the main cycles $M^2, \dots, M^{t-1}$ are pairwise disjoint, and that also the paths $Z_a$ for $a = 0, 1, \dots, k-1$ are chosen as pairwise disjoint subgraphs in $F$. Hence we account for at least $(t-2)k$ distinct edge crossings in $F$ using Claim 4.3.2. ∎

**Lemma 4.4** *Let $q, t$ be integers. Let $\Pi$ be a cyclinder, and let $\varrho_1, \varrho_2$ be two disjoint curves on $\Pi$ both connecting points on the opposite boundaries of $\Pi$. Assume that $D$ is a drawing of the cyclic cubic grid $\mathcal{C}_{t,q}$ on $\Pi$. Moreover, assume the following:*

- *The drawing of $D$ is such that each of the main cycles of $D$ is drawn as a noncontractible closed curve on $\Pi$, intersecting each curve $\varrho_1, \varrho_2$ in exactly one point.*
- *No other edge of the drawing $D$ is intersected by $\varrho_1$ or $\varrho_2$.*
- *There are indices $c, d$, $d > c + 2t$ such that the vertices $v_{1,c}$ and $v_{1,d}$ of the first main cycle $M^1 \subset D$ are drawn inside the same region $\Sigma$ of $\Pi \setminus (\varrho_1 \cup \varrho_2)$.*

*Then all the cycles in the set $\mathcal{X}(c+t, d-t)$ defined as above are drawn inside the region $\Sigma$.*

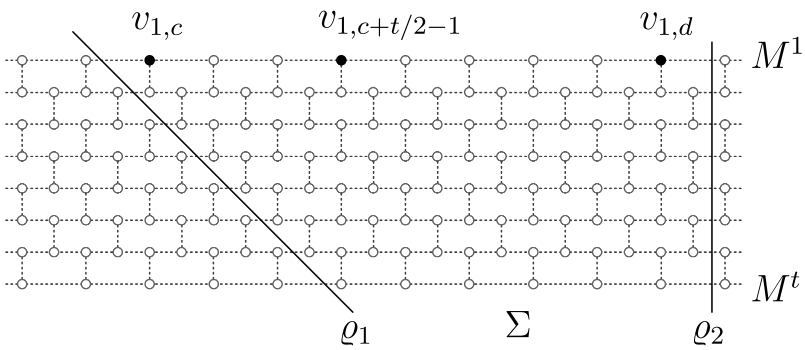


Figure 7: An illustration to Lemma 4.4.

*Proof.* We may assume, without loss of generality, that $\varrho_1$ intersects the edge $v_{1,c-1}v_{1,c}$ of $M^1$. Denote by $P^i \subset M^i$ the subpath of the $i$-th main cycle $M^i$ of $D$ on the vertices $v_{i,c-i}, \ldots, v_{i,c+i-1}$. Recall that $\varrho_1$ intersects $M^i$ exactly once. We establish the following claim by induction on $i \geq 1$:

*Claim 4.4.1.* The curve $\varrho_1$ intersects $M^i$ in an edge from $P^i$.

The claim is true for $i = 1$ by our assumption. Suppose it is true for $i < t$, but false for $i+1$. Then, up to symmetry, the vertices $w_{i+1,c+i}, v_{i+1,c+i}$ of $M^{i+1}$ were drawn outside of $\Sigma$, while the vertices $w_{i,c+i}, v_{i,c+i}$ of $M^i$ were drawn inside $\Sigma$ (or vice versa). But, than the edge $w_{i,c+i}w_{i+1,c+i}$ or $v_{i,c+i}v_{i+1,c+i}$

(depending on parity of $i$) of $D$ would have to be drawn intersecting $\varrho_1 \cup \varrho_2$, a contradiction to the assumptions.

(Actually, Claim 4.4.1 can be established in a stronger form, but we prefer this weak form with a straightforward proof. See Figure 7.) A symmetric statement clearly holds for $\varrho_2$. Since the above paths $P^i$ are disjoint from the cycles in $\mathcal{X}(c+t, d-t)$ by definition, the drawings of these cycles are not intersected by $\varrho_1 \cup \varrho_2$, and hence these cycles are all drawn inside $\Sigma$. ∎

# 5 Proof of the Reduction

Recall the notation from Section 3, and assume that $G$ is a graph on the vertex set $\{1, 2, \dots, n\}$. Let $H_G$ denote the graph constructed along the description on page 8.

**Lemma 5.1** *If an optimal linear arrangement of a graph $G$ has weight $A$, then the crossing number of the graph $H_G$ is at least*

$$\mathrm{cr}(H_G) \geq (s + rn)nt + 2(A + m)t - 8m\,.$$

*(See* (2) *and Lemma 3.2 for details on the notation.)*

We proceed the proof of Lemma 5.1 along the following sequence of claims. (Actually, all technical work has already been done in the previous section.) Assume that we have an optimal drawing of the graph $H_G$ at hand.

**Lemma 5.2** *In the optimal drawing of $H_G$, the boulders $B_1, B_2$ are drawn with no edge crossings.*

*Proof.* We assume, for a contradiction, that the boulder $B_1$ in $H_G$ is drawn with some edge crossings. Notice that $B_1$ has (2) $z$ pairwise disjoint main cycles by definition, and one edge crossing in $H_G$ may involve at most two of them. Since the total number of crossings is less than $z/2$ by Corollary 3.3, we conclude that some main cycle $N \subset B_1$ is drawn with no crossings. Without loss of generality, we may suppose that the subgraph $H_G - V(B_1)$ is drawn in the exterior face of $N$.

In this situation we redraw the whole boulder $B_1$ in the interior face of the original drawing of $N$, such that the first main cycle $N_1 \subset B_1$ coincides with the original cycle $N$. Then we prolong the edges of the original drawing of $H_G - E(B_1)$ that were attached to $N_1$ of $B_1$ along the original drawings of (pairwise disjoint) paths in $B_1$ connecting those edges to $N$. In this way we introduce no new crossings to the drawing of $H_G$, and we eliminate previous crossings on edges of $B_1$, which contradicts optimality of the original drawing. ∎

Hence, in particular, the first main cycles $N_j$ of the boulder $B_j$, $j = 1, 2$ are drawn with no crossings. Then there is a uniquely defined cylinder $\Pi$ with the boundary curves $N_1$ and $N_2$ in the plane. Realize that the whole subgraph $H_G - V(B_1) - V(B_2)$ is drawn on $\Pi$.

**Lemma 5.3** *In the optimal drawing of $H_G$, each main cycle $M$ of every ring $R_i$, $i \in \{1, 2, \dots, n\}$ is drawn as a closed curve separating the subdrawing of the boulder $B_1$ from the subdrawing of $B_2$.*

*Proof.* Suppose, for a contradiction, that the claim is false for a main cycle $M \subset R_h$. Instead of the plane, let us consider the cylinder $\Pi$. Then our contradiction says that $M$ is drawn as a contractible curve on $\Pi$.

We are going to apply Lemma 4.1 in this situation. Let (2) $k = r$ and $\ell = m^3$. For $0 \le i < m^3$ and $0 \le j < r$, we denote by $\sigma_{i+1+j(m^3+1)}$ the drawing of the $(i + jm^3)$-th free spoke – the edge $\epsilon^{\mathrm{s}}_{i+jm^3}$ of $H_{m,n}$ (page 6). Further for $0 \le j < r$, we denote by $\sigma_{j(m^3+1)}$ the drawing of a path $S_j$ associated with the $j$-th ring spoke of the ring $R_h$: $S_j$ consists of the edges $\epsilon^{\mathrm{p1}}_{h,j}, \epsilon^{\mathrm{r1}}_{h,j}$ and $\epsilon^{\mathrm{r3}}_{h,j}, \epsilon^{\mathrm{p3}}_{h,j}$, and of (one of) the shortest path connecting the ends of $\epsilon^{\mathrm{r1}}_{h,j}, \epsilon^{\mathrm{r3}}_{h,j}$ across the ring $R_h$.

One may easily verify that the above collection $\mathcal{S} = \{\sigma_j : 0 \le j < r(m^3+1)\}$ and $\tau = M$ satisfy the assumptions of Lemma 4.1. It follows from Lemma 5.2 that the ends of the curves in $\mathcal{S}$ are ordered on the boundaries of $\Pi$ as required, and that $\tau$ is drawn in the interior of $\Pi$. Naturally, $\tau$ intersects the drawings of each of the paths $S_j$ since $M$ shares a vertex with $S_j$. Moreover, a subcollection of disjoint paths $\mathcal{S}$ in a proper optimal drawing of a graph forms a nice set of curves by definition, and the same applies to the set $(\mathcal{S} \setminus \mathcal{S}_0) \cup \{\tau\}$ as in Lemma 4.1.

If $\tau$ was contractible on $\Pi$, and if the second possibility in Lemma 4.1 happened, then the number of crossings in the drawing of $H_G$ would be, using (3), at least

$$(\frac{2}{25}k^2 - \frac{1}{5}k)\ell = (\frac{2}{25}r^2 - \frac{1}{5}r)m^3 = (\frac{32}{25}m^4n^4 - \frac{4}{5}m^2n^2)m^3 =$$

$$= \frac{32}{25}m^7n^4 - \frac{4}{5}m^5n^2 > m^7n^4 > z\,,$$

which contradicts Corollary 3.3. Hence the first conclusion of the lemma should be true, and there are at least

$$2 \cdot \frac{3}{5}k\ell = \frac{6}{5}rm^3 = \frac{6}{5}s = s + \frac{1}{5}rm^3 > s + rn + \frac{1}{8}rm^3 \tag{4}$$

crossings on the edges of $M$ using (3). (Notice that we have not even considered crossings of $M$ with the ring spokes of other rings than of $R_h$ in this inequality.)

The above inequality (4) applies to every main cycle $M$ in $H_G$ which is drawn contractible on $\Pi$, while the noncontractible main cycles clearly have each at least $s + rn$ crossings with all the spokes in $H_G$. Thus the total number of crossings in our drawing of $H_G$ is at least

$$s + rn + \frac{1}{8}rm^3 + (nt-1)(s+rn) =$$

$$= (m^3+n)rnt + \frac{1}{8}rm^3 > (m^3+n)rnt + r = \frac{1}{2}z\,,$$

which again contradicts Corollary 3.3. Hence, indeed, every main cycle $M$ of $H_G$ must be drawn as a noncontractible closed curve on $\Pi$ in the optimal drawing of $H_G$, and so $M$ separates $B_1$ from $B_2$. ∎

**Corollary 5.4** *In the optimal drawing of $H_G$, there are at least $(s+rn)nt$ crossings between edges of the main cycles of the rings and edges of the free and ring spokes in $H_G$.*

∎

**Lemma 5.5** *There is a selection of main cycles $M_i \subset R_i$, $i = 1, 2, \ldots, n$ of the rings in $H_G$, such that the cycles $M_1, \ldots, M_n$ are drawn as pairwise disjoint closed curves in the above optimal drawing of $H_G$. Moreover, there is a permutation $\pi$ of $\{1, \ldots, n\}$ such that, for each $j = 1, \ldots, n$, the closed curve $M_{\pi(j)}$ separates the subdrawing $B_1 \cup M_{\pi(1)} \cup \ldots \cup M_{\pi(j-1)}$ from the subdrawing $B_2 \cup M_{\pi(j+1)} \cup \ldots \cup M_{\pi(n)}$.*

*Proof.* Combining Corollaries 3.3 and 5.4, we see that there are less than $r = t^2$ crossings between pairs of main cycles of the rings in the optimal drawing of $H_G$. Let us, for $i = 1, \dots, n$, form a collection $\mathcal{M}_i$ of closed curves – the drawings of the $t$ main cycles of the ring $R_i$. Then we apply Lemma 4.2, and hence we find pairwise disjoint representatives $M_i \in \mathcal{M}_i$, as desired.

The second part then naturally follows from Lemma 5.3 and Jordan's curve theorem. ∎

**Lemma 5.6** *For every $k = 0, 1, \dots, 4n^2 - 1$, there is an index $c_k \in C_k = \{km^5 - 2m^4, \dots, km^5 + 2m^4\}$ such that the edge of the $c_k$-th free spoke $\epsilon^s_{c_k}$ is crossed exactly once by each of the main cycles of all the rings, and that $\epsilon^s_{c_k}$ has no more crossings than these in the optimal drawing of $H_G$.*

*Proof.* By Lemma 5.3, each of the main cycles crosses each of the $s + rn$ spokes in the optimal drawing of $H_G$. Suppose, for a contradiction, that for every $j \in C_k$ as above, $|C_k| = 4m^4 + 1$, the $j$-th free spoke has at least two crossings with some main cycle in $H_G$. Then such a drawing of $H_G$ would have at least

$$(s + rn)nt + 4m^4 + 1 > (s + rn)nt + r = z/2$$

edge crossings, which is a contradiction to Corollary 3.3. ∎

Recall that the vertices of $G$ are numbered as $\{1, 2, \dots, n\}$, and that $X_{i,j}$ denotes the subgraph of the handle in the constructed graph $H_G$ corresponding to an edge $ij \in E(G)$ (page 8).

**Lemma 5.7** *Let $\pi$ be the permutation from Lemma 5.5, let $\Pi$ be the cylinder defined after Lemma 5.2 for the optimal drawing of $H_G$, and let $\{i, j\} \in E(G)$ be an edge. For $\ell = i + n(j - 1)$, consider the indices $c_{4\ell-2}$ and $c_{4\ell+2}$ given by Lemma 5.6, and denote by $\Sigma_\ell$ the region on $\Pi$ bounded by the drawings of the $c_{4\ell-2}$, $c_{4\ell+2}$-th free spokes and containing the subdrawing of the handle $X_{i,j}$. Then $\Sigma_\ell$ contains at least*

$$t\left(|\pi^{-1}(i) - \pi^{-1}(j)| - 1\right)$$

*crossings between edges of the subgraph $X_{i,j} \cup R_i \cup R_j$ and edges of the main cycles of other rings $R_k$ in $H_G$ for $k \neq i, j$.*

*Proof.* First, notice that $\Sigma_\ell$ is well defined since the drawings of the $c_{4\ell-2}$-th and of the $c_{4\ell+2}$-th free spokes are disjoint by Lemma 5.6, they do not cross

$X_{i,j}$, and each of them connects two points on the opposite boundaries of $\Pi$. Moreover, by an analogous argument, the drawings of the $c_{4\ell-1}$-th and $c_{4\ell+1}$-th free spokes divide $\Sigma_\ell$ into three topological components $\Sigma_\ell^1, \Sigma_\ell^2, \Sigma_\ell^3$ in this order, such that $X_{i,j}$ is drawn inside $\Sigma_\ell^2$.

We denote by $H'_G$ the subgraph of $H_G$ obtained by deleting the two boulders and all the free and ring spokes. Then, by Corollaries 3.3 and 5.4, the subdrawing of $H'_G$ contains less than $r$ edge crossings. Consider now a ring $R_k$ of $H_G$, for which $\pi^{-1}(i) < \pi^{-1}(k) < \pi^{-1}(j)$ (up to symmetry). By Lemma 5.5, there are main cycles $M_i \subset R_i$, $M_j \subset R_j$, $M_k \subset R_k$ such that the drawing of $M_k$ separates the drawings of $M_i$ and $M_j$ from each other on $\Pi$. Denote by $M_k^b \subset R_k$ the $b$-th main cycle of the ring $R_k$.

Recall that $c_{4\ell-2} \le (4\ell m^2 - 2m^2 + 2m)m^3$, and $c_{4\ell-1} \ge (4\ell m^2 - m^2 - 2m)m^3$. So Lemma 5.6 also implies that the $(4\ell m^2 - 2m^2 + 3m - 1)$-th and $(4\ell m^2 - m^2 - 3m + 1)$-th ring spokes of $R_k$ (page 6) are both drawn inside $\Sigma_\ell^1$. Here we restrict the notation from the definition of a cyclic cubic grid just to the ring $R_k$. We set $c = (4\ell m^2 - 2m^2 + 3m)(m^3 + n + 1)$ and $d = (4\ell m^2 - m^2 - 3m)(m^3 + n + 1)$. The previous argument implies that the vertices $v_{1,c}, v_{1,d}$ of the first main cycle $M_k^1$ of $R_k$ are also drawn inside $\Sigma_\ell^1$. Hence the situation corresponds with the setting of Lemma 4.4, and we conclude that all the cycles of $R_k$ in the set $\mathcal{X}_1 = \mathcal{X}(c+t, d-t)$ (defined in Section 4) are drawn inside $\Sigma_\ell^1$.

Let us now estimate, using $(2,3)$:

$$d - c = (m^2 - 6m)(m^3 + n + 1) > m^5 - 6 \cdot 3m^4 >$$

$$> 100m^4 - 18m^4 > 16m^4 + 4m^4 > 4r + 2t$$

So $|\mathcal{X}_1| = \frac{1}{2}(d-c-2t) > 2r$. Recall that the subdrawing of $H'_G$ has less than $r$ edge crossings. Since one edge crossing may involve at most two of the cycles from $\mathcal{X}_1$, there exists a cycle $C_1 \in \mathcal{X}_1$ which has no edge crossed in the subdrawing of $H'_G$. We analogously find a symmetric cycle $C_2$ drawn inside the region $\Sigma_\ell^3$, and $C_2$ having no edge crossed in the subdrawing of $H'_G$. (See in Figure 8.)

Consider now the drawing of the connected subgraph $R_i \cup R_j \cup X_{i,j} \subset H'_G$ which is disjoint from the drawings of $C_1$ and $C_2$. So it follows from our assumptions, and from $X_{i,j}$ being drawn inside $\Sigma_\ell^2$, that the drawing of $R_i \cup R_j \cup X_{i,j}$ separates the drawings of $C_1$ and of $C_2$ from each other in $\Sigma$. We denote by $Q \subset M_k^b$ the shortest path connecting a vertex on $C_1$ to a vertex on $C_2$ and drawn in $\Sigma$. ($Q$ is uniquely defined by our assumptions.)

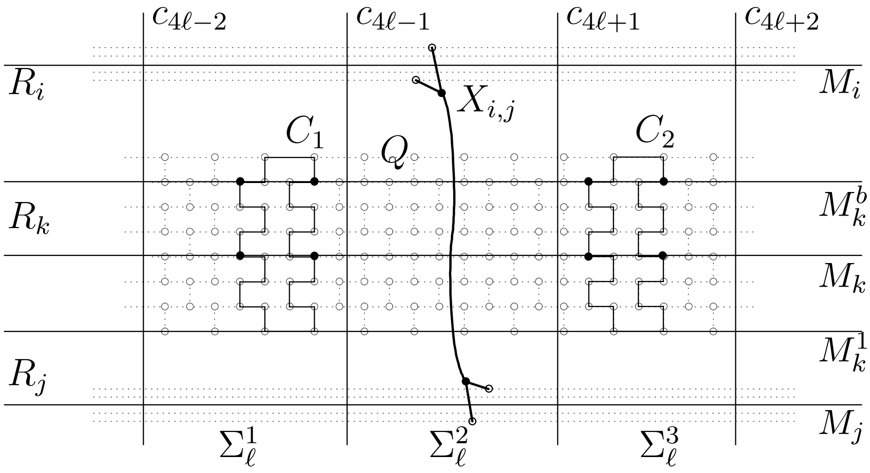


Figure 8: An illustration to the proof of Lemma 5.7.

Then $Q$ crosses the drawing of $R_i \cup R_j \cup X_{i,j}$ by Jordan's curve theorem. In this way we find distinct edge crossings for all choices of $k$ such that $\pi^{-1}(i) < \pi^{-1}(k) < \pi^{-1}(j)$, and for all $t$ choices of the main cycle $M_k^b \subset R_k$. The statement now follows. ∎

And now we are ready to finish the proofs of Lemma 5.1 and of Theorem 2.1:

*Proof of Lemma 5.1.* We are going to count three collections of edge crossings in the optimal drawing of $H_G$. These collections are pairwise disjoint since they involve different pairs of edges of $H_G$, as one may easily check. Firstly, there are (at least) $(s + rn)nt$ crossings described in Corollary 5.4.

Secondly, denote by $d_i$ the degree of the vertex $i$ in $G$. Let us consider the subgraph $F_i$ of $H_G$ formed by the ring $R_i$ and by $2d_i$ pairs of incident edges from all handles which are attached to $R_i$ in $H_G$. Then, by Lemma 4.3, the subgraph $F_i$ itself has at least $2d_i(t-2)$ edge crossings in any drawing of $H_G$.

Thirdly, the permutation $\pi$ from Lemma 5.5 defines a linear arrangement $\alpha = \pi^{-1}$ of the vertices of $G$. (An edge $\{i, j\} \in E(G)$ contributes with $|\alpha(i) - \alpha(j)|$ to the total weight of the arrangement $\alpha$ on $G$ (1).) Recall the notation and conclusion of Lemma 5.7: An edge $\{i, j\}$ of $G$ contributes (via its two handles in $H_G$) with at least $2t(|\alpha(i) - \alpha(j)| - 1)$ crossings in $H_G$ which are contained in the regions $\Sigma_\ell$ and $\Sigma_{\ell'}$, where $\ell = i - 1 + n(j - 1)$ and $\ell' = j - 1 + n(i - 1)$. So in particular, the sets of crossings accounted here for distinct edges of $G$ are pairwise disjoint, and also disjoint from the crossings contributed by the subgraphs $F_i$ above.

Altogether, we have found at least this many distinct edge crossings in the optimal drawing of $H_G$:

$$(s+rn)nt + \sum_{i\in V(G)} 2d_i(t-2) + \sum_{\{i,j\}\in E(G)} 2t(|\alpha(i)-\alpha(j)|-1) =$$

$$= (s+rn)nt + 2t \sum_{\{i,j\}\in E(G)} |\alpha(i)-\alpha(j)| - 2tm + 4tm - 8m =$$

$$= (s+rn)nt + 2tA + 2tm - 8m$$

∎

*Proof of Theorem 2.1.* Assume that $G$, $a$ is an input instance of the OptimalLinearArrangement problem, and that $G$ is sufficiently large (3). The above described graph $H_G$ is clearly cubic, it has polynomial size in $n = |V(G)|$, and $H_G$ has been constructed efficiently. We now ask the problem CrossingNumber on the input $H_G$, $(s+rn)nt + 2t(a+m)$, and give the same answer to OptimalLinearArrangement on $G, a$.

If there is a linear arrangement of $G$ of weight at most $a$, then the correct answer is YES according to Lemma 3.2. Conversely, if the optimal linear arrangement of $G$ has weight greater than $a$, then the crossing number of $H_G$ is by Lemma 5.1

$$\mathrm{cr}(H_G) \geq (s+rn)nt + 2t(a+1+m) - 8m >$$

$$> (s+rn)nt + 2t(a+m)\,,$$

and so the correct answer is NO. Since the OptimalLinearArrangement problem is known to be $NP$-complete [2], the statement of Theorem 2.1 follows. ∎

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