EPIMORPHISMS OF UNIFORM FRAMES

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Abstract. It is shown that some familiar properties of epimorphisms in the category of frames cary over to the category of uniform frames. This is achieved by suitably enriching certain frame homomorphisms to uniform frame homomorphisms.

This note deals with the natural question, apparently never considered so far, whether certain familiar facts concerning epimorphisms, and specifically epi-extensions, of frames also hold for uniform frames. We shall show this is indeed the case by establishing the following results.

There are uniform frames with arbitrarily large epi-extensions.
Whenever the underlying frame of a uniform frame has arbitrarily large epi-extensions, the same holds for the uniform frame itself.
A uniform frame has no proper epi-extensions iff it is a Boolean frame with its largest uniformity.

Of course, the first of these assertions may readily be obtained as a consequence of the second but since its proof is considerably more direct than that of the latter it seemed worthwhile to include it.

For general background of frames we refer to Johnstone [4] or Vickers [8], and for uniform frames to the original paper by Isbel [3] or the more recent Banaschewski [1].

We begin by recalling the relevant basic facts concerning epimorphisms of frames. The crucial construction here is the embedding $L \to \mathfrak{C}L$ of any frame L into the frame $\mathfrak{C}L$ of its *congruences*, these being the equivalence relations on L which are subframes of $L \times L$, otherwise characterized as the kernel relations of the homomorphisms

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$L \to M$. $\mathfrak{C}L$ is *generated* by the congruences

$$\nabla_a = \{(x,y) \in L \times L \mid x \vee a = y \vee a\} \quad \text{and}$$
$$\Delta_a = \{(x,y) \in L \times L \mid x \wedge a = y \wedge a\},$$

for each $a \in L$, and ∇_a and Δ_a are complements of each other in $\mathfrak{C}L$. In particular, then, $\mathfrak{C}L$ is zero-dimensional. Further, the map $\gamma_L : L \to \mathfrak{C}L$ taking $a \in L$ to ∇_a is a frame homomorphism, evidently one-one and epic – the latter since $f(\nabla_a) = g(\nabla_a)$ implies $f(\Delta_a) = g(\Delta_a)$ for any frame homomorphisms $f, g : \mathfrak{C}L \to M$ since complements are unique and preserved by homomorphisms. We note that $\mathfrak{C}L$ with the embedding $\gamma_L : L \to \mathfrak{C}L$ is characterized as the universal extension of L in which each element of L is complemented (Joyal - Tierney [5]).

The correspondence $L \mapsto \mathfrak{C}L$ is functorial and the functor $\mathfrak{C}$ can be iterated transfinitely such that

$$\mathfrak{C}^0 L = L, \quad \mathfrak{C}^{\alpha+1} L = \mathfrak{C}(\mathfrak{C}^\alpha L) \text{ for any } \alpha,$$
$$\mathfrak{C}^\lambda L = \varinjlim_{\alpha<\lambda} \mathfrak{C}^\alpha \text{ for any limit ordinal } \lambda.$$

Moreover, as simple induction shows, all the further frames resulting here are zero-dimensional. Similarly, the original embeddings $\gamma_L : L \to \mathfrak{C}L$ determine corresponding $\gamma_L^\alpha : L \to \mathfrak{C}^\alpha L$ in the obvious way, and each of these is an epi-embedding.

Turning now to uniform frames, a uniform frame homomorphism $L \to M$ which is one-one (that is, just as a set map) will be called a *uniform extension* of L; further, if it is an epimorphism in the category **UFrm** of uniform frames as well it will be referred to as a uniform epi-extension.

Now we have

Proposition 1. *There are uniform frames which have arbitrarily large epi-extensions.*

Proof. Let L be any zero-dimensional frame and view each $\mathfrak{C}^\alpha L$ as a uniform frame with the uniformity generated by its *finite partitions*, that is, the finite covers consisting of pairwise disjoint elements. Then, evidently, each $\gamma_L^\alpha : L \to \mathfrak{C}^\alpha L$ is a uniform extension of L, trivially epic since this is already the case at the frame level. In particular, if $L = \mathfrak{C}F$ for any infinite free frame F then, as is familiar, the transfinite sequence $\mathfrak{C}^\alpha L$ is strictly increasing because the category of complete Boolean algebras and complete homomorphisms has no infinite free object. $\square$

In the following, we shall make use of a certain characterization of the existence of uniformities on a frame.

For this, recall the relation $\prec\!\!\shortmid$ which is the interpolative part of the familiar rather below (= well inside) relation $\prec$ where $x \prec a$ iff $x^* \vee a = e$, the unit of the frame, for the pseudocomplement x^* of x, and the interpolative part S of any binary relation R is the largest relation $S \subseteq R$ such that $S \circ S \subseteq S$. With this, a frame L is called *strongly regular* iff

$$a = \bigvee\{x \in L \mid x \prec\!\!\shortmid a\}$$

for each $a \in L$, and a frame has this property iff it has a uniformity (Banaschewski - Pultr [2]).

We note in passing that, with the Axiom of Countable Dependent Choice, strong regularity coincides with complete regularity, and the corresponding characterization of the existence of uniformities is a long-established fact (Pultr [7]). The advantage of the present notion is that it modifies this characterization so that it becomes constructively valid.

Regarding uniformities on a frame L, it is clear that any set of these generates a further uniformity and consequently any strongly regular frame has a largest uniformity, referred to as its *fine uniformity*, in line with the terminology for topological spaces.

Lemma 1. *For any uniform frame L, if $h : L \to M$ is a homomorphism of its underlying frame to a strongly regular frame then h is uniform with respect to the fine uniformity $\mathfrak{W}$ of M.*

Proof. The image covers $h[C]$, C any uniform cover of L, may not define a uniformity on M but by the properties which they do inherit from the uniformity of L, the corresponding covers

$$h[C] \wedge D = \{h(c) \wedge d \mid c \in C, d \in D\}$$

for $D \in \mathfrak{W}$ generate such a uniformity (which must be $\mathfrak{W}$ again), showing that $h[C] \in \mathfrak{W}$ for each uniform cover C of L, as claimed. $\square$

Proposition 2. *It the underlying frame of a uniform frame L has arbitrarily large epi-extensions then the same holds for L itself.*

Proof. By Madden - Molitor [6] the transfinite sequence $\mathfrak{C}^\alpha L$ (allowing notational confusion of L with its underlying frame) is strictly increasing, and if $\mathfrak{C}^\alpha L$, $\alpha \geq 1$, is taken as uniform frame with its fine uniformity (zero-dimensional implies strongly regular!) then $\gamma_L^\alpha : L \to \mathfrak{C}^\alpha L$ is a uniform epi-extension by Lemma 1. $\square$

In order to deal with the third assertion stated at the beginning we first have to relate the epimorphisms of uniform frames to those of frames.

Trivially, as was already used, a uniform frame homomorphism which is epic as frame homomorphism is also epic as uniform frame homomorphism. On the other hand, the somewhat less obvious converse also holds so that we have the following

Lemma 2. *A uniform frame homomorphism is epic iff it is epic as frame homomorphism.*

Proof. To show the missing ($\Rightarrow$), let $h : L \to M$ be any epimorphism of uniform frames and $f, g : M \to N$ any frame homomorphisms such that $fh = gh$. Now, the underlying frame of M is strongly regular, and since taking coproducts and quotients of frames preserves this property (Banaschewski - Pultr [2]) the subframe K of N generated by $\mathrm{Im}(f) \cup \mathrm{Im}(g)$ is also strongly regular. Consequently, Lemma 1 shows that the corestrictions $\overline{f}, \overline{g} : M \to K$ of f and g, respectively, are uniform homomorphisms for K taken with its fine uniformity. Further, $\overline{f}h = \overline{g}h$ so that $\overline{f} = \overline{g}$ by hypotesis, and hence $f = g$ as desired. $\square$

In line with common terminology, a uniform frame L will be called *epi-complete* if any uniform epi-extension $L \to M$ is an isomorphism. Then we have the following counterpart to a familiar result for frames (Madden - Molitor [6]).

Proposition 3. *A uniform frame is epicomplete iff it is a Boolean frame with its fine uniformity.*

Proof. ($\Rightarrow$) As before, $\gamma_L : L \to \mathfrak{C}L$ is a uniform epi-extension of the uniform frame L if $\mathfrak{C}L$ is equipped with its fine uniformity; this makes γ_L an isomorphism, and therefore L is of the stated kind.

($\Leftarrow$) Any epi-extension $L \to M$ of uniform frames is epic as frame homomorphism by Lemma 2, hence an isomorphism for the underlying frames by Madden - Molitor [6], and then a uniform isomorphism because the uniformity of L is fine. $\square$

Remark. The fine uniformity of a Boolean frame L consists of all covers of L provided the Axiom of Choice is assumed; given this, any cover of L is refined by a partition which, in turn, is its own star refinement. We do not know what happens without this assumption.

In closing we note by way of contrast that, in the case of *metric frames*, the epimorphisms have quite different properties. Thus, in the category of these frames and their contractive homomorphisms, any epicomplete object is *atomic* Boolean, with the metric diameter which has value $+\infty$ for all non-atoms different from 0. Moreover, it is conjectured that the converse of this also holds, but that remains as yet to be proved. We omit the details.

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References

[1] B. Banaschewski, *Frames and compactifications,*
[2] B. Banaschewski and A. Pultr, *A constructive view of complete regularity,* Kyungpook Math. Journal **43** (2003), 257-262.
[3] J.R. Isbell, *Atomless parts of spaces,* Math. Scand. **31** (1972), 5-32.
[4] P.T. Johnstone, *Stone spaces,* Cambridge Sudies in Advanced Math. No 3 (Camb. Univ. Press 1983).
[5] A. Joyal and M. Tierney, *An extension of the Galois theory of Grothendieck,* Memoirs of the AMS **309** (1984).
[6] J.J. Madden and A. Molitor, *Epimorphisms of frames,* J. Pure Appl. Alg. **70** (1991), 129-132.
[7] A. Pultr, *Pointless uniformities I,* Comment. Math. Univ. Carolinae **25** (1984), 91-104.
[8] S. Vickers, *"Topology via Logic",* Cambridge Tracts in Theoretical Computer Science, No. 5, Cambridge University Press, Cambridge, 1985.

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