Metric Spaces are Ramsey

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Abstract

We prove that the class of all finite ordered metric spaces is a Ramsey class. This solves a problem of Kechris, Pestov and Todorćevic.

1 Ramsey Classes

Let $\mathcal{K}$ be a class of objects which is isomorphism closed and endowed with subobjects. Given two objects $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ we denote by $\binom{\mathbf{B}}{\mathbf{A}}$ the set of all subobjects $\mathbf{A}'$ of $\mathbf{B}$ which are isomorphic to $\mathbf{A}$. (Thus in this notation the rôle of $\mathcal{K}$ is suppressed. It should be always clear from the context.) We say that the class $\mathcal{K}$ has $\mathbf{A}$-*Ramsey property* if the following statement holds:

For every positive integer k and for every $\mathbf{B} \in \mathcal{K}$ there exists $\mathbf{C} \in \mathcal{K}$ such that $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_k$. Here the last symbol (*Erdős-Rado partition arrow*) has the following meaning:

For every partition $\binom{\mathbf{C}}{\mathbf{A}} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \ldots \cup \mathcal{A}_k$ there exists $\mathbf{B}' \in \binom{\mathbf{C}}{\mathbf{B}}$ and an $i, 1 \leq i \leq k$ such that $\binom{\mathbf{B}'}{\mathbf{A}} \subset \mathcal{A}_i$.

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In the extremal case that a class $\mathcal{K}$ has **A**-Ramsey property for every its object **A** we say that $\mathcal{K}$ is a *Ramsey class.*

These notions crystallized in the early seventies, see e.g. [9, 18, 3]. This formalism and the natural questions it motivated essentially contributed to create establish Ramsey theory as a "theory" (as nicely put in the introduction to [4]).

The notion of a Ramsey class is highly structured and in a sense it is the top of the line of the Ramsey notions ("one can partition everything in any number of classes to get anything homogeneous"). Consequently there are not many (essentially different) examples of Ramsey classes known.

Examples of Ramsey classes include

i. The class of all finite ordered graphs;

ii. The class of all finite partially ordered sets (with a fixed linear extension);

iii. The class of all finite vector spaces (over a fixed field F).

iv. The class of all (labeled) finite partitions.

For these results see [3, 4, 16, 12]. We formulate explicitly one of the most general results (for relational structures) which will be needed in the sequel:

Let I be a finite set of real numbers and $\Delta = (\delta_i; i \in I)$ be a sequence of natural numbers. Δ is called the type (or signature). We consider objects *ordered relational structures* of the form $\mathbf{A} = (X, (R_i; i \in I))$ where X is a non-empty ordered set and $R_i \subseteq X^{\delta_i}$ (i.e. R_i is a δ_i-nary relation). We also denote the type of **A** by $\Delta(\mathbf{A}) = \Delta = (\delta_i; i \in I)$, the underlying set (vertices) of **A** by $\underline{\mathbf{A}} = X$ (sometimes we simply denote the set of vertices as **A**) the relations by $R_i(\mathbf{A}) = R_i$.

We denote by *Rel* the class of all such ordered relational structures **A** of all possible (finite) types Δ. The class *Rel* will be considered with embeddings (corresponding to induced substructures): Given relational structures $\mathbf{A} = (X, (R_i; i \in I))$ and $\mathbf{A}' = (X', (R'_{i'}; i' \in I'))$ of types Δ and $\Delta' = (\delta'_{i'}; i' \in I')$ (note that the types Δ and Δ' may be different) a mapping $f : X \longrightarrow X'$ is called an *embedding* of A into A' if $I \subset I'$ and $\delta_i = \delta'_i$ for every $i \in I$ (i.e. Δ is a subtype of Δ') and if f is a monotone injection of X into X' satisfying $(f(x_j); j = 1, \dots, \delta_i') \in R'_i$ iff $(x_j; j = 1, \dots, \delta_i) \in R_i$.

As usual, an inclusion (or bijective) embedding is called *substructure* (or *isomorphism*). Given two ordered relational structures $\mathbf{A}, \mathbf{B}$ we denote by $\binom{\mathbf{B}}{\mathbf{A}}$ the class of all substructures $\mathbf{A}'$ of $\mathbf{B}$ which are isomorphic to $\mathbf{A}$. One more definition: For real numbers $d, D, 0 < d < D$, we denote by $Rel(d, D)$ the subclass of Rel induced by all systems $\mathbf{A} = (X, (R_i; i \in I))$ where I is a subset of the interval $[d, D]$. We have the following

Theorem 1.1 ([14]) *For every choice of reals $d, D, 0 < d < D$ the class $Rel(d, D)$ is a Ramsey class.*

Explicitly: For every choice of a natural number k and of structures $\mathbf{A}, \mathbf{B} \in Rel(d, D)$ there exists a structure $\mathbf{C} \in Rel(d, D)$ with the following property: For every partition $\binom{\mathbf{C}}{\mathbf{A}} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \ldots \cup \mathcal{A}_k$ there exists $i, 1 \le i \le k$, and a substructure $\mathbf{B}' \in \binom{\mathbf{C}}{\mathbf{B}}$ such that $\binom{\mathbf{B}'}{\mathbf{A}} \subset \mathcal{A}_i$.

In [14, 13, 12] (and elsewhere) Theorem1.1 is stated in its equivalent form as a Ramsey theorem for classes $Rel(\Delta)$ of all ordered relational structures of a fixed type Δ. We shall make use of Theorem 1.1 in our proof of the following theorem which is the main result of this paper:

Theorem 1.2 *The class of all finite ordered metric spaces is a Ramsey class.*

Let us formulate Theorem 1.2 explicitly:

Denote by $\mathcal{M}$ the class of all finite ordered metric spaces (i.e. metric spaces where the set of points is linearly ordered). An embedding will be monotone isometry. We claim that $\mathcal{M}$ is a Ramsey class: For every choice of ordered metric spaces $(X, \rho), (Y, \sigma)$ (ordering is not indicated) there exists a metric space (Z, λ) such that

$$(Z, \lambda) \longrightarrow (Y, \sigma)_2^{(X,\rho)}.$$

(To keep the notation simple from now on we shall consider only 2-colorings. Colorings using more colors can be reduced to 2-colorings by iterating.)

Theorem 1.2 solves a problem of Kechris, Pestov and Todorćevic, see [6]. The paper [6] lists several consequences of Theorem 1.2 to dynamical systems and topological groups (extremal amenable groups, minimal flows). This also implies a remarkable property of the *Urysohn Space* which is defined as a completion of the homogeneous universal rational metric space, see e.g. [21, 22]. (The author, himself student of Katětov, cannot resist mentioning that this construction was one of the last results of both Urysohn [20] and Katětov [5].) Theorem 1.2 also generalizes Ramsey theorem for pairs in metric spaces stated in [13].

Theorem 1.2 will be proved as a consequence of a more technical form stated in Sections 2 and 3. Here is the outline of our proof:

We view any metric space (X, ρ) as a labeled complete graph and this in turn may be viewed as a *binary* relational system of type $\Delta = (\delta_i; i \in I)$ where $\delta_i = 2$: we put $(x, y) \in R_i$ iff $\rho(x, y) = i$. (Thus I is the set of all possible distances in (X, ρ), an $i \in I$ may be viewed as the length (or weight) of edge (x, y).) Clearly not every binary relational system corresponds to a metric space (we need symmetry and triangle inequality). But every binary relational system $\mathbf{A}$ may be converted to a metric space $(\underline{\mathbf{A}}, \sigma_{\mathbf{A}})$ by defining $\rho_{\mathbf{A}}(x, y)$ as the minimal (weighted) length of a path from x to y in $\mathbf{A}$. $\rho_{\mathbf{A}}$ is also called *free metric* generated by $\mathbf{A}$. We denote by $F(\mathbf{A})$ the binary relational system corresponding to the metric space $(\underline{\mathbf{A}}, \rho_{\mathbf{A}})$. Clearly it can (only too often) happen that a binary relational system $\mathbf{B}$ satisfies $\mathbf{B} \to \mathbf{A}$ while $\mathbf{B} \not\to F(\mathbf{A})$. Thus we shall introduce the notion of *ℓ-approximation system.*We then prove by induction on ℓ a Ramsey type theorem for ordered ℓ- approximation systems (Theorem 2.1). On the other hand, for each (fixed) $\mathbf{B}$ there exists ℓ such that $\mathbf{B} \to F(\mathbf{A})$ iff there exists an ℓ-approximation embedding of $\mathbf{B}$ into $\mathbf{A}$. This can then be used to prove that Ramsey theorems for ℓ-approximation systems implies Theorem 1.2.

The paper is organized as follows: In Section 2 we state the Theorem 1.2 in a more technical form and introduce classes $Rel_{(\ell)}(d, D)$ of ℓ-approximation systems (and a given range of edge lengths). In Sections 3 and 4 we further refine the classes $Rel_{(\ell)}(d, D)$ to classes $PartiRel_{(\ell)}(d, D)$ and prove $\mathbf{A}$-Ramsey property by a variant of *amalgamation technique* (known also as *Partite Construction*) see [19, 17, 16, 12]. This then implies Theorem 1.2. Section 5 contains concluding remarks and some related results.

2 Metric Approximation

Let $d < D$ be positive real numbers, ℓ a positive integer. Before defining objects and morphisms of our classes we take time out for a definition: We say that $(x, y) \in R_i$ is *ℓ-metric edge* in $\mathbf{A} = (X, (R_i; i \in I))$ if for any path $x = x_0, x_1, \ldots, x_t, t \leq \ell$, with lengths of edges $i_1, i_2, \ldots, i_t$ (i.e. we assume $\rho(x_{j-1}, x_j) = i_j$) holds $i \leq i_1 + i_2 + \ldots + i_t$. A pair (x, y) which is ℓ-metric for every ℓ is called a *metric edge*.

We shall define the class $Rel_{(\ell)}(d, D)$ as follows:

Objects of $Rel_{(\ell)}(d, D)$ (called *approximative systems* and usually denoted by $\mathbf{A}, \mathbf{B}, \ldots$) are those objects $\mathbf{A} = (X, (R_i; i \in I))$ of the class $Rel(d, D)$ which satisfy the following additional properties:

i. $R_i \subseteq X^2$ for each $i \in I$;

ii. all relations R_i are symmetric and antireflexive and $R_i \cap R_j = \emptyset$ whenever $i \neq j \in I$.

iii. every edge of $\mathbf{A}$ is ℓ-metric.

Thus the objects are relational structures of the type Δ where $\Delta = (2, 2, \ldots, 2)$ (a sequence of 2's indexed by a set of real numbers which we may interpret as lengths (or weights) of edges; these lengths will be denoted by ρ: if $(x, y) \in R_i$ we also write $\rho(x, y) = i$).

Embeddings of $Rel_{(\ell)}(d, D)$ are inherited from $Rel(d, D)$.

An edge (x, y) which is ℓ-metric for every ℓ is called a *metric edge*. If all pairs of vertices of a system $\mathbf{A}$ are edges and they are metric (and in this case it suffices that they are 2-metric) then of course $\mathbf{A}$ corresponds to a metric space $(\underline{\mathbf{A}}, \rho)$.

Note that the objects $\mathbf{A}, \mathbf{A}'$ of $Rel_{(\ell)}(d, D)$ need not correspond to metric spaces. However lengths of edges of an ℓ-approximate systems cannot be "shortened" by paths of length $\leq \ell$. Thus the larger ℓ-approximate system we have the better approximation of an isometry we get.

The class $Rel_{(\ell)}(d, D)$ will be considered with embeddings and given objects $\mathbf{A}, \mathbf{B}$ we denote again by $\binom{\mathbf{B}}{\mathbf{A}}$ the class of all subobjects of $\mathbf{B}$ which are isomorphic to $\mathbf{A}$. Note also that for $\ell = 1$ the notion of an ℓ-approximative system (and their embeddings) coincide with the notion of relational structures (and their embeddings) – it is $Rel_{(1)}(d, D) = Rel(d, D)$. Thus the following generalizes Theorem 1.1:

Theorem 2.1 *For every metric systems $\mathbf{A}$ and $\mathbf{B}$ in $Rel(d, D)$ and for every positive integer ℓ and every pair of real numbers $0 < d < D$ there exists $\mathbf{C} \in Rel_{(\ell)}(d, D)$ such that in the class $Rel_{(\ell)}(d, D)$ holds*

$$\mathbf{C} \longrightarrow (\mathbf{B})_2^{\mathbf{A}}.$$

We postpone the proof to Section 4. Here we show that Theorem 2.1 implies Theorem 1.2.

Proof. Let $(X,\rho),(Y,\sigma)$ be finite ordered metric spaces. We may assume that (Y,σ) contains an isometric copy of (X,ρ). Put $d=\min\{\sigma(x,y)\}$ and $D=\max\{\sigma(x,y)\}$. Let $\ell\geq D/d$. Let $\mathbf{A}=(X,(R_i;i\in I))$ and $\mathbf{B}=(Y,(S_j;j\in J))$ be binary relational systems corresponding to the metric spaces (X,ρ) and (Y,σ) (thus both systems have all edges metric). By Theorem 2.1 there exists a binary relational system $\mathbf{C}=(Z,(T_k;k\in K))$ which is Ramsey for $\mathbf{A}$ and $\mathbf{B}$ in the class $Rel_{(\ell)}(d,D)$. Let us write this explicitly:

For every partition $\binom{\mathbf{C}}{\mathbf{A}}=\mathcal{A}_1\cup\mathcal{A}_2$ there exists an ℓ-approximation embedding $g:\mathbf{B}\longrightarrow\mathbf{C}$ and $\iota\in\{1,2\}$ such that for all ℓ-approximation embeddings $f:\mathbf{A}\longrightarrow\mathbf{B}$ we have $g\circ f\in\mathcal{A}_\iota$.

In this situation consider the metric space (Z,θ) freely generated by the binary relational system $\mathbf{C}$: we put $\theta(x,y)=\min\{D,\min\{i_1+i_2+\ldots+i_t\}\}$ where the second minimum is taken over all paths $x=x_0,x_1,\ldots,x_t$ where (x_{r-1},x_r) has length i_r. We note that all the θ distances are in the interval $[d,D]$ and thus the corresponding binary system $F(\mathbf{C})$ belongs to $Rel_{(\ell)}(d,D)$. As $\ell\geq D/d$ we have that for every edge (x,y) of $\mathbf{C}$ holds $(x,y)\in R_i$ iff $\theta(x,y)=i$.

$f:A\longrightarrow C$ is an ℓ-embedding iff $f:A\longrightarrow F(\mathbf{C})$ is an embedding iff $f:(X,\rho)\longrightarrow(Z,\theta)$ is an isometry.

Similarly, $f:B\longrightarrow C$ is an ℓ-embedding iff $g:B\longrightarrow F(\mathbf{C})$ is an embedding iff $g:(Y,\sigma)\longrightarrow(Z,\theta)$ is an isometry.

Thus $F(\mathbf{C})$ is a Ramsey (for $\mathbf{A}$ and $\mathbf{B}$) ordered binary relational system which corresponds to a metric space. Thus $F(\mathbf{C})\longrightarrow(\mathbf{B})_2^{\mathbf{A}}$ (in the class $Rel_{(\ell)}(d,D)$) and also $(Z,\theta)\longrightarrow(Y,\sigma)_2^{(X,\rho)}$. This proves Theorem 1.2.

∎

3 Partite Approximative Classes.

Our proof proceeds by a double induction and towards this end we introduce a version of Partite Construction (see Introduction).

We define the class $PartiRel_{(\ell)}(d,D)$ of structures as follows:

An object is a triple $(\mathbf{B},\mathbf{A},\iota)$ where $\mathbf{A},\mathbf{B}$ are ordered binary relational structures $\mathbf{A}\in Rel_{(\ell)}(d,D)$, $\mathbf{B}\in Rel_{(\ell-1)}(d,D)$. Put explicitly
$\mathbf{A}=(X,(R_i;i\in I))$, $\mathbf{B}=(Y,(S_j;j\in J))$, I,J are finite set of reals $I\subset[d,D]$. ι is a monotone homomorphism $\iota:\mathbf{B}\to\mathbf{A}$. Let us define explicitly the properties of ι:

i. If $(x,y) \in S_j$ the $(f(x), f(y)) \in R_j$ (thus $J \subset I$);

ii. For simplicity we shall assume that $\iota : Y \to X$ is not only monotone but also each set $\iota^{-1}(x)$ is an interval in (the ordering of) Y.

We also call $\mathbf{B}$ an $\mathbf{A}$-*partite* (binary relational) system. This looks as a little change. But in fact considering partite ("leveled") systems is the key fact which allows us to derive more complex Ramsey type statements from simpler ones and to start the induction procedure in our case. And for this the key is the definition of morphisms which we define as follows:

Let $(\mathbf{B}, \mathbf{A}, \iota)$ and $(\mathbf{B}', \mathbf{A}', \iota')$ be objects of $PartiRel_{(\ell)}(d,D)$. An embedding is a pair (f, α) with the following properties:

i. $\alpha : \mathbf{A} \to \mathbf{A}'$ is an embedding (in the class $Rel_{(\ell-1)}(d,D)$);

ii. $f : \mathbf{B} \to \mathbf{B}'$ is an embedding (in the class $Rel_{(\ell)}(d,D)$);

iii. $\iota' \circ f = \alpha \circ \iota$.

This means that the mappings f and g commute with ι's as indicated by the following diagram.

$$\begin{tikzcd} \mathbf{B} \arrow[r, "\iota"] \arrow[d, "f"'] & \mathbf{A} \arrow[d, "\alpha"] \\ \mathbf{B}' \arrow[r, "\iota'"] & \mathbf{A}' \end{tikzcd}$$

(Thus an embedding has to preserve parts of $\mathbf{B}$ and $\mathbf{B}'$.)

Consider an object $(\mathbf{A}, \mathbf{B}, \iota) \in PartiRel_{(\ell)}(d,D), \iota : \mathbf{B} \to \mathbf{A}$. If ι is an injective mapping then we say that $\mathbf{B}$ is a *transversal* system. Clearly any $\mathbf{B} \in Rel_{(\ell)}(d,D)$ can be regarded as a transversal system $(\mathbf{B}, \mathbf{B}, \mathbf{1}) \in PartiRel_{(\ell)}(d,D)$ where $\mathbf{1} : \mathbf{B} \longrightarrow \mathbf{B}$) is the identity mapping. Thus we may regard $Rel_{(\ell)}(d,D)$ as a subcategory of $PartiRel_{(\ell)}(d,D)$.

We shall prove the following technical result :

Theorem 3.1 *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be metric systems in* $Rel(d,D)$*. Then for every* ℓ *there exists* $C \in PartiRel_{(\ell)}(d,D)$ *such that (in the class* $PartiRel_{(\ell)}(d,D)$ *holds*

$$\mathbf{C} \longrightarrow (\mathbf{B})_2^{\mathbf{A}}.$$

We could also prove that the classes $Rel_{(\ell)}(d,D)$ and $PartiRel_{(\ell)}(d,D)$ are Ramsey classes. (We want to keep generalities at the minimum and concentrate on the proof of Theorem 1.2 only; we shall publish generalizations of the proof elsewhere.)

4 Proofs

As stated above we apply Partite Construction in the heart of which lies the amalgamation property.

The amalgamation property now takes the following technical form. (To simplify the notation the symbol $\mathbf{1}$ will denote an inclusion mapping or identity mapping.)

Lemma 4.1 *(Amalgamation Lemma)*
Let $\mathbf{C} \in Rel_{(\ell)}(d,D)$, and let $\mathbf{A}$ be a metric subsystem of $\mathbf{C}$ (in $Rel_{(\ell)}(d,D)$), denote by $\mathbf{1} : \mathbf{A} \longrightarrow \mathbf{C}$ the inclusion map. Let for $i = 1,2$ be given systems $(\mathbf{B}_i, \mathbf{C}, \iota_i : \mathbf{B}_i \longrightarrow \mathbf{C}) \in PartiRel_{(\ell+1)}(d,D)$. Let $(\mathbf{B}_0, \mathbf{A}, \iota_0 : \mathbf{B}_0 \longrightarrow \mathbf{A})$ be a system with embeddings $(f_i, \mathbf{1}) : (\mathbf{B}_0, \mathbf{A}, \iota_0) \longrightarrow (\mathbf{B}_i, \mathbf{A}, \iota_i), i = 1,2$ in $PartiRel_{(\ell)}(d,D)$. Then there exists $(\mathbf{B}_3, \mathbf{C}, \iota_3) \in PartiRel_{(\ell+1)}(d,D)$ and embedding $(g_i, \mathbf{1}) : (\mathbf{B}_i, \mathbf{C}, \iota_i) \longrightarrow (\mathbf{B}_3, \mathbf{C}, \iota_3) \in PartiRel_{(\ell+1)}(d,D)$ such that $(g_i, \mathbf{1})$ is an amalgam of $(f_i, \mathbf{1}), i = 1,2$. Explicitly, we have $g_1 \circ f_1 = g_2 \circ f_2$ while the embeddings g_i commute with homomorphisms ι_i, see Fig. 2.

Proof. We let $(\mathbf{B}_3, \mathbf{C}, \iota_3)$ with embeddings $(g_i, \mathbf{1}) : (\mathbf{B}_i, \mathbf{C}, \iota_i) \longrightarrow (\mathbf{B}_3, \mathbf{C}, \iota_3)$ be the (free) amalgamation of $(\mathbf{B}_i, \mathbf{C}, \iota_i)$ with respect to the embeddings $(f_i, \mathbf{1})$. We only have to justify our claim that $(\mathbf{B}_3, \mathbf{C}, \iota_3)$ belongs to the class $PartiRel_{(\ell+1)}(d,D)$. Let $\{x,y\}$ be an edge of $\mathbf{B}_3$ and let $P = (x = x_0, x_1, \ldots, x_t = y)$ be a path in $\mathbf{C}$ from x to y of length $\leq \ell + 1$. We have to prove that the length $\rho(x,y)$ of the edge $\{x,y\}$ satisfies $\rho(x,y) \leq \rho(P) = \sum_{i=1}^{t} \rho(x_{i-1}, x_i)$. Towards this end consider the image $\iota_3(P) = (\iota_3(x_0), \iota_3(x_1), \ldots, \iota_3(x_t))$. Note that it is $\rho(x_i, x_{i+1}) = \rho(\iota_3(x_i), \iota_3(x_{i+1})$ (as ι_3 is a homomorphisms of binary relational systems). The sequence $\iota_3(P) = (\iota_3(x_0), \iota_3(x_1), \ldots, \iota_3(x_t))$ induces a trail in $\mathbf{C}$ and some vertices and edges may be identified by ι_3. However if this really happens then the length $\rho(P) = \rho(\iota_3(P)$ is bounded by $\rho(P')$ where P' is a path (a subpath of $\iota(P)$) from $\iota(x)$ to $\iota(y)$ of length $\leq \ell$ and thus (as $\mathbf{C} \in Rel_{(\ell)}(d,D)$ we have that $\rho(\iota(x), \iota(y)) = \rho(x,y) \leq \rho(P')$ is an ℓ-metric edge. Thus we can assume that $\iota(P)$ is a path of length $\ell + 1$ in $\mathbf{C}$. We distinguish three cases:

$$
\begin{tikzcd}
\mathbf{B}_0 \arrow[rr, "\iota_0"] \arrow[dr, "f_2"] \arrow[dd, "f_1"'] & & \mathbf{A} \arrow[dr, "\mathbf{1}"] \arrow[dd, "\mathbf{1}"'] & \\
& \mathbf{B}_2 \arrow[rr, "\iota_2"] \arrow[dd, "g_2"'] & & \mathbf{C} \arrow[dd, "\mathbf{1}"] \\
\mathbf{B}_1 \arrow[rr, "\iota_1"] \arrow[dr, "g_1"'] & & \mathbf{C} \arrow[dr, "\mathbf{1}"] & \\
& \mathbf{B}_3 \arrow[rr, "\iota_3"] & & \mathbf{C}
\end{tikzcd}
$$

If $\iota(P)$ is a subset of $\mathbf{A}$ then clearly $(\iota(x), \iota(y))$ is a metric pair as $\mathbf{A}$ is a metric space (this holds for any length t).

If P is a subset of either $\mathbf{B}_1$ or $\mathbf{B}_2$ then again $\rho(x,y) \leq \rho(P) = \sum_{i=1}^{t} \rho(x_{i-1}, x_i)$ (as $(\mathbf{B}_i, \mathbf{C}, \iota_i) \in PartiRel_{(\ell+1)}(d, D)$.

Thus assume that there exist $x_{j(i)}$ such that $x_{j(i)} \in \underline{\mathbf{B}}_i, \iota(x_{j(i)}) \notin \underline{\mathbf{A}}, i = 1, 2$. But as $\mathbf{B}_3$ is a free amalgamation there are no edges outside $\iota^{-1}(\underline{\mathbf{A}})$ and thus there are at least two vertices $x_{k(i)}, i = 1, 2, k(1) < k(2)$, for which $\iota(x_{k(i)}) \in \underline{\mathbf{A}}$. But then the path between $\iota(x_{k(1)})$ and $\iota(x_{k(2)})$ has length at least $\rho(\iota(x_{k(1)}), \iota(x_{k(2)}))$ and thus $\rho(P) \geq \rho(P')$ where $P' = (\iota(x) = \iota(x_0), \iota(x_1), \ldots, \iota(x_{k(1)}), \iota(x_{k(2)}), \ldots, \iota(x_t) = \iota(y))$ and we have $\rho(P') \geq \rho(\iota(x), \iota(y)) = \rho(x, y)$ again by the assumption on $\mathbf{C}$. ∎

Proof. We are now in position to prove Theorem 2.1. We shall proceed by induction on ℓ. As explained above, for $\ell = 1$ Theorem 2.1 reduces to Theorem 1.1.

In the induction step ($\ell \Rightarrow \ell + 1$) we assume that Theorem 2.1 holds ℓ. Let $\mathbf{A}, \mathbf{B}$ be metric binary systems considered as transversal systems in $PartiRel_{(\ell+1)}(d, D)$. Let $\mathbf{R} \in Rel_{(\ell)}(d, D)$ be a system satisfying $\mathbf{R} \longrightarrow (\mathbf{B})_2^{\mathbf{A}}$

in the class $Rel_{(\ell)}(d,D)$. $\mathbf{R}$ will be fixed from now on and it will be considered as transversal system (in $PartiRel_{\ell}(d,D)$). We shall construct $\mathbf{R}$-partite systems $\mathbf{P}^0, \mathbf{P}^1, \ldots, \mathbf{P}^b$ where $a = |\binom{\mathbf{R}}{\mathbf{A}}|$. The system $\mathbf{C} = \mathbf{P}^b$ will satisfy (as we shall show below) all the required properties of Theorem 2.1.

Put explicitly $\binom{\mathbf{R}}{\mathbf{A}} = \{\mathbf{A^1}, \mathbf{A^2}, \ldots, \mathbf{A}^a\}$ and also $\binom{\mathbf{R}}{\mathbf{B}} = \{\mathbf{B^1}, \mathbf{B^2}, \ldots, \mathbf{B}^b\}$.

Let the system $(\mathbf{P}^0, \mathbf{R}, \iota^0)$ be any system in the class $PartiRel_{(\ell+1)}(d,D)$ for which the mapping ι^0) satisfies:

For any $i = 1, \ldots, b$ the set $(\iota^0)^{-1}(\mathbf{B}^i)$ contains a subsystem isomorphic to $\mathbf{B}^i$ (in $PartiRel_{(\ell+1)}(d,D)$).

Such a system is easy to construct: we can take the disjoint union of b copies of $\mathbf{B}$ and define mapping ι^0 such that the above condition holds.

In the induction step ($i \Rightarrow i+1$) let be given an $\mathbf{R}$-partite system $(\mathbf{P}^i, \mathbf{R}, \iota^i) \in PartiRel_{(\ell+1)}(d,D)$. Consider the system $\mathbf{B}$ and let $(\mathbf{D}^i, \mathbf{A}, \iota^i)$ denote the subsystem of $(\mathbf{P}^i, \mathbf{R}, \iota^i)$ induced by the set $(\iota^i)^{-1}(\mathbf{B}^i)$ (we denoted the restriction of ι^i to the subset by the same symbol). We have $(\mathbf{D}^i, \mathbf{R}, \iota^i) \in PartiRel_{(\ell+1)}(d,D)$ thus by the induction hypothesis there exits a system $(\mathbf{E}^i, \mathbf{A}, \lambda^i)$ such that

$$\mathbf{E}^i \longrightarrow (\mathbf{D}^i)^{\mathbf{A}}_2$$

(in the class $PartiRel_{(\ell)}(d,D)$).

Let $(\mathbf{P}^{i+1}, \mathbf{R}, \iota^{i+1})$ be a free amalgamation of copies of $(\mathbf{P}^i, \mathbf{R}, \iota^i)$ such that every copy of $(\mathbf{D}^i, \mathbf{A}, \iota^i)$ in $(\mathbf{E}^i, \mathbf{A}, \lambda^i)$ is extended to unique copy of $(\mathbf{P}^i, \mathbf{R}, \iota^i)$. (Such a free amalgamation we obtain by repeatedly using amalgamation of pairs defined above.) According to Lemma 4.1 we know that $(\mathbf{P}^{i+1}, \mathbf{R}, \iota^{i+1}) \in PartiRel_{(\ell+1)}(d,D)$.

Thus let $(\mathbf{C}, \mathbf{R}, \iota) = (\mathbf{P}^a, \mathbf{R}, \iota^a) \in PartiRel_{(\ell+1)}(d,D)$. It remains to show that

$$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2.$$

However this is the underlying idea of every application of the Partite Construction and this follows by a backward induction for $i = a$, $a-1, \ldots, 1, 0$. Let $\binom{\mathbf{C}}{\mathbf{A}} = \mathcal{A}_1 \cup \mathcal{A}_2$ be arbitrary partition (coloring). By induction for $i = a, a-1, \ldots, 1, 0$ we prove that there exists a subsystem $(\widetilde{\mathbf{P}^i}, \mathbf{R}, \tilde{\iota})$ (in $PartiRel_{(\ell+1)}(d,D)$) isomorphic to $\mathbf{P}^i$ such that for all $j > i$ all copies $\tilde{\mathbf{A}} \in \binom{\mathbf{C}}{\mathbf{A}}$ for which $\tilde{\iota}(\tilde{\mathbf{A}}) = \mathbf{A}^j$ get the same color, say $c(j)$.

In the induction step (as for $i = a$ the statement clearly holds) we consider a copy $\widetilde{\mathbf{P}^i}$ of $\mathbf{P}^i$ with the stated properties. In the set $\binom{\widetilde{\mathbf{P}^i}}{\mathbf{A}}$ consider those

$\tilde{\mathbf{A}}$ for which $\tilde{\iota}(\tilde{\mathbf{A}}) = \mathbf{A}^i$. These copies of lie in a copy of $\mathbf{A}$-partite system which is isomorphic to $\mathbf{E}^i$ and thus by $\mathbf{E}^i \longrightarrow (\mathbf{D}^i)^{\mathbf{A}}_2$ we get that there exists a subsystem $(\widetilde{\mathbf{P}^{i-1}}, \mathbf{R}, \tilde{\iota}))$ of $(\widetilde{\mathbf{P}^i}, \mathbf{R}, \tilde{\iota})$ which is isomorphic to $(\mathbf{P}^i, \mathbf{R}, \iota^i)$ with the stated properties.

Finally, we obtain a copy $(\widetilde{\mathbf{P}^0}, \mathbf{R}, \tilde{\iota})$ of $(\mathbf{P}^0, \mathbf{R}, \iota^0)$ such that for every $\tilde{\mathbf{A}} \in \binom{\widetilde{\mathbf{P}^0}}{\mathbf{A}}$ its color depends only on $\tilde{\iota}(\tilde{\mathbf{A}})$. But this in turn induces a coloring $\widetilde{\mathcal{A}}_1 \cup \widetilde{\mathcal{A}}_2$ of the set $\binom{\mathbf{R}}{\mathbf{A}}$ defined by $\mathbf{A}_j \in \widetilde{\mathcal{A}}_i$ iff $c(j) = i$. Thus there exists $\widetilde{\mathbf{B}} \in \binom{\mathbf{R}}{\mathbf{B}}$ such that $\binom{\widetilde{\mathbf{B}}}{\mathbf{A}} \subset \widetilde{\mathcal{A}}_{i(0)}$ and thus by the construction of $\mathbf{P}^0$ any $\mathbf{B}' \in \binom{\widetilde{\mathbf{P}^0}}{\mathbf{B}}$ with $\tilde{\iota}(\mathbf{B}') = \widetilde{\mathbf{B}}$ satisfies $\binom{\mathbf{B}'}{\mathbf{A}} \subset \mathcal{A}_{i(0)}$ which we wanted to prove. ∎

5 Remarks and Open Problems

1. Theorem 1.2 also implies the following (*ordering property of finite metric spaces*:

Theorem 5.1 *For every metric space there exists a metric space (Y, σ) such that for any linear orderings $\leq_X$ and $\leq_Y$ of X and Y there exists a monotone isometry $(X, \rho) \longrightarrow (Y, \sigma)$.*

This result explains why we are considering ordered metric spaces. Theorem 5.1 may be derived either from Theorem 1.2 (applied for a A with just 2 vertices; compare [18] or directly, see [11].

2. One can prove results analogous to Theorem 1.2 for other classes of metric spaces: for example one can consider only rational, or integer or graph-metrics. We only have to check that the amalgamation property holds for these classes. Rational metrics then applies to the Urysohn space.

3. Perhaps in the spirit of [10, 11] one could ask for a characterization of all Ramsey classes of metric spaces. However this seems to be beyond the reach as the corresponding characterization of homogeneous metric spaces (and thus equivalently (Fraissé classes) seems not be known, compare [2, 7, 8].

4. It is interesting that the amalgamation technique is (almost) necessary as we have the following easy but important result observed already in [10], see recent [11, 6]. Particularly one can prove that every hereditary Ramsey class of structures with the joint embedding property is amalgamation class. This shows the relevance of the classification programme of Ramsey classes [11] and the classification programme for homogeneous structures [2].

5. It would be interesting to investigate the "simple" Ramsey properties (such as the vertex- and edge-partitions of the Urysohn space (in the analogy of a similar results for the Random graph, [1].

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