On Brown's Conjecture on Accessible Sets

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Abstract

In this note we use a sequence constructed by H. Furtsenberg in 1981 to disprove the following conjecture posted by T. Brown: If a set of positive numbers L is such that for any finite coloring of $\mathbb{N}$ there are arbitrarily long monochromatic sequences with all gaps in L, then for any finite coloring of $\mathbb{N}$ there are arbitrarily long monochromatic arithmetic progressions whose common differences belong to L.

1 Introduction

Let $\mathbb{N}$ be the set of positive integers. For $r \in \mathbb{N}$, an r-coloring of $\mathbb{N}$ is a function $f : \mathbb{N} \to A$, with $|A| = r$. A finite coloring is an r-coloring for some r. If f is a finite coloring and if $B \subseteq \mathbb{N}$ satisfies $|f(B)| = 1$, we say that B is f-monochromatic. An arithmetic progression of length k and common difference d, $k, d \in \mathbb{N}$, is a set of the form $\{a + (i-1)\,d : i \in [1, k]\}$, for some $a \in \mathbb{N}$.

Van der Waerden's theorem [5] on arithmetic progressions says that for any finite coloring f and any $k \in \mathbb{N}$ there is an f-monochromatic arithmetic progression of length k. Brown, Graham, and Landman in [2] study subsets L of $\mathbb{N}$ such that van der Waerden's theorem can be strengthened to guarantee the existence of arbitrarily long monochromatic arithmetic progressions having common differences in L. Sets with this property are called *large*. Somehow surprisingly there are many large sets. For example, by the Polynomial van der Waerden's Theorem [1], if p is a polynomial with rational coefficients

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taking integer values on the integers and satisfying $p(0) = 0$ then $|p(\mathbb{N})|$ is large.

For a given set D of positive integers Landman and Robertson ([4], Definition 10.12) define a k-term D-diffsequence as a sequence $x_1 < x_2 < \ldots < x_k$ such that $x_i - x_{i-1} \in D$ for all $i = 2, 3, \ldots, k$. D is said to be *accessible* if for any finite coloring f of positive integers there are arbitrarily long f-monochromatic D-diffsequences.

It is known ([4], Theorem 10.27) that for any infinite set T of positive integers, the difference set $T - T = \{|t - s| : \ s, t \in T\}$ is accessible.

T. Brown conjuctured ([4], Research Problem 10.9) that every accessible set is large.

We use a sequence of positive numbers constructed by H. Furstenberg [3] to disprove Brown's conjecture. In [3] this sequence is used to show that there is a set that intersects each IP-set of $\mathbb{Z}$, but does not intersect each difference set of $\mathbb{Z}$. A set $Q \subseteq \mathbb{Z}$ is an IP-set of $\mathbb{Z}$ if there is sequence $\{a_i\}_{i\in\mathbb{Z}}$ of not necessarily distinct integers so that $Q = \{\sum_{i\in F} a_i : F \subseteq \mathbb{N} \text{ and } |F| < \infty\}$ for some $S \subseteq \mathbb{Z}$. (IP stands for *infinite-dimensional parallepiped.*)

2 Not All Accessible Sets Are Large

In this section we show that there is an accessible set that is not large.

It is not difficult to check that

$$\|x\| = \min\{|x + n| : n \in \mathbb{Z}\}$$

is a norm on $\mathbb{R}$. It is known ([3], page 22) that for any $\alpha, a \in (0, 1)$, with α irrational, and any $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that

$$\max\{\|n\alpha\|, \|n^2\alpha - a\|\} < \varepsilon.$$

Let $\alpha \in (0, 1)$ be irrational and let $\varepsilon \in (0, \frac{1}{8})$. We define the set $S = \{s_i\}_{i\in\mathbb{N}}$ inductively in the following way. Let $s_1 \in \mathbb{N}$ be such that

$$\max\left\{\|s_1\alpha\|, \left\|s_1^2\alpha - \frac{1}{4}\right\|\right\} < \varepsilon.$$

If $s_1, \ldots, s_k$ are defined, then $s_{k+1} \in \mathbb{N}$ is such that

$$\max\left\{\|s_{k+1}\alpha\|, \left\|s_{k+1}^2\alpha - \frac{1}{4}\right\|\right\} < \frac{\varepsilon}{\prod_{i=1}^{k} s_i}.$$

We note that for all $n \in \mathbb{N}$

$$\left\| s_n^2 \alpha - \frac{1}{4} \right\| < \varepsilon$$

and, for all $m, n \in \mathbb{N}$ such that $m < n$,

$$\|s_m s_n \alpha\| \leq s_m \|s_n \alpha\| < \frac{\varepsilon}{\prod_{i \neq m} s_i} < \varepsilon.$$

Thus, for $m \neq n$ we have that

$$\left\| (s_m - s_n)^2 \alpha - \frac{1}{2} \right\| \leq \left\| s_m^2 \alpha - \frac{1}{4} \right\| + 2 \|s_m s_n \alpha\| + \left\| s_n^2 \alpha - \frac{1}{4} \right\| < 4\varepsilon.$$

In particular, since $\varepsilon < \frac{1}{8}$, we have that $s_n \neq s_m$ if $n \neq m$. Hence, S is an infinite set and by ([4], Theorem 10.27), $L = S - S = \{|s_m - s_n| : m \neq n\}$ is accessible.

We claim that there is a finite coloring of $\mathbb{N}$ with no monochromatic 3-term arithmetic progression having its common difference in L.

For $m \in \mathbb{N}$ we define an m-coloring $f_m : \mathbb{N} \to \{1, \ldots, m\}$ in the following way. By definition $f_m(n) = i$ if and only if $\left\| \frac{n(n-1)}{2} \alpha \right\| \in \left(\frac{i-1}{2m}, \frac{i}{2m} \right)$.

Suppose that $n, p \in \mathbb{N}$ are such that $\{n, n+p, n+2p\}$ is f_m-monochromatic and suppose that i is such that $f_m(\{n, n+p, n+2p\}) = \{i\}$. Then, for all $k \in \{n, n+p, n+2p\}$, $\left\| \frac{k(k-1)}{2} \alpha - \frac{i-1}{2m} \right\| \in \left(0, \frac{1}{2m}\right)$.

Particularly, for $k = n + p$

$$\begin{aligned} \frac{1}{2m} &> \left\| \frac{n(n-1)}{2} \alpha - \frac{i-1}{2m} + pn\alpha + \frac{p(p-1)}{2} \alpha \right\| \geq \\ &\geq \left\| pn\alpha + \frac{p(p-1)}{2} \alpha \right\| - \left\| \frac{n(n-1)}{2} \alpha - \frac{i-1}{2m} \right\|. \end{aligned}$$

It follows that $\left\| pn\alpha + \frac{p(p-1)}{2} \alpha \right\| < \frac{1}{m}$, or, equivalently, $\|2pn\alpha + p(p-1)\alpha\| < \frac{2}{m}$.

From

$$\begin{aligned} \frac{1}{2m} &> \left\| \frac{(n+2p)(n+2p-1)}{2} \alpha - \frac{i-1}{2m} \right\| \geq \\ &\geq \|p^2 \alpha\| - \left\| \frac{n(n-1)}{2} \alpha - \frac{i-1}{2m} \right\| - \|2pn\alpha + p(p-1)\alpha\| > \\ &> \|p^2 \alpha\| - \frac{1}{2m} - \frac{2}{m} \end{aligned}$$

we have that if p is the common difference of a f_m-monochromatic 3-term arithmetic progression, then $\|p^2\alpha\| < \frac{3}{m}$.
Let $k \in N$ be such that $\frac{1}{k} < \frac{1}{3}\left(\frac{1}{2} - 4\varepsilon\right)$ and let $l, n \in \mathbb{N}$, $l \neq n$. From

$$\frac{1}{2} - \left\|(s_l - s_n)^2 \alpha\right\| \leq \left\|(s_l - s_n)^2 \alpha - \frac{1}{2}\right\| < 4\varepsilon$$

we have that

$$\left\|(s_l - s_n)^2 \alpha\right\| > \frac{1}{2} - 4\varepsilon > \frac{3}{k}.$$

Thus, f_k is a finite coloring of $\mathbb{N}$ such that there is no f_k-monochromatic 3-term arithmetic progression having common difference in L.

Therefore, L is not large.

3 Brown-Graham-Landman Conjecture

For $k \in \mathbb{N}$, a set of positive integers L is said to be chromatically k-intersective if for any coloring f of positive integers there is an f-monochromatic k-term arithmetic progression whose common difference belongs to L. Clearly, L is large if it is chromatically k-intersective for all k. The difference set of Furstenberg's sequence from the previous section is an example of a set that is chromatically 2-intersective, but not chromatically 3-intersective.

Another way to define large sets is to start by fixing the number of colors and then to vary the length of monochromatic arithmetic progressions. For $r \in \mathbb{N}$, a set of positive integers L is said to be r-large if for any r-coloring f of positive integers there are arbitrarily long f-monochromatic arithmetic progressions whose common differences belong to L. L is large if it is r-large for all r. It is not known if there is an r-large set that is not large.

Brown, Graham, and Landman posted the following conjecture [2].

Conjecture 1 *Every 2-large set is large.*

References

[1] V. Bergelson and A Leibman. Polynomial extension of van der Waerden's and Szemerédi's theorems. *J. Amer. Math. Soc*, 9:725–753, 1996.

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