On the maximal order of numbers in the "factorisatio numerorum" problem

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Abstract

Let $m(n)$ be the number of ordered factorizations of n in factors larger than 1. In this paper, we show that the inequality

$$m(n) < \frac{n^\rho}{\exp\left((\log n)^{1/\rho+o(1)}\right)}$$

holds for all positive integers n, while the inequality

$$m(n) > \frac{n^\rho}{\exp\left((\log n)^{\rho/(\rho^2-1)+o(1)}\right)}$$

holds for infinitely many positive integers n, where $\rho = 1.72864\ldots$ is the real solution to $\zeta(\rho) = 2$. We investigate also arithmetic properties of $m(n)$ and numbers of distinct values of $m(n)$.

1 Introduction

Let $m(n)$ be the number of ordered factorizations of a positive integer n such that every factor is > 1. For example, $m(12) = 8$ because of the factorizations

12, $2\cdot 6$, $6\cdot 2$, $3\cdot 4$, $4\cdot 3$, $2\cdot 2\cdot 3$, $2\cdot 3\cdot 2$, and $3\cdot 2\cdot 2$. By the definition, $m(1)=0$ but we will see that sometimes it is useful to set $m(1)=1$ or $m(1)=1/2$. It was proved by Kalmár [10] that for $x\to\infty$

$$M(x)=\sum_{n<x} m(n) = cx^{\rho}(1+o(1)), \tag{1}$$

where $\rho\approx 1.72864$ is the real solution to $\zeta(\rho)=2$ and $c\approx 0.31817$ is given by $c=-1/\rho\zeta'(\rho)$. (As usual, $\zeta(s)=\sum_{n\geq 1} n^{-s}$ is Euler–Riemann zeta function.) Erdős [2] claimed (in the end of his article) that there exist positive constants c_1 and c_2 such that

$$m(n) < \frac{n^{\rho}}{\exp\left((\log n)^{c_1}\right)}$$

holds for all n, while

$$m(n) > \frac{n^{\rho}}{\exp\left((\log n)^{c_2}\right)}$$

holds for infinitely many n. He gave no details. Here, we show that the first inequality holds with $c_1=1/\rho-\varepsilon$ for all $n>n_0(\varepsilon)$, and that the second inequality holds for infinitely many positive integers n with $c_2=\rho/(\rho^2-1)+\varepsilon$, for any $\varepsilon>0$. Note that $1/\rho\approx 0.57849$ and $\rho/(\rho^2-1)\approx 0.86945$.

We prove the upper bound on the maximal order of $m(n)$ in Section 2 and the lower bound in Section 3. In Section 4 we give further references and comments on the history of $m(n)$ and some related problems. We will also investigate arithmetical properties of $m(n)$. For example, we prove that $m(n)$ is not eventually periodic modulo k for any integer $k>1$, and we also show that $m(n)$ is not a polynomially recursive sequence.

For a positive integer n we write $\omega(n)$ and $\Omega(n)$ for the number of distinct prime factors of n and the total number of prime factors of n; i.e., including multiplicities, respectively. We put $P(n)$ for the largest prime factor of n. We write log for the natural logarithm. We will let x be a large positive real number and we will assume that $\varepsilon>0$ is fixed. We use the letters p and q with or without subscripts to denote prime numbers. We use the Vinogradov symbols $\ll$ and $\gg$ and the Landau symbols O and o with their usual meanings. The constants implied by these symbols may depend on some other data like ε, α, β, γ, δ, etc.

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2 The upper bound

The following estimate is well-known and its proof is elementary (i.e., it does not use the Prime Number Theorem).

Lemma 1. *If $\delta > \delta_0 > 1$, then the estimate*

$$\sum_{p>t} \frac{1}{p^\delta} = \frac{(\delta-1)^{-1}}{t^{\delta-1}\log t} + O\left(\frac{1}{t^{\delta-1}(\log t)^2}\right) \tag{2}$$

holds uniformly for $t > 2$.

Let p_k be the kth prime, $\mathcal{P}_k$ be the set (including 1) of positive integers composed only of the primes $p_1 = 2, p_2, \ldots, p_k$, and $m_k(n)$ be the number of ordered factorizations of n in factors lying in $\mathcal{P}_k \backslash \{1\}$. Let, for real $s > 1$,

$$\zeta_k(s) = \prod_{p \le p_k} \left(1 - \frac{1}{p^s}\right)^{-1} = \sum_{n \in \mathcal{P}_k} \frac{1}{n^s}$$

and ρ_k be the real solution to $\zeta_k(\rho_k) = 2$. Chor, Lemke and Mador proved in [1, Theorem 5] that $m_k(n) < n^{\rho_k}$ for every $n \ge 1$. For the sake of completeness we reprove their result. Using a small improvement in their argument we obtain, in fact, a better inequality.

Lemma 2. *For every $n \ge 1$,*

$$m_k(n) \le \frac{1}{\sqrt{2}}\, n^{\rho_k}.$$

Proof. It is easy to see that for $s > \rho_k$ we have (now $m_k(1) = 1$)

$$\sum_{n \ge 1} \frac{m_k(n)}{n^s} = \sum_{k \ge 0} (\zeta_k(s) - 1)^k = \frac{1}{2 - \zeta_k(s)}$$

and this identity implies that $m_k(n) = o(n^\sigma)$ for every fixed $\sigma > \rho_k$.

For every $r, s \geq 1$ we have

$$m_k(rs) \geq 2m_k(r)m_k(s). \tag{3}$$

To show this inequality, we assume that $r, s \geq 2$ (for $r = 1$ or $s = 1$ it holds trivially) and consider the set X of all pairs (u, v) where u (v) is an ordered factorization of r (s) in factors lying in $\mathcal{P}_k \backslash \{1\}$ and the set Y of the same factorizations of rs. If u is $r = d_1 \cdot d_2 \cdot \ldots \cdot d_i$ and v is $s = e_1 \cdot e_2 \cdot \ldots \cdot e_j$, we define the factorizations of rs

$$\begin{aligned} F((u, v)) &= d_1 \cdot d_2 \cdot \ldots \cdot d_i \cdot e_1 \cdot e_2 \cdot \ldots \cdot e_j \\ G((u, v)) &= d_1 \cdot d_2 \cdot \ldots \cdot d_{i-1} \cdot (d_i e_1) \cdot e_2 \cdot \ldots \cdot e_j. \end{aligned}$$

The inequality (3) follows from the fact that the mappings F and G are injections from X to Y which moreover have disjoint images. We leave a simple verification of this fact to the reader.

Suppose now that $m_k(n_0) > n_0^{\rho_k}/\sqrt{2}$ for some $n_0 \geq 2$. By (3) we have $m_k(n_0^2) \geq 2m_k(n_0)^2 > n_0^{2\rho_k}$ and hence we can take a $\sigma > \rho_k$ so that $m_k(n_0^2) \geq (n_0^2)^\sigma$. Then, again by (3), $m_k(n_0^{2i}) \geq (n_0^{2i})^\sigma$ for every $i = 1, 2, \ldots$ which is in contradiction with $m_k(n) = o(n^\sigma)$. □

It follows from the proof that the previous lemma and inequality (3) hold for $k = \infty$ as well (i.e., for $m(n)$ in place of $m_k(n)$ and ρ in place of ρ_k). Since for $r = p^a$ one has $m(r) = 2^{a-1}$, for $r = p^a$ and $s = p^b$ we have $m(rs) = 2m(r)m(s)$ and inequality (3) is tight for such r, s.

Let $n = q_1^{a_1} q_2^{a_2} \ldots q_k^{a_k}$ be a prime decomposition of n where $q_1 < q_2 < \ldots < q_k$. We denote by $\bar{n}$, $\bar{n} \leq n$, the number obtained from n by replacing q_i in the decomposition by p_i, the ith smallest prime. From the fact that $m(n)$ depends only on the exponents a_i and from the previous lemma we get that

$$m(n) = m(\bar{n}) = m_k(\bar{n}) < \bar{n}^{\rho_k} \leq n^{\rho_k}, \tag{4}$$

where $k = \omega(n)$.

It is clear that $\rho_k < \rho$ and that $\rho_k \to \rho$ when $k \to \infty$. The next result gives an upper bound for the speed of convergence of ρ_k to ρ.

Lemma 3. *Let $\varepsilon > 0$. There exists $k_0 = k_0(\varepsilon)$ such that if $k > k_0$, then*

$$\rho - \rho_k > \frac{1}{k^{\rho - 1 + \varepsilon}}.$$

Proof. The equation $\zeta_k(\rho_k)^{-1} = \zeta(\rho)^{-1} = 1/2$ implies that

$$\prod_{2 \le p \le p_k} \left(1 - \frac{1}{p^{\rho_k}}\right) = \prod_{p \ge 2} \left(1 - \frac{1}{p^{\rho}}\right).$$

Taking logarithms and regrouping, we get

$$\sum_{2 \le p \le p_k} \left(\log\left(1 - \frac{1}{p^{\rho}}\right) - \log\left(1 - \frac{1}{p^{\rho_k}}\right)\right) = -\sum_{p > p_k} \log\left(1 - \frac{1}{p^{\rho}}\right). \quad (5)$$

Clearly,

$$-\sum_{p > p_k} \log\left(1 - \frac{1}{p^{\rho}}\right) = \sum_{p > p_k} \frac{1}{p^{\rho}} + O\left(\sum_{p > p_k} \frac{1}{p^{2\rho}}\right) \gg \frac{1}{p_k^{\rho-1} \log(p_k)}, \quad (6)$$

where in the above inequality we used estimate (2).

Since ρ_k converges to ρ, there exists k_1, which is absolute, such that $\rho_k > 1.5$ for $k > k_1$. The derivative of the function $x \mapsto \log(1 - 1/p^x)$ is $(\log p)/(p^x - 1)$. By Lagrange's Mean-Value Theorem we have (for $k > k_1$ and with some number $\sigma_k \in (1.5, \rho)$)

$$\log\left(1 - \frac{1}{p^{\rho}}\right) - \log\left(1 - \frac{1}{p^{\rho_k}}\right) = (\rho - \rho_k) \frac{\log p}{p^{\sigma_k} - 1} \le (\rho - \rho_k) \frac{\log p}{p^{1.5} - 1}. \quad (7)$$

Equation (5) together with estimates (6) and (7) implies that

$$(\rho - \rho_k) \sum_{p \ge 2} \frac{\log p}{p^{1.5} - 1} \gg \frac{1}{p_k^{\rho-1} \log(p_k)}.$$

Since $p_k \sim k \log k$, this leads to the conclusion of the lemma if $k_0 > k_1$ is sufficiently large (depending on $\varepsilon > 0$). □

As a warm up for the upper bound, we first prove an upper bound of the same shape as stated in the introduction but with the smaller constant $c_1 = 2 - \rho \approx 0.27136$.

Theorem 1. *For every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that*

$$m(n) < \frac{n^{\rho}}{\exp\left((\log n)^{c_1 - \varepsilon}\right)}$$

for all $n > n_0(\varepsilon)$, where $c_1 = 2 - \rho$.

Proof. Let $\varepsilon > 0$ be fixed and let k be a positive integer. Assume that $\omega(n) = k$. If $k \le k_0$, where $k_0 = k_0(\varepsilon)$ is the positive integer appearing in Lemma 3, then, by inequality (4),

$$m(n) < n^{\rho_k} \le n^{\rho_{k_0}} < \frac{n^\rho}{\exp\left((\log n)^{c_1-\varepsilon}\right)},$$

if $n > n_0$ is large enough. Assume therefore that $k > \max(k_0, 2)$. Using inequality (4), Lemma 3 and the inequality $k = \omega(n) < \log n$ that holds for $k > 2$, we get that

$$\begin{aligned} m(n) &< n^{\rho_k} \\ &< n^{\rho - 1/k^{\rho-1+\varepsilon}} \\ &= \frac{n^\rho}{\exp\left(\frac{\log n}{k^{\rho-1+\varepsilon}}\right)} \\ &< \frac{n^\rho}{\exp\left((\log n)^{c_1-\varepsilon}\right)}. \end{aligned}$$

□

To do better, we need the following combinatorial fact which will be used in the proof of the lower bound as well.

Lemma 4. *Suppose that $q_1, \ldots, q_k$ are primes, not necessarily distinct, such that $q_1 \ldots q_k$ divides n. Then, with $m(1) = 1$,*

$$m(n) < (2\Omega(n))^k m(n/q_1 \ldots q_k). \tag{8}$$

Proof. It suffices to prove only the case $k = 1$, i.e., the inequality

$$m(n) < 2\Omega(n)m(n/p), \tag{9}$$

where p is a prime dividing n, because the general case follows easily by iteration. Let X be the set of all pairs (u, i) where u is an ordered factorization of n/p in r parts bigger than 1 and i, $1 \le i \le 2r + 1$, is an integer. Let Y be the set of all ordered factorizations of n in parts biger than 1. We shall define a surjection F from X onto Y. This will prove (9) because $r \le \Omega(n/p) = \Omega(n) - 1$ and therefore for every u we have $2r+1 < 2\Omega(n)$ pairs (u, i) and $|X| < 2\Omega(n)m(n/p)$. For $(u, i) \in X$, where u is $n/p = d_1 \cdot d_2 \cdot \ldots \cdot d_r$, we define $j = i - r$ and set $F((u, i))$ to be the factorization

$$n = d_1 \cdot \ldots \cdot d_{i-1} \cdot (pd_i) \cdot d_{i+1} \cdot \ldots \cdot d_r$$

if $1 \le i \le r$ and

$$n = d_1 \cdot \ldots \cdot d_{j-1} \cdot p \cdot d_j \cdot \ldots \cdot d_r$$

if $r+1 \le i \le 2r+1$ (for $j = 1$ p is the first part and for $j = r+1$ it is the last one). It is clear that F is a surjection. $\square$

We can now prove the announced upper bound.

Theorem 2. *For every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that*

$$m(n) < \frac{n^\rho}{\exp\left((\log n)^{c_1-\varepsilon}\right)}$$

for all $n > n_0(\varepsilon)$, where $c_1 = 1/\rho$.

Proof. We put $d_n \in (0,1)$ to be determined later and $k(n) = \lfloor (\log n)^{d_n} \rfloor$. Assume first that $\omega(n) = k \le k(n)$. If $k \le k_0$, where k_0 is the positive integer appearing in Lemma 3, then, by inequality (4),

$$m(n) < n^{\rho_k} \le n^{\rho_{k_0}} < \frac{n^\rho}{\exp\left((\log n)^{c_1-\varepsilon}\right)},$$

if $n > n_0$ is large enough. Assume therefore that $k > \max(k_0, 2)$. Using the same argument as in the proof of Theorem 1, we get

$$\begin{aligned} m(n) &< n^{\rho_k} \\ &< n^{\rho - 1/k^{\rho-1+\varepsilon}} \\ &= \frac{n^\rho}{\exp\left(\frac{\log n}{k^{\rho-1+\varepsilon}}\right)} \\ &< \frac{n^\rho}{\exp\left((\log n)^{1-d_n(\rho-1)+\varepsilon}\right)}. \end{aligned} \tag{10}$$

Assume now that $\omega(n) = k > k(n)$, and let $\ell(n)$ be the squarefree divisor of n which is the product of the first (smallest) $k(n)$ prime factors of n. By Lemma 4, Lemma 2 with $k = \infty$, the fact that

$$2\Omega(n) \le (2/\log 2)\log n \le 3\log n,$$

and the known estimates

$$\sum_{p \le x} \log p = x + O(x/\log x), \tag{11}$$

and

$$p_k = k\log k + O(k\log\log k), \tag{12}$$

which hold for all positive real numbers x and positive integers k, we have that if n is large then

$$\begin{aligned}
m(n) &< (2\Omega(n))^{k(n)} m(n/\ell(n)) \\
&< (3\log n)^{k(n)} \frac{n^\rho}{\ell(n)^\rho} \\
&\le (3\log n)^{k(n)} \frac{n^\rho}{(p_1\dots p_{k(n)})^\rho} \\
&= n^\rho \exp\big(k(n)\log(3\log n) - \rho p_{k(n)} + O(p_{k(n)}/\log p_{k(n)})\big) \\
&= n^\rho \exp\big(k(n)\log\log n - \rho p_{k(n)} + O(k(n))\big) \\
&= n^\rho \exp\big(k(n)\log\log n - \rho k(n)\log k(n) + O(k(n)\log\log k(n))\big) \\
&= \frac{n^\rho}{\exp\big((\rho d_n - 1 + o(1))k(n)\log\log n\big)} \\
&= \frac{n^\rho}{\exp\big((\rho d_n - 1 + o(1))(\log n)^{d_n}\log\log n\big)} \qquad (13)
\end{aligned}$$

where the last two $o(1)$'s are in fact $O(\log\log\log n/\log\log n)$. The above computations (10) and (13) show that if we choose

$$d_n = 1/\rho + C\log\log\log n/\log\log n$$

for some sufficiently large constant $C > 0$, then the inequality

$$m(n) < \frac{n^\rho}{\exp((\log n)^{c_1-\varepsilon})}$$

holds for all $n > n_0(\varepsilon)$ with $c_1 = 1/\rho$. □

3 The lower bound

To obtain the lower bound in the next theorem we first prove two lemmas.

Theorem 3. *For every $\varepsilon > 0$ there exist infinitely many positive integers n such that*

$$m(n) > \frac{n^\rho}{\exp\left((\log n)^{c_2-\varepsilon}\right)},$$

where $c_2 = \rho/(\rho^2 - 1)$.

Lemma 5. *Let $\alpha = 1/(\rho-1)$, $y = (\log x)^\alpha$, and $\mathcal{A}_1(x) = \{n \le x \ : \ P(n) > y\}$. Then*

$$\sum_{n\in\mathcal{A}_1(x)} m(n) = o(x^\rho) \tag{14}$$

as $x \to \infty$.

Proof. If $n \in \mathcal{A}_1(x)$, then there exists a prime $p > y$ such that $p|n$. Hence, by (9), we have

$$m(n) \ll \Omega(n)m(n/p).$$

Fix p and write $m = n/p$, then $m \le x/p$. Summing up the above inequality over all the possible values of m when p is fixed, and using the fact that $\Omega(n) \ll \log n \le \log x$, we get

$$\sum_{m\le x/p} m(mp) \ll \log x \sum_{m\le x/p} m(m).$$

Summing up the above inequalities over all possible values of $p > y$, and using (1), we get

$$\begin{aligned}
\sum_{n\in\mathcal{A}_1(x)} m(n) &\le \sum_{y<p\le x}\sum_{m\le x/p} m(mp) \\
&\ll \log x \sum_{y<p\le x}\sum_{m\le x/p} m(m) \\
&= \log x \sum_{y<p\le x} M(x/p) \\
&\ll x^\rho \log x \sum_{p>y} \frac{1}{p^\rho} \\
&\ll x^\rho \left(\frac{\log x}{y^{\rho-1}}\right)\frac{1}{\log y} \\
&= o(x^\rho),
\end{aligned}$$

which proves (14). In the above inequalities, we used again estimate (2). □

Lemma 6. *Let $\beta \in (0,\rho)$ be any fixed constant. Assume that $\varepsilon > 0$, and put $\gamma = 1/(\rho-\beta)+\varepsilon$, and $z = (\log x)^\gamma$. Then there exists $k_0 = k_0(\beta,\varepsilon)$, such that if we write*

$$\begin{aligned}
\mathcal{A}_2(x) \ = \ &\{n \le x \ : \ pq_1\ldots q_k|n \textit{ for some primes} \\
&z < p \le q_1 \le \cdots \le q_k \le p+p^\beta, \ \textit{and } k > k_0\},
\end{aligned}$$

then

$$\sum_{n\in\mathcal{A}_2(x)} m(n) = o(x^\rho) \tag{15}$$

as $x \to \infty$.

Proof. Let $k = k_0 + 1$. If $n \in \mathcal{A}_2(x)$, then there exists a prime $p > z$ and a k-tuple of primes $q_1 \le \cdots \le q_k$ in $[p,\ p+p^\beta]$ such that $pq_1\ldots q_k | n$. Hence, by (8), we have

$$m(n) \ll \Omega(n)^{k+1} m(n/pq_1\ldots q_k).$$

Fix $p,\ q_1,\ \ldots,\ q_k$ and write $m = n/pq_1\ldots q_k$, then $m \le x/pq_1\ldots q_k$. Summing up the above inequality over all the possible values of m when p and $q_1,\ldots,q_k$ are fixed, and using the fact that $\Omega(n) \ll \log n \le \log x$, we get

$$\sum_{m\le x/pq_1\ldots q_k} m(mpq_1\ldots q_k) \ll (\log x)^{k+1} \sum_{m\le x/pq_1\ldots q_k} m(m).$$

Summing up the above inequalities over all possible values of $p > z$ and $q_1 \le \cdots \le q_k$ in $[p,\ p+p^\beta]$, and using (1), we get

$$\begin{aligned}
\sum_{n\in\mathcal{A}_2(x)} m(n) &\le \sum_{z<p\le x}\ \sum_{q_1\le\cdots\le q_k\in[p,p+p^\beta]}\ \sum_{m\le x/pq_1\ldots q_k} m(mpq_1\ldots q_k)\\
&\ll (\log x)^{k+1} \sum_{z<p\le x}\ \sum_{q_1\le\cdots\le q_k\in[p,p+p^\beta]}\ \sum_{m\le x/pq_1\ldots q_k} m(m)\\
&= (\log x)^{k+1} \sum_{z<p\le x}\ \sum_{q_1\le\cdots\le q_k\in[p,p+p^\beta]} M(x/pq_1\ldots q_k)\\
&\ll x^\rho(\log x)^{k+1}\sum_{p>z}\frac{S_p}{p^\rho}, \qquad (16)
\end{aligned}$$

where

$$S_p = \sum_{q_1\le\cdots\le q_k\in[p,p+p^\beta]} \frac{1}{(q_1\ldots q_k)^\rho}.$$

Since $q_i \ge p$ for $i = 1,\ldots,k$, and since there are at most $\left(p^\beta\right)^k$ possibilities to choose k integers $q_1 \le \cdots \le q_k$ from $[p,\ p+p^\beta]$, we conclude that

$$S_p \le \frac{p^{k\beta}}{p^{k\rho}} = \frac{1}{p^{k\rho-k\beta}}. \tag{17}$$

Inserting estimate (17) into (16) and using estimate (2) again, we get that if k is sufficiently large such that

$$(k+1)\rho - k\beta > 1, \tag{18}$$

then

$$\begin{aligned}\sum_{n\in\mathcal{A}_2(x)} m(n) &\le x^{\rho}(\log x)^{k+1}\sum_{p>z}\frac{1}{p^{(k+1)\rho-k\beta}}\\ &\ll x^{\rho}\left(\frac{\log x}{z^{\rho-\beta-\frac{1-\beta}{k+1}}}\right)^{k+1}\frac{1}{\log z}.\end{aligned} \tag{19}$$

Since $z=(\log x)^{\frac{1}{\rho-\beta}+\varepsilon}$, for $\beta\ge 1$ the denominator in the bracket is always $\ge \log x$ and for $\beta<1$ one checks that if

$$k+1 > \frac{1-\beta}{\varepsilon(\rho-\beta)}\left(\frac{1}{\rho-\beta}+\varepsilon\right), \tag{20}$$

then again

$$z^{\rho-\beta-\frac{1-\beta}{k+1}} > \log x.$$

Hence, if we choose k_0 to be the smallest positive integer such that both inequalities (18) and (20) hold, then estimate (19) implies estimate (15). □

Proof of Theorem 3. Let $\mathcal{B}(x)$ be the set of all those positive integers $n\le x$ not in $\mathcal{A}_1(x)\cup\mathcal{A}_2(x)$ for some constant β to be found later. Write $n=ab$, where a and b are coprime, $P(a)\le z$, and every prime factor of b is $>z$. Clearly, $\omega(a)\le\pi(z)<z$ and $P(b)\le y$. (Here y and z are as in Lemmas 5 and 6, respectively.) To find $\omega(b)$, we note that if p is any prime factor of b, then the interval $[p,\ p+p^{\beta}]$ contains at most k_0+1 prime factors of b (including p itself). Since $p>z$, the length of this interval is at least z^{β}. We now claim that every interval of length z^{β} contained in $[z,\ y)$ contains at most k_0+1 prime factors of b. Indeed, assume that this is not the case, and let $\mathcal{I}$ be such an interval containing k_0+2 prime factors of b. Let p be the smallest one in $\mathcal{I}$. Then $p^{\beta}>z^{\beta}$ therefore all the primes in $\mathcal{I}$ are also in $[p,\ p+p^{\beta}]$, which contradicts the fact that n is not in $\mathcal{A}_2(x)$. Since $[z,\ y]$ can be partitioned into at most $\lfloor y/z^{\beta}\rfloor+1$ intervals of length z^{β}, it follows that

$$\omega(b)\ll y/z^{\beta}=(\log x)^{\frac{1}{\rho-1}-\frac{\beta}{\rho-\beta}-\beta\varepsilon}.$$

Hence,

$$\omega(n) = \omega(a) + \omega(b) \le z + \omega(b) \ll (\log x)^{\frac{1}{\rho-\beta}+\varepsilon} + (\log x)^{\frac{1}{\rho-1}-\frac{\beta}{\rho-\beta}-\beta\varepsilon}.$$

The above argument suggests that in order to make $\omega(n)$ as small as possible, we should choose β such that

$$\frac{1}{\rho-\beta} + \varepsilon = \frac{1}{\rho-1} - \frac{\beta}{\rho-\beta} - \beta\varepsilon,$$

which leads to

$$\beta = \frac{1}{\rho} + O(\varepsilon),$$

where the constant understood in the above O is absolute. This shows that

$$\omega(n) \le (\log x)^{\eta+O(\varepsilon)},$$

where $\eta = 1/(\rho-\beta) = \rho/(\rho^2-1)$.

We now count the number of such integers. Assume that $\{q_1, \ldots, q_\ell\}$ are all the prime factors of n. Since $P(n) < y = (\log x)^\alpha$, we have that this set of prime factors of n can be chosen in at most

$$\sum_{\ell \le (\log x)^{\eta+O(\varepsilon)}} \binom{\lfloor y \rfloor}{\ell} \ll (\log x)^{\eta+O(\varepsilon)} y^{(\log x)^{\eta+O(\varepsilon)}} = \exp\left((\log x)^{\eta+O(\varepsilon)}\right)$$

ways. Furthermore, once the primes factors $q_1, \ldots, q_\ell$ have been chosen, we have that $n = q_1^{a_1} \ldots q_\ell^{a_\ell}$, where $a_i \le \log x/\log 2$. Thus, the exponents a_i for $i = 1, \ldots, \ell$, can be chosen in at most

$$(\log x/\log 2)^\ell = \exp\left((\log x)^{\eta+O(\varepsilon)}\right)$$

ways. In conclusion,

$$\#\mathcal{B}(x) \le \exp\left((\log x)^{\eta+O(\varepsilon)}\right),$$

and since by estimate (1) and Lemmas 5 and 6, we have that

$$\sum_{n\in\mathcal{B}(x)} m(n) = c(1+o(1))x^\rho,$$

we get that there exists $n_x \in \mathcal{B}(x)$ such that

$$\begin{aligned} m(n_x) &= \max\{m(n) \ : \ n \in \mathcal{B}(x)\} \\ &\gg \frac{x^\rho}{\#\mathcal{B}(x)} \\ &\geq \frac{x^\rho}{\exp\left((\log x)^{\eta+O(\varepsilon)}\right)} \\ &\geq \frac{n_x^\rho}{\exp\left((\log n_x)^{\eta+O(\varepsilon)}\right)}, \end{aligned}$$

which immediately implies the conclusion of the theorem. □

4 Historical remarks and arithmetical properties of $m(n)$

Kalmár proved in [11] for the error term $o(1)$ in (1) the bound

$$O(\exp(-\alpha \log\log x \log\log\log x)), \text{ with } \alpha < \frac{1}{2(\rho-1)\log 2} \approx 1.97996.$$

Ikehara devoted three papers to the estimates of $M(x)$. In [7], he gave weak bounds of the type $M(x) > x^{\rho-\varepsilon}$ on a sequence of x tending to infinity, and $M(x) < x^{\rho+\varepsilon}$ for all large enough x. In the review of [7], Kalmár pointed out a gap in the proof and sketched a correct argument. In [8], Ikehara gave a proof of (1) with an error bound $O(\exp(q \log\log x))$ for some constant $q < 0$, which is slightly weaker than Kalmár's result. Finally, in [9], he succeeded to get a stronger error bound

$$O(\exp(-\alpha(\log\log x)^\gamma)), \text{ with } \alpha > 0 \text{ and } \gamma < 4/3.$$

Hwang [6] obtained an improvement of Ikehara's last bound by replacing 4/3 with 3/2.

Rieger proved in [18], besides other results, that for every positive integers k, l with $(k, l) = 1$ one has

$$\sum_{n \leq x,\, n \equiv l\,(k)} m(n) = \frac{1+o(1)}{\varphi(k)} M(x) = \frac{-1}{\varphi(k)\rho\zeta'(\rho)} \cdot x^\rho (1 + o(1)).$$

Warlimont investigated in [22] variants of $m(n)$ counting ordered factorizations with distinct parts and with coprime parts and estimated their summatory functions. Hille in [5] proved that $m(n) = O(n^\rho)$ and that $m(n) > n^{\rho-\varepsilon}$ for infinitely many n. We already mentioned in Section 1 the remark of Erdős on $m(n)$ in [2] and in Section 2 we mentioned, used, and improved the result of Chor, Lemke and Mador [1] that $m(n) < n^\rho$ for all n.

We now turn to recurrences and explicit formulas. The recurrence $m(1) = 1$ and

$$m(n) = \sum_{d|n,\, d<n} m(d) \quad \text{for } n > 1 \tag{21}$$

is immediate from fixing the first part in a factorization. If we set $m^*(1) = 1/2$ and $m^*(n) = m(n)$ for $n > 1$, then $2m^*(n) = \sum_{d|n} m(d)$ holds for all $n \geq 1$ and Möbius inversion gives

$$m(n) = 2\left(\sum_i m\left(\frac{n}{q_i}\right) - \sum_{i<j} m\left(\frac{n}{q_i q_j}\right) + \cdots + (-1)^{r-1} m\left(\frac{n}{q_1 q_2 \ldots q_r}\right)\right), \tag{22}$$

where $n = q_1^{a_1} q_2^{a_2} \ldots q_r^{a_r} > 1$ and we must set $m(1) = 1/2$. Formulas (21) and (22) are from Hille's paper [5]. In fact, (22) is stated there incorrectly with $m(1) = 1$, as was pointed out by Kühnel [12] and Sen [19].

Clearly, $m(p^a) = 2^{a-1}$ because ordered factorizations of p^a in parts > 1 are in bijection with (additive) compositions of a in parts > 0. If $p \neq q$ are primes and $a \geq b \geq 0$ are integers, we have the formula

$$m(p^a q^b) = 2^{a+b-1} \sum_{k=0}^{b} \binom{a}{k} \binom{b}{k} 2^{-k}$$

that was derived in [1] and before by Sen [19] and MacMahon [16]. In particular,

$$m(p^a q) = (a+2)2^{a-1} \quad \text{and} \quad m(p^a q^2) = (a^2 + 7a + 8)2^{a-2}. \tag{23}$$

In general, for $n = q_1^{a_1} q_2^{a_2} \ldots q_r^{a_r}$, and $a = a_1 + a_2 + \cdots + a_r$, MacMahon [16] derived the formula

$$m(q_1^{a_1} q_2^{a_2} \ldots q_r^{a_r}) = \sum_{j=1}^{a} \sum_{i=0}^{j-1} (-1)^i \binom{j}{i} \prod_{k=1}^{r} \binom{a_k + j - i - 1}{a_k}.$$

A more complicated summation formula for $m(q_1^{a_1}q_2^{a_2}\dots q_r^{a_r})$ but involving only nonnegative summands was obtained by Kühnel in [12] and [13]. Let $d_k(n)$ be the number of solutions of $n = n_1n_2\dots n_k$, where $n_i \geq 1$ are positive integers; so $d_2(n)$ is the number of divisors of n. Sklar [20] mentions the formula

$$m(n) = \sum_{k=1}^{\infty} \frac{d_k(n)}{2^{k+1}}. \tag{24}$$

Somewhat surprisingly, $m(n)$ has an additive definition in terms of integer partitions. We say that a partition $(1^{a_1}, 2^{a_2}, \dots, k^{a_k})$ of n is *perfect*, if for every $m < n$ there is exactly one k-tuple $(b_1, \dots, b_k)$, $0 \leq b_i \leq a_i$ for all i, such that $(1^{b_1}, 2^{b_2}, \dots, k^{b_k})$ is a partition of m. MacMahon [14] proved the identity

$$m(n) = \#\,\text{perfect partitions of } (n-1).$$

For example, since $m(12) = 8$, we have 8 perfect partitions of 11, namely $(1^2, 3, 6)$, $(1, 2^2, 6)$, $(1^5, 6)$, $(1, 2, 4^2)$, $(1^3, 4^2)$, $(1^2, 3^3)$, $(1, 2^5)$, and (1^{11}).

In conclusion of the survey of previous results we should remark that from enumerative point of view it is natural to consider $m(n)$ as a function of the partition $\lambda = (a_1, a_2, \dots, a_k)$ of $\Omega(n)$, where $n = q_1^{a_1}q_2^{a_2}\dots q_k^{a_k}$ with $a_1 \geq a_2 \geq \dots \geq a_k$, rather than n. Then $m(\lambda)$ is defined as the number of ways to write $\lambda = v_1 + v_2 + \dots + v_t$ where each v_i is a k-tuple of nonnegative integers, the order of summands matters, and no v_i is a zero vector. So $m(\lambda)$ is naturally understood as the number of k-dimensional compositions of λ. This approach pursued MacMahon in his memoirs [14], [15], and [16], see also [17].

The sequence

$$(m(n))_{n\geq 1} = (1, 1, 1, 2, 1, 3, 1, 4, 2, 3, 1, 8, 1, 3, 3, 8, 1, 8, 1, 8, 3, 3, 1, 20, 2, \dots)$$

forms entry A074206 of the database [21]. It follows immediately from the recurrence (22) that $m(n)$ is odd if and only if n is squarefree. Continuing the sequence a little further, we notice that $m(48) = 48$ and that $n = 48 = 2^4 \cdot 3$ is the smallest $n > 1$ such that $m(n) = n$. The first formula in (23) produces infinitely many n with this property: setting $n = 2^{2q-2}q$ with a prime $q > 2$, we get $m(n) = n$.

We record this observation as follows:

Proposition 1. *There exist infinitely many positive integers n such that $m(n) = n$.*

We now look at periodicity properties of the numbers $m(n)$. Recall that an integer valued function $f(n)$ defined on the set of positive integers is called *eventually periodic* modulo k if there exist integers n_0 and T such that $f(n) \equiv f(n+T) \pmod{k}$ for all $n > n_0$.

Proposition 2. *The function $m(n)$ is not eventually periodic modulo k for any positive integer $k \geq 2$.*

Proof. It suffices to prove the proposition when $k = p$ is a prime number. Assume, for the contradiction, that there are positive integers n_0 and T such that $m(n) \equiv m(n+T) \pmod{p}$ whenever $n > n_0$. Take a prime q such that $q^2 > n_0$ and $(q, T) = 1$. By Dirichlet's theorem on primes in arithmetic progressions, the progression $q^2, q^2 + T, q^2 + 2T, \ldots$ contains a prime r. But then $2 = m(q^2) \equiv m(r) = 1 \pmod{p}$ which is a contradiction. □

Recall now that a sequence $(f(n))_{n\geq 1}$ is polynomially recursive if there exist positive integers polynomials $g_0, \ldots, g_k$, not all zero, such that

$$g_k(n)f(n+k) + g_{k-1}(n)f(n+k-1) + \cdots + g_0(n)f(n) = 0 \qquad \text{for all } n \geq 1. \tag{25}$$

Proposition 3. *The sequence $m(n)$ is not polynomially recursive.*

Proof. Dividing (25) by one of the (nonzero) coefficients g_j with the largest degree, we obtain the relation

$$f(n+j) = \sum_{0\leq i\leq k, i\neq j} h_i(n)f(n+i)$$

where the h_i's are rational functions such that each $h_i(x)$ goes to a finite constant c_i as $x \to \infty$ (we may even assume that $|c_i| \leq 1$ for every i). Hence there is a constant $C > 0$ (depending only on k and the polynomials g_i) such that

$$|f(n)| \leq C \max\{|f(n+i)| : -k \leq i \leq k, i \neq 0\} \quad \text{for every } n \geq k+1.$$

We show that $(m(n))_{n\geq 1}$ violates this property.

We fix two integers $k, a \geq 1$ with the only restriction that a is coprime to each of the numbers $1, 2, \ldots, k$. It is an easy consequence of the Fundamental Lemma of the Combinatorial Sieve (see [3]) that there is a constant $K > 0$ depending only on k so that

$$\Omega((an-k)(an-k+1)\ldots(an-1)(an+1)\ldots(an+k)) \leq K$$

holds for infinitely many integers $n \geq 1$. For each of these n's the $2k$ values $m(an+i)$, $-k \leq i \leq k$ and $i \neq 0$, are bounded by a constant (depending only on k) while the value $m(an)$ is at least $m(a)$ and can be made arbitrarily large by an appropriate selection of a. This contradicts the above property of polynomially recursive sequences. □

Remark 1. *The above proof can be adapted in a straightforward way to show that other number theoretical functions $f(n)$ such as $\omega(n)$, $\Omega(n)$ and $\tau(n)$, where $\tau(n)$ is the number of divisors of n, have the property that $f(n)$ is not polynomially recursive.*

In what follows, we present some more estimates related to the function $m(n)$.

Proposition 4. *The estimate*

$$\#\{m(n):\ n \leq x\} \leq \exp\left(\pi\sqrt{2/\log 8}(1+o(1))(\log x)^{1/2}\right)$$

holds as $x \to \infty$.

Proof. Because $m(n)$ depends only on the partition $a_1 + \cdots + a_k = \Omega(n)$, where $n = q_1^{a_1} \ldots q_k^{a_k}$ ($q_1, \ldots, q_k$ are distinct primes and $a_1 \geq a_2 \geq \cdots \geq a_k > 0$ are integers), we have that

$$\#\{m(n) : n \leq x\} \leq p(1) + p(2) + \cdots + p(r) \leq rp(r)$$

where $p(n)$ denotes the number of partitions of n and $r = \max_{n \leq x} \Omega(n)$. The result follows from $r \leq \log x/\log 2$ and the classic asymptotics $p(n) \sim \exp(\pi\sqrt{2n/3})/(4n\sqrt{3})$ due to Hardy and Ramanujan [4]. □

We show that the same bound on the number of distinct values of $m(n)$ holds when the condition $n \leq x$ is replaced with $m(n) \leq x$. We need a lemma.

Lemma 7. *If $n_1, n_2, \ldots, n_k$ are positive integers such that for no $i \neq j$ we have $n_i | n_j$, then*

$$m(n_1 n_2 \ldots n_k) \geq k! \cdot m(n_1) m(n_2) \ldots m(n_k).$$

This implies that for every $n \geq 1$ we have

$$m(n) \geq \omega(n)! \cdot 2^{\Omega(n)-\omega(n)} \quad \text{and} \quad m(n) \geq 2^{\Omega(n)-1}.$$

Proof. Let X be the set of all k-tuples $(u_1, u_2, \ldots, u_k)$ where u_i is an ordered factorization of n_i in parts bigger than 1 and let Y be the set of these factorizations for $n_1 n_2 \ldots n_k$. For every permutation σ of $1, 2, \ldots, k$ we define a mapping $F_\sigma : X \to Y$ by

$$F_\sigma((u_1, u_2, \ldots, u_k)) = u_{\sigma(1)} \cdot u_{\sigma(2)} \cdot \ldots \cdot u_{\sigma(k)},$$

i.e., we concatenate factorizations u_i in the order prescribed by σ. It is clear that each F_σ is an injection. Suppose that $F_\sigma((u_1, u_2, \ldots, u_k)) = F_\tau((v_1, v_2, \ldots, v_k))$ for some permutations σ, τ and factorizations u_i and v_i. It follows that $u_{\sigma(1)}$ is an initial segment of $v_{\tau(1)}$ or vice versa and hence $n_{\sigma(1)}$ divides $n_{\tau(1)}$ or vice versa. This implies that $\sigma(1) = \tau(1)$ and $u_{\sigma(1)} = v_{\tau(1)}$. Applying the same argument we obtain that $\sigma(j) = \tau(j)$ and $u_{\sigma(j)} = v_{\tau(j)}$ also for $j = 2, \ldots, k$. Thus $\sigma = \tau$ and $u_j = v_j$ for $j = 1, 2, \ldots, k$. We have proved that the $k!$ mapings F_σ have mutually disjoint images. Therefore $k! \cdot m(n_1) m(n_2) \ldots m(n_k) = k! |X| \leq |Y| = m(n_1 n_2 \ldots n_k)$.

If $n = q_1^{a_1} q_2^{a_2} \ldots q_k^{a_k}$ is the prime factorization of n, applying the first inequality to the k numbers $n_i = q_i^{a_i}$ and using that $m(p^a) = 2^{a-1}$, we obtain

$$m(n) \geq k! \prod_{i=1}^{k} 2^{a_i - 1} = k! \cdot 2^{\Omega(n)-k}$$

which is the second inequality. Using that $k!/2^k \geq 1/2$ for every $k \geq 1$, we get the third inequality. □

Note that $m(n) \geq 2^{\Omega(n)-1}$ is tight for every $n = p^a$.

Proposition 5. *The estimate*

$$\#\{m(n) : \ m(n) \leq x, n \geq 1\} \leq \exp\left(\pi\sqrt{2/\log 8}(1 + o(1))(\log x)^{1/2}\right)$$

holds as $x \to \infty$.

Proof. As in Proposition 4 we have

$$\#\{m(n): \ m(n) \leq x, n \geq 1\} \leq p(1) + p(2) + \cdots + p(r) \leq rp(r)$$

where now $r = \max_{m(n) \leq x} \Omega(n)$. By the third inequality in the previous Lemma, $2^{r-1} = 2^{\Omega(n)-1} \leq m(n) \leq x$ for some n. Thus $r \leq 1 + \log x / \log 2$ and the result follows as in the proof of Proposition 4 using the asymptotics of $p(n)$. □

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