On identities concerning the numbers of crossings and nestings of two edges in matchings

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Abstract

Let M, N be two matchings on $[2n]$ (possibly $M = N$) and for an integer $l \geq 0$ let $\mathcal{T}(M, l)$ be the set of those matchings on $[2n + 2l]$ which can be obtained from M by successively adding l times in all ways the first edge, and similarly for $\mathcal{T}(N, l)$. Let $s, t \in \{cr, ne\}$ where cr is the statistic of the number of crossings (in a matching) and ne is the statistic of the number of nestings (possibly $s = t$). We prove that if the statistics s and t coincide on the sets of matchings $\mathcal{T}(M, l)$ and $\mathcal{T}(N, l)$ for $l = 0, 1$, they must coincide on these sets for every $l \geq 0$; similar identities hold for the joint statistic of cr and ne. These results are instances of a general identity in which crossings and nestings are weighted by elements from an abelian group.

1 Introduction and formulation of the main result

In this article we investigate distributions of the numbers of crossings and nestings of two edges in matchings. For example, it is known that for each k and n there are as many matchings M on $\{1, 2, \ldots, 2n\}$ with k crossings as those with k nestings. All matchings form an infinite tree $\mathcal{T}$ rooted in the

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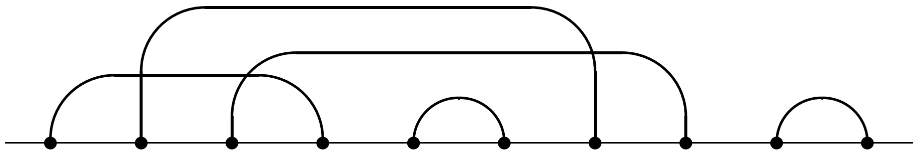

Figure 1: Matching with 3 crossings and 2 nestings.

empty matching $\emptyset$, in which the children of M are the matchings obtained from M by adding to M in all possible ways new first edge. The problem we address is this: Given two (not necessarily distinct) matchings M and N on $\{1, 2, \ldots, 2n\}$, when is it the case that the numbers of crossings (or nestings, or crossings versus nestings) have the same distributions on the levels of the two subtrees of $\mathcal{T}$ rooted in M and N. Our main result is Theorem 1.1 that determines when this happens, in fact in a more general setting. Before formulating it we give definitions and fix notation.

We denote the set $\{1, 2, 3, \ldots\}$ by $\mathbf{N}$, the set $\mathbf{N} \cup \{0\}$ by $\mathbf{N}_0$, and (for $n \in \mathbf{N}$) the set $\{1, 2, \ldots, n\}$ by $[n]$. The cardinality of a set A is denoted $|A|$. By a *multiset* we understand a "set" in which repetitions of elements are allowed. This can be modeled by a pair $H = (X, m)$ where X is a set, the *groundset* of the multiset H, and the mapping $m: X \to \mathbf{N}$ determines the multiplicities of the elements in H. However, we will not need this formalism and will record multiplicities by repetitions. A *matching* M on $[2n]$ is a set partition of $[2n]$ in n two-element *blocks* which we also call *edges*. The set of all matchings on $[2n]$ is denoted $\mathcal{M}(n)$; we define $\mathcal{M}(0) = \{\emptyset\}$. Two distinct blocks A and B of M form a *crossing* (they *cross*) if $\min A < \min B < \max A < \max B$ or $\min B < \min A < \max B < \max A$. Similarly, they form a *nesting* (they are *nested*) if $\min A < \min B < \max B < \max A$ or $\min B < \min A < \max A < \max B$. We draw a diagram of M in which we put the elements $1, 2, \ldots, 2n$ as points on a line, from left to right, and connect by a semicircular arc lying above the line the two points of each block. For two crossing blocks the corresponding arcs intersect and for two nested blocks one of the arcs covers the other, see Figure 1. By $cr(M)$, respectively $ne(M)$, we denote the number of crossings, respectively nestings, in M. The n edges of $M \in \mathcal{M}(n)$ are naturally ordered by their first elements. The first edge of M is $\{1, x\}$ and the last edge is the one whose first vertex is the last one among the n first vertices.

We investigate distribution of the numbers $cr(M)$ and $ne(M)$ on $\mathcal{M}(n)$

and on the subsets of $\mathcal{M}(n)$ defined by prescribing the matching formed by the last k edges of M. The total number of matchings in $\mathcal{M}(n)$ is

$$|\mathcal{M}(n)| = (2n-1)!! = 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1).$$

It is known that the number of matchings on $[2n]$ with no crossing equals the number of matchings with no nesting and that it is the n-th Catalan number, see Stanley [9, Problems 6.19o and 6.19ww]:

$$|\{M \in \mathcal{M}(n) : \; cr(M) = 0\}| = |\{M \in \mathcal{M}(n) : \; ne(M) = 0\}| = \frac{1}{n+1}\binom{2n}{n}.$$

The more general result that for each k and n

$$|\{M \in \mathcal{M}(n) : \; cr(M) = k\}| = |\{M \in \mathcal{M}(n) : \; ne(M) = k\}|$$

was derived by M. de Sainte-Catherine in [7]. Even more is true because the joint statistic is symmetric:

$$|\{M \in \mathcal{M}(n) : \; cr(M) = k, ne(M) = l\}| = \\ = |\{M \in \mathcal{M}(n) : \; cr(M) = l, ne(M) = k\}|$$

for every $k, l \in \mathbf{N}_0$ and $n \in \mathbf{N}$. A simple proof for this symmetry can be given by adapting the Touchard-Riordan method ([10], [6]) that encodes matchings and their numbers of crossings by weighted Dyck paths, see Klazar and Noy [5]. Here we put these results in a more general framework.

By the *tree of matchings* $\mathcal{T} = (\mathcal{M}, E, r)$ we understand the infinite rooted tree with the vertex set

$$\mathcal{M} = \bigcup_{n=0}^{\infty} \mathcal{M}(n),$$

which is rooted in the empty matching $r = \emptyset$ and in which directed edges in E are the pairs (M, N) such that $M \in \mathcal{M}(n)$, $N \in \mathcal{M}(n+1)$, and N arises from M by adding a new first edge, that is, we relabel the vertices of M as $\{2, 3, \dots, 2n+2\} \backslash \{x\}$ for some $x \in \{2, 3, \dots, 2n+2\}$ and add to M the block $\{1, x\}$, see Figure 2. Each vertex $N \in \mathcal{M}(n)$ has $2n+1$ children and, if $n > 0$, is a child of a unique vertex $M \in \mathcal{M}(n-1)$. A *level* in a rooted tree is the set of vertices with the same distance from the root. In $\mathcal{T}$ the levels are the sets $\mathcal{M}(n)$. The *subtree* $\mathcal{T}(M)$ of $\mathcal{T}$ rooted in $M \in \mathcal{M}(n)$ is the rooted subtree on the vertex set $\mathcal{N} \subset \mathcal{M}$ consisting of M and all

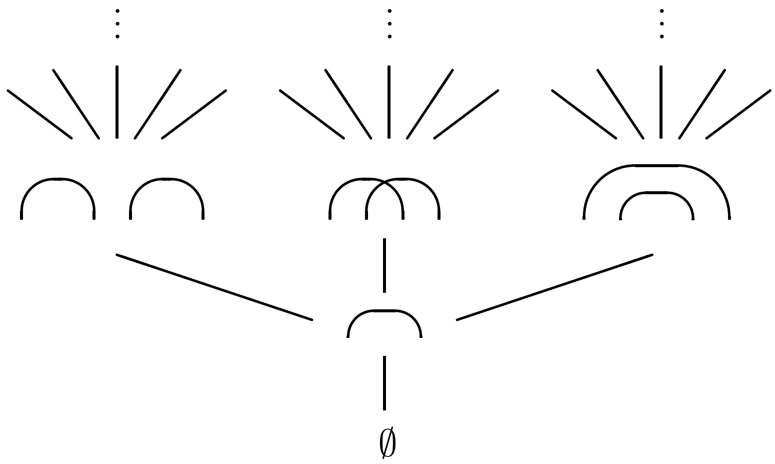

Figure 2: Tree of matchings $\mathcal{T}$.

its descendants, that is, $\mathcal{N}$ contains M and all matchings obtained from M by successively adding new first edge. In other words, $\mathcal{T}(M)$ consists of all $N \in \mathcal{M}$ in which the last n edges form a matching (order-isomorphic to) M. Clearly, $\mathcal{T}(\emptyset) = \mathcal{T}$. We denote the l-th level of $\mathcal{T}(M)$ by $\mathcal{T}(M, l)$. For $M \in \mathcal{M}(n)$ we have $\mathcal{T}(M, 0) = \{M\}$ and $\mathcal{T}(M, 1)$ is the set of children of M in $\mathcal{T}$. Also, $|\mathcal{T}(M, l)| = (2n+1)(2n+3)\dots(2n+2l-1)$.

Besides the statistics $cr(M) \in \mathbf{N}_0$ and $ne(M) \in \mathbf{N}_0$ on $\mathcal{M}$ we consider the joint statistics $cn(M) = (cr(M), ne(M)) \in \mathbf{N}_0^2$ and $nc(M) = (ne(M), cr(M)) \in \mathbf{N}_0^2$. Two statistics s, u on two subsets $\mathcal{N}_1, \mathcal{N}_2 \subset \mathcal{M}$ *coincide (have the same distribution)* if $s(\mathcal{N}_1) = u(\mathcal{N}_2)$ as multisets, that is, if for every element e we have

$$|\{M \in \mathcal{N}_1 : \ s(M) = e\}| = |\{M \in \mathcal{N}_2 : \ u(M) = e\}|.$$

Notational convention. If $f : \ X \to Y$ is a mapping and $Z \subset Y$, the symbol $f(Z)$ usually denotes the image $Im(f|Z) = \{f(z) : \ z \in Z\}$. In this article we use $f(Z)$ to denote the multiset whose ground set is $Im(F|Z)$ and in which each element $y = f(z)$, $z \in Z$, appears with the multiplicity $|f^{-1}(y) \cap Z|$. So in our $f(Z)$ each element y has the proper multiplicity in which it is attained as a value of f on Z.

Let $A = (A, +)$ be an abelian group and $\alpha, \beta \in A$ be its two elements. The most general statistic on matchings that we consider is $s_{\alpha,\beta} : \ \mathcal{M} \to A$

given by

$$s_{\alpha,\beta}(M) = cr(M)\alpha + ne(M)\beta.$$

Our main result is the next theorem.

Theorem 1.1 *Let $M, N \in \mathcal{M}(n)$ be two (not necessarily distinct) matchings and, for $\alpha, \beta \in A$, $s_{\alpha,\beta}$ be the above statistic.*

1. *If $s_{\alpha,\beta}(\mathcal{T}(M,l)) = s_{\alpha,\beta}(\mathcal{T}(N,l))$ for $l = 0,1$ then $s_{\alpha,\beta}(\mathcal{T}(M,l)) = s_{\alpha,\beta}(\mathcal{T}(N,l))$ for all $l \geq 0$.*

2. *If $s_{\alpha,\beta}(\mathcal{T}(M,l)) = s_{\beta,\alpha}(\mathcal{T}(N,l))$ for $l = 0,1$ then $s_{\alpha,\beta}(\mathcal{T}(M,l)) = s_{\beta,\alpha}(\mathcal{T}(N,l))$ for all $l \geq 0$.*

In words, for the statistic $s_{\alpha,\beta}$ to coincide level by level on the subtrees $\mathcal{T}(M)$ and $\mathcal{T}(N)$ it suffices if it coincides on the first two levels, and similarly for the pair of statistics $s_{\alpha,\beta}, s_{\beta,\alpha}$.

Specializing, we obtain identities for the statistics cr, ne, cn, and nc.

Theorem 1.2 *Let $M, N \in \mathcal{M}(n)$ be two (not necessarily distinct) matchings and $s, t \in \{cr, ne\}$, $u, v \in \{cn, nc\}$ be statistics on matchings (we allow $s = t$ and $u = v$).*

1. *If $s(\mathcal{T}(M,l)) = t(\mathcal{T}(N,l))$ for $l = 0,1$ then $s(\mathcal{T}(M,l)) = t(\mathcal{T}(N,l))$ for all $l \geq 0$.*

2. *If $u(\mathcal{T}(M,l)) = v(\mathcal{T}(N,l))$ for $l = 0,1$ then $u(\mathcal{T}(M,l)) = v(\mathcal{T}(N,l))$ for all $l \geq 0$.*

Proof. 1. Let $A = (\mathbf{Z}, +)$. Setting $\alpha = 1, \beta = 0$ and $\alpha = 0, \beta = 1$ and using 1 and 2 of Theorem 1.1, we obtain the identities for cr and ne.

2. Let $A = (\mathbf{Z}^2, +)$. Setting $\alpha = (1,0), \beta = (0,1)$ and $\alpha = (0,1), \beta = (1,0)$ and using 1 and 2 of Theorem 1.1, we obtain the identities for cn and nc. $\square$

We illustrate the last theorem by four examples. We mentioned the first two already, it is the result of de Sainte-Catherine and the symmetry $cn = nc$.

Corollary 1.3 *For every $k \in \mathbf{N}_0$ and $n \in \mathbf{N}$ there are as many matchings on $[2n]$ with k crossings as those with k nestings.*

Proof. Set $M = N = \emptyset$ and $s = cr, t = ne$. The assumption of the theorem is satisfied because $cr(\emptyset) = ne(\emptyset) = 0$ and $cr(\mathcal{M}(1)) = ne(\mathcal{M}(1)) = \{0\}$. $\square$

Corollary 1.4 *For every $k, l \in \mathbf{N}_0$ and $n \in \mathbf{N}$ there are as many matchings on $[2n]$ with k crossings and l nestings, as those with l crossings and k nestings; the joint statistic is symmetric.*

Proof. Set $M = N = \emptyset$ and $s = cn, t = nc$. The assumption of the theorem is satisfied because $cn(\emptyset) = nc(\emptyset) = (0, 0)$ and $cn(\mathcal{M}(1)) = nc(\mathcal{M}(1)) = \{(0, 0)\}$. $\square$

Corollary 1.5 *For every $k \in \mathbf{N}_0$ and $n \in \mathbf{N}$ there are as many matchings on $[2n]$ which have k crossings and have the last two edges nested, as those which have k nestings and have the last two edges separated (neither crossing nor nested).*

Proof. Set $M = \{\{1, 4\}, \{2, 3\}\}$, $N = \{\{1, 2\}, \{3, 4\}\}$, $s = cr$, and $t = ne$. The assumption of the theorem is satisfied because $cr(M) = ne(N) = 0$ and the values of cr on the five children of M are $0, 0, 1, 1, 2$, which coincides with the values of ne on the five children of N. $\square$

Corollary 1.6 *Let $M = \{\{1, 2\}, \{3, 5\}, \{4, 6\}\}$ and $N = \{\{1, 3\}, \{2, 4\}, \{5, 6\}\}$. For every $k, n \in \mathbf{N}$ there are as many matchings on $[2n]$ with k crossings in which the last three edges form a matching order-isomorphic to M, as those in which the last three edges form a matching order-isomorphic to N.*

Proof. Set the matchings M, N as given and $s = t = cr$. Then $cr(M) = cr(N) = 1$ and $cr(\mathcal{T}(M, 1)) = cr(\mathcal{T}(N, 1)) = \{1, 1, 1, 2, 2, 2, 3\}$. $\square$

We call two matchings $M, N \in \mathcal{M}(n)$ *crossing-similar* and write $M \sim_{cr} N$ if $cr(\mathcal{T}(M, l)) = cr(\mathcal{T}(N, l))$ for all $l \geq 0$. Similarly we define the *nesting-similarity* $\sim_{ne}$. These two relations are equivalences and partition $\mathcal{M}(n)$ in equivalence classes. We use Theorem 1.2 to characterize these classes and to count them. In Theorems 3.3 and 3.5 we prove that the numbers of classes in $\mathcal{M}(n)/\!\sim_{cr}$ and $\mathcal{M}(n)/\!\sim_{ne}$ are, respectively,

$$2^{n-2}\left(\binom{n}{2} + 2\right) \text{ and } 2 \cdot 4^{n-1} - \frac{3n-1}{2n+2}\binom{2n}{n}.$$

These two numbers differ, the latter is roughly a square of the former. On the first level of description of the enumerative complexity of crossings and nestings, that of the numbers $cr(M)$ and $ne(M)$, symmetry reigns as shown in Corollaries 1.3 and 1.4. On the next level of description, that of the similarity classes, symmetry is broken because $|\mathcal{M}(n)/\sim_{ne}|$ is much bigger than $|\mathcal{M}(n)/\sim_{cr}|$. From this point of view nestings are definitely more complicated than crossings, see also Theorem 4.4.

We prove Theorem 1.1 in Section 2. The method we employ is induction on the number of edges. In Section 3 we prove Theorems 3.3 and 3.5 enumerating the crossing-similarity and nesting-similarity classes. In the last Section 4 we give further applications of the main theorem in Proposition 4.1 that characterizes the matchings M, N such that $cr(\mathcal{T}(M,l)) = ne(\mathcal{T}(N,l))$ for every $l \geq 0$, in Corollary 4.3 that concerns the statistic of pairs of separated edges, and in Theorem 4.4 that enumerates the classes of mod 2 crossing-similarity and mod 2 nesting-similarity. We also give some concluding comments.

2 The proof of Theorem 1.1

For a set X let $\mathcal{S}(X)$ be the set of all *finite multisets* with elements in X. By the sum

$$X_1 + X_2 + \cdots + X_r = \sum_1^r X_i$$

of the multisets $X_1, X_2, \ldots, X_r \in \mathcal{S}(X)$ we mean the union of their groundsets with multiplicities of the elements added. Any function $f\colon X \to \mathcal{S}(Y)$ naturally extends to

$$f:\ \mathcal{S}(X) \to \mathcal{S}(Y) \ \text{ by } \ f(U) = \sum_{x \in U} f(x)$$

where the summand $f(x)$ appears with the multiplicity of x in U. Now if $Z \subset X$, we can understand the symbol $f(Z)$ in two ways—as the image of $f|Z$ or as the value of the extended f on Z. Due to the convention on image we get in both cases the same result.

In this section A shall denote an abelian group $(A, +)$ and A^* will be the set of finite sequences over A. We shall work with functions from A^* to $\mathcal{S}(A)$ or to $\mathcal{S}(A^*)$ which we will extend in the mentioned way, often without explicit notice, to functions defined on $\mathcal{S}(A^*)$. If $u = x_1x_2\dots x_t \in A^*$ and $y \in A$, by $x_1x_2\dots x_t + y$ we denote the sequence $(x_1+y)(x_2+y)\dots(x_t+y)$ obtained by adding y to each term of u.

Definition 2.1 *For $\alpha, \beta \in A$ and $i \in \mathbf{N}$ we define the mapping $R_{\alpha,\beta,i} : \bigcup_{l\ge i} A^l \to \bigcup_{l\ge i+2} A^l$ by*

$$R_{\alpha,\beta,i}(x_1x_2\dots x_l) = x_i(x_1x_2\dots x_i + x_i - x_1 + \alpha)(x_ix_{i+1}\dots x_l + x_i - x_1 + \beta)$$

and the mapping $R_{\alpha,\beta} :\ A^* \to \mathcal{S}(A^*)$ by

$$R_{\alpha,\beta}(x_1x_2\dots x_l) = \{R_{\alpha,\beta,i}(x_1x_2\dots x_l) :\ 1 \le i \le l\}.$$

So $R_{\alpha,\beta}(x_1x_2\dots x_l)$ is an l-element multiset of sequences which have length $l+2$.

Let $M \in \mathcal{M}(n)$ be a matching. The *gaps* of M are the first gap before $[2n]$, the $2n-1$ gaps between the elements in $[2n]$, and the last $(2n+1)$-th gap after $[2n]$; M has $2n+1$ gaps. For $\alpha, \beta \in A$ we assign to every matching $N \in \mathcal{M}(n)$, $n \in \mathbf{N}_0$, a sequence $seq_{\alpha,\beta}(N) \in A^*$ with length $2n+1$. If $n = 0$, we set $seq_{\alpha,\beta}(\emptyset) = 0 = 0_A$. Let $n \ge 1$ and $(M, N) \in E(\mathcal{T})$, $M \in \mathcal{M}(n-1)$, which means that N is obtained from M by adding a new first edge $e = \{1, x\}$ where x is inserted in the i-th gap of M for some i lying between 1 and $2n-1$. We set

$$seq_{\alpha,\beta}(N) = R_{\alpha,\beta,i}(seq_{\alpha,\beta}(M)).$$

For example, if $M = \{\{1, 3\}, \{2, 4\}\}$ then $seq_{\alpha,\beta}(M) = \alpha, 2\alpha, 3\alpha, 2\alpha + \beta, \alpha + 2\beta$.

For $u \in A^*$ we denote by $R^l_{\alpha,\beta}(u) = R_{\alpha,\beta}(R_{\alpha,\beta}(\dots(R_{\alpha,\beta}(u))\dots))$ the l-th iteration of the mapping $R_{\alpha,\beta}$ (which we extend to $\mathcal{S}(A^*)$). The next lemma is immediate from the definitions.

Lemma 2.2 *For every $\alpha, \beta \in A$, $M \in \mathcal{M}$, and $l \in \mathbf{N}_0$ we have*

$$R^l_{\alpha,\beta}(seq_{\alpha,\beta}(M)) = seq_{\alpha,\beta}(\mathcal{T}(M, l)).$$

The next lemma relates the sequences $seq_{\alpha,\beta}(M)$ and the statistic $s_{\alpha,\beta}$ on $\mathcal{M}$.

Lemma 2.3 *For every $\alpha, \beta \in A$ and $N \in \mathcal{M}(n)$ the first term of the sequence $seq_{\alpha,\beta}(N)$ equals $s_{\alpha,\beta}(N) = cr(N)\alpha + ne(N)\beta$.*

Proof. For $n = 0$ this holds. For $n \geq 1$ we proceed by induction on n. Suppose that $(M, N) \in E(\mathcal{T})$ and that N arises by adding new first edge $\{1, x\}$ to M, where x is inserted in the i-th gap. Let $seq_{\alpha,\beta}(M) = a_1 a_2 \ldots a_{2n-1}$.

We claim that in

$$a_j - a_1 = u_j \alpha + v_j \beta$$

the number u_j counts the edges in M covering the j-th gap and v_j counts the edges in M lying to the left of the j-th gap.

Suppose that this claim holds. Then $cr(N) = cr(M) + u_i$ and $ne(N) = ne(M) + v_i$. Since $cr(M)\alpha + ne(M)\beta = a_1$ (by induction), the first term of $seq_{\alpha,\beta}(N)$ is $a_i = a_i - a_1 + a_1 = u_i\alpha + v_i\beta + cr(M)\alpha + ne(M)\beta = cr(N)\alpha + ne(N)\beta$, as we wanted to show.

It suffices to prove by induction on n the claim. For $n = 0$ it holds trivially. We assume that it holds for $seq_{\alpha,\beta}(M)$ and deduce it for $seq_{\alpha,\beta}(N)$; M, N, and i are as before. Let $seq_{\alpha,\beta}(N) = b_1 b_2 \ldots b_{2n+1}$. We first describe the changes in gaps caused by the addition of $\{1, x\}$ to M. A new first gap appears; it is of course covered by no edge and has no edge to its left. For $1 \leq j \leq i$ the j-th gap turns in the $(j+1)$-th one; these gaps get covered by one more edge and have the same numbers of edges to their left as before. The i-th gap is split in two which creates a new gap, the $(i+2)$-th one; it is covered by as many edges as the i-th gap in M but it has one more edge to its left. For $i + 1 \leq j \leq 2n - 1$ the j-th gap turns in the $(j+2)$-th one; these gaps are covered by as may edges as before but they have one more edge to their left.

By the definition of $R_{\alpha,\beta,i}$, $b_1 = a_i$, $b_j = a_{j-1} + a_i - a_1 + \alpha$ for $2 \leq j \leq i+1$, and $b_j = a_{j-2} + a_i - a_1 + \beta$ for $i + 2 \leq j \leq 2n + 1$. Thus $b_1 - b_1 = 0$, $b_j - b_1 = a_{j-1} - a_1 + \alpha = (u_{j-1} + 1)\alpha + v_{j-1}\beta$ for $2 \leq j \leq i + 1$, and $b_j - b_1 = a_{j-2} - a_1 + \beta = u_{j-2}\alpha + (v_{j-2} + 1)\beta$ for $i + 2 \leq j \leq 2n + 1$. This agrees with the described changes in gaps and so the claim holds for $seq_{\alpha,\beta}(N)$. □

Let us denote by $f_0^0 : A^* \to A$ the function taking the first term of a sequence and by $f_0^1 : A^* \to \mathcal{S}(A)$ the function creating the multiset of all terms of a sequence. By the definitions and Lemmas 2.2 and 2.3, if $seq_{\alpha,\beta}(M) = a_1 a_2 \ldots a_{2n+1}$ then

$$s_{\alpha,\beta}(\mathcal{T}(M, 1)) = f_0^0(R_{\alpha,\beta}(seq_{\alpha,\beta}(M))) = \{a_1, a_2, \ldots, a_{2n+1}\} = f_0^1(seq_{\alpha,\beta}(M)).$$

For the induction argument we will need more complicated functions besides f^0_0 and f^1_0. For an integer $r \geq 0$ and $\gamma \in A$ we define the function $f^r_\gamma \colon A^* \to \mathcal{S}(A)$ by

$$f^r_\gamma(x_1x_2\dots x_l) = \{x_{a_1}+x_{a_2}+\dots+x_{a_r}-(r-1)x_1+\gamma : 1 \leq a_1 \leq a_2 \leq \dots \leq a_r \leq l\}.$$

So $f^0_0(x_1x_2\dots x_l) = \{x_1\}$ and $f^1_\gamma(x_1x_2\dots x_l)$ is the multiset $\{x_1+\gamma, x_2+\gamma, \dots, x_l+\gamma\}$.

Lemma 2.4 *Let $X, Y \in \mathcal{S}(A^*)$ (possibly $X = Y$) be two multisets such that $f^r_\gamma(X) = f^r_\gamma(Y)$ for every $r \geq 0$ and $\gamma \in A$. Then for every mapping $R_{\alpha,\beta}$ of Definition 2.1 we have*

1. $f^r_\gamma(R_{\alpha,\beta}(X)) = f^r_\gamma(R_{\alpha,\beta}(Y))$
2. $f^r_\gamma(R_{\alpha,\beta}(X)) = f^r_\gamma(R_{\beta,\alpha}(Y))$

for every $r \geq 0$ and $\gamma \in A$.

Proof. We prove only the second identity with $R_{\alpha,\beta}$ and $R_{\beta,\alpha}$; the proof of the first one is similar and easier. We proceed by induction on r. The case $r = 0$ is clear since $f^0_\gamma(R_{\alpha,\beta}(X)) = f^1_\gamma(X)$ for every $X \in \mathcal{S}(A^*)$ and $\gamma \in A$. We assume that $r \geq 1$ and that for every s, $0 \leq s < r$, and $\gamma \in A$ we have $f^s_\gamma(R_{\alpha,\beta}(X)) = f^s_\gamma(R_{\beta,\alpha}(Y))$. We consider only the function f^r_0, the proof for general γ is similar.

We split the multisets $U = f^r_0(R_{\alpha,\beta}(X))$ and $V = f^r_0(R_{\beta,\alpha}(Y))$, which arise by summation, in several contributions and show that after rearranging, the corresponding contributions to U and V are equal. U is the multiset of elements $y_{a_1} + y_{a_2} + \dots + y_{a_r} - (r-1)y_1$ where the sequence $y_1y_2\dots y_l$ runs through $R_{\alpha,\beta}(X)$ and the indices a_i run through the r-tuples $1 \leq a_1 \leq a_2 \leq \dots \leq a_r \leq l$, and similarly for V. The first contribution C is defined by the condition $a_1 = 1$. C contributes to U the elements

$$y_1 + y_{a_2} + \dots + y_{a_r} - (r-1)y_1 = y_{a_2} + \dots + y_{a_r} - (r-2)y_1,$$

where $y_1y_2\dots y_l$ runs through $R_{\alpha,\beta}(X)$ and the indices a_i run through the $(r-1)$-tuples $1 \leq a_2 \leq a_3 \leq \dots \leq a_r \leq l$. Thus C contributes $f^{r-1}_0(R_{\alpha,\beta}(X))$. To V it contributes $f^{r-1}_0(R_{\beta,\alpha}(Y))$. Hence C contributes equally to U and V because $f^{r-1}_0(R_{\alpha,\beta}(X)) = f^{r-1}_0(R_{\beta,\alpha}(Y))$ by the inductive assumption.

Each $v = y_1y_2\dots y_l \in R_{\alpha,\beta}(X)$ is in $R_{\alpha,\beta}(u)$ for some $u = x_1x_2\dots x_{l-2} \in X$ and (by the definition of $R_{\alpha,\beta}$) consists of three segments: it starts with a term x_i of u, then it comes $x_1\dots x_i$ termwise incremented by $x_i - x_1 + \alpha$, and the third segment of v is $x_i\dots x_{l-2}$ termwise incremented by $x_i - x_1 + \beta$; similarly for $v \in R_{\beta,\alpha}(Y)$. We split the rest of U and V (in which $a_1 > 1$, i.e., every y_{a_i} lies in the second or in the third segment) in $r+1$ disjoint contributions C_t according to the number t, $0 \le t \le r$, of the y_{a_i}'s lying in the second segment. By the definition of $R_{\alpha,\beta}$, C_t contributes to U the elements

$$\begin{aligned} & x_{b_1} + \dots + x_{b_r} + t(x_i - x_1 + \alpha) + (r-t)(x_i - x_1 + \beta) - (r-1)x_i \\ = \; & x_{b_1} + \dots + x_{b_r} + x_i - rx_1 + t\alpha + (r-t)\beta \end{aligned}$$

where $u = x_1x_2\dots x_{l-2}$ runs through X, the indices b_j run through the r-tuples satisfying $1 \le b_1 \le \dots \le b_t \le i \le b_{t+1} \le \dots \le b_r \le l-2$, and i runs through $1 \le i \le l-2$. (The length $l-2$ depends on u.) Effectively the indices b_j and i run through all weakly increasing $(r+1)$-tuples of numbers from $[l-2]$. Thus C_t contributes to U the elements $f_\gamma^{r+1}(X)$ where $\gamma = t\alpha + (r-t)\beta$. By the definition of $R_{\beta,\alpha}$, C_t contributes to V the elements $f_{\gamma'}^{r+1}(Y)$ where $\gamma' = t\beta + (r-t)\alpha$. So C_t contributes to U and V in general differently but (by the assumption on X and Y) the contributions of C_t to U and C_{r-t} to V are equal. By symmetry, $\sum_0^r C_i$ contributes the same amount to U and V. Since U and V are covered by equal and disjoint contributions C and $\sum_0^r C_i$, we conclude that $U = V$, i.e., $f_0^r(R_{\alpha,\beta}(X)) = f_0^r(R_{\beta,\alpha}(Y))$.

The proof of 1 is similar and easier, because now C_t contributes equally to $U = f_0^r(R_{\alpha,\beta}(X))$ and $V = f_0^r(R_{\alpha,\beta}(Y))$. $\square$

Next we show that for the equality of all functions f_γ^r on two one-element sets it in fact suffices that f_0^0 and f_0^1 are equal. We prove it in two lemmas. Let $g^r\colon A^* \to \mathcal{S}(A)$ be defined by

$$g^r(x_1x_2\dots x_l) = \{x_{a_1} + x_{a_2} + \dots + x_{a_r} \colon 1 \le a_1 \le a_2 \le \dots \le a_r \le l\}.$$

Lemma 2.5 *If $u, v \in A^*$ are such that $g^1(u) = g^1(v)$ then $g^r(u) = g^r(v)$ for all $r \ge 1$.*

Proof. Let $g^1(u) = g^1(v)$ and $r \in \mathbf{N}$. For $\bar{a} = (a_1, \dots, a_s) \in A^s$ we denote $(\bar{a})$ the multiset $\{a_1, \dots, a_s\}$ and if $\bar{n} = (n_1, \dots, n_s) \in \mathbf{N}^s$ then $\bar{n} \cdot \bar{a} =$

$n_1a_1 + \cdots + n_sa_s \in A$. For $s \in \mathbf{N}$, $X \in \mathcal{S}(A)$, and $u = x_1x_2\ldots x_l \in A^*$ we denote

$$S(s, X, u) = \{\bar{x} = (x_{a_1}, \ldots, x_{a_s})\colon 1 \le a_1 < a_2 < \cdots < a_s \le l, (\bar{x}) = X\}.$$

For $r, s \in \mathbf{N}$ we denote

$$N(r, s) = \{(n_1, \ldots, n_s) \in \mathbf{N}^s\colon n_1 + \cdots + n_s = r\}.$$

Now we can rewrite $g^r(u)$ and $g^r(v)$ as

$$\begin{aligned} g^r(u) &= \{\bar{n} \cdot \bar{a} :\ s \in [r], X \in \mathcal{S}(A), \bar{n} \in N(r, s), \bar{a} \in S(s, X, u)\} \\ g^r(v) &= \{\bar{n} \cdot \bar{a} :\ s \in [r], X \in \mathcal{S}(A), \bar{n} \in N(r, s), \bar{a} \in S(s, X, v)\}. \end{aligned}$$

We claim that (i) for every fixed $s \in [r]$ and $X \in \mathcal{S}(A)$ the multiset

$$m(\bar{a}) = \{\bar{n} \cdot \bar{a} :\ \bar{n} \in N(r, s)\}$$

is the same for all $\bar{a} \in A^s$ with $(\bar{a}) = X$ and that (ii) for every fixed $s \in [r]$ and $X \in \mathcal{S}(A)$ we have $|S(s, X, u)| = |S(s, X, v)|$. This will prove that $g^r(u) = g^r(v)$.

To show (i), we take $\bar{a}, \bar{b} \in A^s$ with $(\bar{a}) = (\bar{b}) = X$. Then $\bar{a}$ can be obtained from $\bar{b}$ by permuting coordinates: $\bar{a} = \pi(\bar{b})$ for some $\pi \in \mathcal{S}_s$, and $\bar{n}\cdot\bar{b} = \pi(\bar{n})\cdot\bar{a}$. If $\bar{n}$ runs through $N(r, s)$, so does $\pi(\bar{n})$. Hence $m(\bar{a}) = m(\bar{b})$. To show (ii), we suppose that X consists of the distinct elements $x_1, \ldots, x_t$ with multiplicities $n_1, \ldots, n_t$ where $n_1 + \cdots + n_t = s$ (else $|S(s, X, u)| = |S(s, X, v)| = 0$) and denote by $m_a(u)$ and $m_a(v)$ the numbers of occurrences of $a \in A$ in u and v. Because $m_a(u) = m_a(v)$ for every $a \in A$, we have indeed

$$|S(s, X, u)| = \prod_{i=1}^{t} \binom{m_{x_i}(u)}{n_i} = \prod_{i=1}^{t} \binom{m_{x_i}(v)}{n_i} = |S(s, X, v)|.$$

$\square$

Lemma 2.6 *If $X, Y \in \mathcal{S}(A^*)$ are one-element sets such that $f_0^0(X) = f_0^0(Y)$ and $f_0^1(X) = f_0^1(Y)$, then $f_\gamma^r(X) = f_\gamma^r(Y)$ for every $r \ge 0$ and $\gamma \in A$.*

Proof. We need to prove that if $u, v \in A^*$ are two sequences beginning with the same term and having equal numbers of occurrences of each $a \in A$, then

$f_\gamma^r(u) = f_\gamma^r(v)$ for every $r \geq 0$ and $\gamma \in A$. It suffices to consider functions f_0^r, the proof with general γ is similar. Since u and v start with the same term, by the definition of f_0^r it suffices to prove that $g^r(u) = g^r(v)$ for every $r \geq 1$. This is true by Lemma 2.5. $\square$

Proof of Theorem 1.1. We prove only 2, the proof of 1 is very similar and easier. Let $s_{\alpha,\beta}(\mathcal{T}(M,l)) = s_{\beta,\alpha}(\mathcal{T}(N,l))$ for $l = 0, 1$. By Lemma 2.3 and the following remark, this means that $f_0^0(seq_{\alpha,\beta}(M)) = f_0^0(seq_{\beta,\alpha}(N))$ and $f_0^1(seq_{\alpha,\beta}(M)) = f_0^1(seq_{\beta,\alpha}(N))$. By Lemma 2.6,

$$f_\gamma^r(seq_{\alpha,\beta}(M)) = f_\gamma^r(seq_{\beta,\alpha}(N))$$

for every $r \in \mathbf{N}_0$ and $\gamma \in A$. By repeated application of 2 of Lemma 2.4 we get

$$f_\gamma^r(R_{\alpha,\beta}^l(seq_{\alpha,\beta}(M))) = f_\gamma^r(R_{\beta,\alpha}^l(seq_{\beta,\alpha}(N)))$$

for every $l, r \in \mathbf{N}_0$ and $\gamma \in A$. In particular,

$$f_0^0(R_{\alpha,\beta}^l(seq_{\alpha,\beta}(M))) = f_0^0(R_{\beta,\alpha}^l(seq_{\beta,\alpha}(N))).$$

But by Lemma 2.2 we have

$$R_{\alpha,\beta}^l(seq_{\alpha,\beta}(M)) = seq_{\alpha,\beta}(\mathcal{T}(M,l)) \quad \text{and} \quad R_{\beta,\alpha}^l(seq_{\beta,\alpha}(N)) = seq_{\beta,\alpha}(\mathcal{T}(N,l)).$$

Thus, by Lemma 2.3,

$$s_{\alpha,\beta}(\mathcal{T}(M,l)) = s_{\beta,\alpha}(\mathcal{T}(N,l))$$

for every $l \geq 0$, which we wanted to prove. $\square$

We give a formulation of Theorem 1.1 in terms of the sequences $seq_{\alpha,\beta}(M)$.

Theorem 2.7 *Let $M, N \in \mathcal{M}(n)$ be two (not necessarily distinct) matchings and $\alpha, \beta \in A$ be two elements of the abelian groups.*

1. *We have $s_{\alpha,\beta}(\mathcal{T}(M,l)) = s_{\alpha,\beta}(\mathcal{T}(N,l))$ for all $l \geq 0$ iff $s_{\alpha,\beta}(M) = s_{\alpha,\beta}(N)$ and the sequences $seq_{\alpha,\beta}(M)$ and $seq_{\alpha,\beta}(N)$ are equal as multisets (when order is neglected).*

2. *We have $s_{\alpha,\beta}(\mathcal{T}(M,l)) = s_{\beta,\alpha}(\mathcal{T}(N,l))$ for all $l \geq 0$ iff $s_{\alpha,\beta}(M) = s_{\beta,\alpha}(N)$ and the sequences $seq_{\alpha,\beta}(M)$ and $seq_{\beta,\alpha}(N)$ are equal as multisets.*

3 The numbers of similarity classes

In this section we determine the cardinalities $|\mathcal{M}(n)/\sim_{cr}|$ and $|\mathcal{M}(n)/\sim_{ne}|$. Let $A = (\mathbf{Z}, +)$. For $M \in \mathcal{M}$ we define its *crossing sequence* $crs(M)$ by $crs(M) = seq_{1,0}(M) - a_1$, where a_1 is the first term of $seq_{1,0}(M)$, and its *nesting sequence* $nes(M)$ by $nes(M) = seq_{0,1}(M) - b_1$, where b_1 is the first term of $seq_{0,1}(M)$. Recall that (by the proof of Lemma 2.3) the i-th term of $crs(M)$ is the number of edges in M covering the i-th gap and the i-th term of $nes(M)$ is the number of edges lying to the left of the i-th gap. For example, $M = \{\{1,4\},\{2,5\},\{3,6\}\}$ has $crs(M) = (0,1,2,3,2,1,0)$ and $nes(M) = (0,0,0,1,2,3)$. By Theorems 1.2 and 2.7, $M \sim_{cr} N \iff cr(M) = cr(N)$ & $f_0^1(crs(M)) = f_0^1(crs(N))$, that is, M and N are crossing-similar iff they have the same numbers of crossings and their crossing sequences are equal as multisets; an analogous result holds for the nesting-similarity.

Let $e = \{a,d\}, f = \{b,c\} \in M$, $1 \le a < b < c < d \le 2n$, be a nesting in $M \in \mathcal{M}(n)$. We define its *width* as $\min(b-a, d-c)$. We define the width of a crossing in the same way, only $\{a,d\}$ is replaced with $\{a,c\}$ and $\{b,c\}$ with $\{b,d\}$. Suppose the nesting e, f has the minimum width among all nestings in M and its width is realized by $b-a$. Switching the first vertices of the edges e and f, we obtain another matching N. If the width of e, f is realized by $d-c$, we switch the second vertices of e and f. This transformation $M \rightsquigarrow N$ is called the *n-c transformation.* In the same way, by switching the first or the second vertices of the edges in a crossing with minimum width, we define the *c-n transformation.*

Lemma 3.1 *Let $M, N \in \mathcal{M}(n)$ where N is obtained from M by the n-c (c-n) transformation. Then N has the same sets of first and second vertices of the edges as M and $ne(N) = ne(M) - 1, cr(N) = cr(M) + 1$ ($ne(N) = ne(M) + 1, cr(N) = cr(M) - 1$).*

Proof. The first claim about N is obvious. Let $e = \{a,c\}, f = \{b,d\} \in M$, $1 \le a < b < c < d \le 2n$, be a crossing in M with the minimum width which is equal to $b-a$ (if it is equal to $d-c$, the argument is similar). The c-n transformation replaces e by $e' = \{b,d\}$ and f by $f' = \{a,d\}$. Because of the minimality of the width, every edge of M that has one endpoint between a and b must have the other endpoint between a and b as well. It follows that e' crosses the same edges distinct from f as e does and similarly for f'

and f. The edge e' is covered by the same edges different from f' as e and similarly for f' and f. The edge e' does not cover the edges lying between a and b which were covered by e but these are now covered by f' and were not covered by f. If we do not consider the pairs e, f and e', f', M and N have the same numbers of crossings and the same numbers of nestings. Since e, f is a crossing and e', f' is a nesting, in total N has one less crossing and one more nesting than M. The argument for the n-c transformation is similar and is left to the reader. $\Box$

We use *Dyck paths* to encode $crs(M)$ and $nes(M)$. Recall that a Dyck path D with semilength $n \in \mathbf{N}$ is a lattice path $D = (d_0, d_1, \ldots, d_{2n})$, where $d_i \in \mathbf{Z}^2$, from $d_0 = (0, 0)$ to $d_{2n} = (2n, 0)$ that makes n up-steps $d_i - d_{i-1} = (1, 1)$, n down-steps $d_i - d_{i-1} = (1, -1)$, and never gets below the x axis (so in fact $d_i \in \mathbf{N}_0^2$). We denote the set of Dyck paths with semilength n by $\mathcal{D}(n)$; $|\mathcal{D}(0)| = 1$. We think of $D \in \mathcal{D}(n)$ also as a broken line in the plane that connects $(0, 0)$ with $(2n, 0)$ and consists of $2n$ straight segments $s_i = d_i d_{i+1}$, see Figure 3. A *tunnel* in D is a horizontal segment t that has altitude $n + \frac{1}{2}$ for some $n \in \mathbf{N}_0$, lies below D, and intersects D only in its endpoints. Each $D \in \mathcal{D}(n)$ has exactly n tunnels. Note that projections of two tunnels on the x axis are either disjoint or they are in inclusion (as in the example in Figure 3). If the latter happens, we say that the tunnel with larger projection *covers* the other tunnel.

Deleting from $D \in \mathcal{D}(n)$, $n \geq 1$, the first up step and the first downstep at which D visits again the x axis, we obtain, shifting appropriately the resulting two parts of D, a unique decomposition of D in a pair of Dyck paths E, F, where $E \in \mathcal{D}(m)$ for $0 \leq m < n$ and $F \in \mathcal{D}(n - 1 - m)$. This *decomposition of Dyck paths* can be used for inductive proofs of their properties.

We associate with every Dyck path $D = (d_0, d_1, \ldots, d_{2n})$ its *sequence of altitudes* $als(D) = (d_0^y, d_1^y, \ldots, d_{2n}^y) \in \mathbf{N}_0^{2n+1}$, where $d_i = (d_i^x, d_i^y)$, and its *profile* $pr(D) = (a_1, a_2, \ldots, a_m) \in \mathbf{N}^m$, where m is the maximum term of $als(D)$ and a_i is half of the number of segments s_i of D that lie in the horizontal strip $i-1 \leq y \leq i$. It follows that $a_1 + a_2 + \cdots + a_m = n$ and $pr(D)$ is a composition of n. It follows easily by induction on m that for every composition $a = (a_1, a_2, \ldots, a_m)$ of n there is a $D \in \mathcal{D}(n)$ with $pr(D) = a$. For example, the Dyck path in Figure 3 has $als(D) = (0, 1, 0, 1, 2, 1, 2, 3, 2, 3, 2, 1, 2, 1, 0)$ and $pr(D) = (2, 3, 2)$.

There is a natural surjective mapping $F: \mathcal{M}(n) \to \mathcal{D}(n)$ defined as

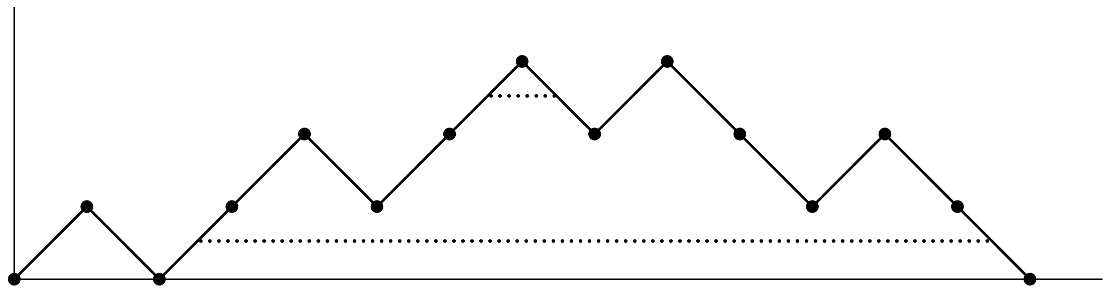

Figure 3: Dyck path with semilength 7 and two tunnels.

follows. We take the diagram of $M \in \mathcal{M}(n)$ and travel the baseline l from $-\infty$ to ∞. Simultaneously we construct, step by step, a lattice path D. We start D at $(0, 0)$ and when we encounter on l the first (second) vertex of an edge, we make in D an up-step (down-step). In the end we get a Dyck path $D \in \mathcal{D}(n)$ and set $F(M) = D$. Using the decomposition of Dyck paths and induction, it is easy to prove that F is surjective. Clearly, the preimages $F^{-1}(D)$ consist exactly of the matchings sharing the same sets of first and second vertices. Another important property of F is that for every $D \in \mathcal{D}(n)$ there is exactly one *noncrossing* (i.e., with $cr(M) = 0$) $M \in F^{-1}(D)$, namely the M whose edges correspond in the obvious way to the tunnels in D. This follows by the decomposition of Dyck paths.

Lemma 3.2 *Let $n \in \mathbf{N}$ and $F : \mathcal{M}(n) \to \mathcal{D}(n)$ be the above mapping.*

1. *For every $M \in \mathcal{M}(n)$ we have $crs(M) = als(F(M))$.*
2. *For every $M, N \in \mathcal{M}(n)$ we have $f_0^1(crs(M)) = f_0^1(crs(N))$ if and only if $pr(F(M)) = pr(F(N))$.*
3. *For every composition $a = (a_1, a_2, \ldots, a_m)$ of n and every $i \in \mathbf{N}_0$, $0 \le i \le \sum_{i=1}^{m}(i-1)a_i$, there is an $M \in \mathcal{M}(n)$ such that $pr(F(M)) = a$ and $cr(M) = i$. There exist no a and no M such that $pr(F(M)) = a$ and $cr(M) > \sum_{i=1}^{m}(i-1)a_i$.*

Proof. 1. This is clear from the definitions of $crs(M)$ and $als(D)$.

2. Using 1, we look at $f_0^1(als(D))$ where $D = F(M)$. Let $pr(D) = (a_1, a_2, \ldots, a_m)$ and r_i be the multiplicity of $i \in \mathbf{N}_0$ in $als(D)$. It is clear that $r_0 = a_1 + 1$ and $r_m = a_m$. We claim that for $0 < i < m$ we have

$r_i = a_i + a_{i+1}$. In the strip $i-1 \le y \le i$ we have $v = 2a_i$ segments $s_1, s_2, \ldots, s_v$ of D and in the strip $i \le y \le i+1$ we have $w = 2a_{i+1}$ segments $t_1, t_2, \ldots, t_w$. The occurrences of i in $als(D)$ are due to the upper endpoints of the s_j's and due to the lower endpoints of the t_j's. But for each s_j its upper endpoint coincides with the upper endpoint of s_{j-1} or with that of s_{j+1} or with the lower endpoint of some t_k, and similarly for the lower endpoints of the t_j's. So i appears $(v+w)/2 = a_i + a_{i+1}$ times. On the other hand, $a_i = r_{i-1} - r_{i-2} + \cdots + (-1)^i r_1 + (-1)^{i+1}(r_0 - 1)$ for every $1 \le i \le m$. Therefore the r_i's are completely determined by the composition $pr(D)$ and vice versa.

3. Let a composition $a = (a_1, a_2, \ldots, a_m)$ of n be given. We take an arbitrary $D \in \mathcal{D}(n)$ with $pr(D) = a$. It follows by the decomposition of Dyck paths and induction that the sum

$$S(a) = \sum_{i=1}^{m} (i-1)a_i$$

counts the ordered pairs t_1, t_2 of distinct tunnels in D where t_1 covers t_2. For the unique noncrossing $M \in F^{-1}(D)$ we have $ne(M) = S(a)$ because nestings in M are in 1-1 correspondence with the pairs of tunnels, one of them covering the other. So $cr(M) = 0, ne(M) = S(a), F(M) = D, pr(F(M)) = a$. For any given $i \in \{0, 1, \ldots, S(a)\}$, using repeatedly the n-c transformation of Lemma 3.1, we transform M into N such that $cr(N) = i, ne(N) = S(a) - i$, and $F(N) = F(M) = D$. Now suppose that there is an $M \in F^{-1}(D)$ with $cr(M) = c > S(a)$. Using the c-n transformation of Lemma 3.1 we transform it into $N \in F^{-1}(D)$ with $cr(N) = 0$ and $ne(N) = ne(M) + c > S(a)$. This contradicts the unicity of the noncrossing matching in $F^{-1}(D)$. $\square$

Theorem 3.3 *For $n \in \mathbf{N}$ the set $\mathcal{M}(n)/\sim_{cr}$ of crossing-similarity classes has*

$$2^{n-2}\left(\binom{n}{2} + 2\right)$$

elements.

Proof. By the previous lemma, $|\mathcal{M}(n)/\sim_{cr}|$ equals

$$\sum_a (1 + a_2 + 2a_3 + \cdots + (m-1)a_m) = 2^{n-1} + \sum_a (a_2 + 2a_3 + \cdots + (m-1)a_m)$$

where we sum over all compositions $a_1 + a_2 + \cdots + a_m = n$, which are 2^{n-1} in number. The last sum is the coefficient of x^n in the expansion of

$$\left(\frac{d}{dy} \sum_{m \geq 0} \frac{x}{1-x} \cdot \frac{xy}{1-xy} \cdot \frac{xy^2}{1-xy^2} \cdot \cdots \cdot \frac{xy^m}{1-xy^m}\right)\Bigg|_{y=1}.$$

Differentiating the product in the summand by the Leibniz rule and using that

$$\left(\frac{d}{dy} \frac{xy^i}{1-xy^i}\right)\Bigg|_{y=1} = \frac{ix}{(1-x)^2},$$

we obtain that the expansion equals

$$\frac{1}{1-x} \sum_{m \geq 0} \binom{m+1}{2} \left(\frac{x}{1-x}\right)^{m+1}.$$

Using the binomial expansion $(1-z)^{-r} = \sum_{n \geq 0} \binom{r+n-1}{n} z^n$, we simplify this to

$$\frac{x^2}{(1-2x)^3} = \sum_{n \geq 0} \binom{n+2}{2} 2^n x^{n+2}$$

and the result follows. $\square$

The values of $|\mathcal{M}(n)/\sim_{cr}|$ form the sequence $(1, 3, 10, 32, 96, 276, \ldots)$. Subtracting 2^{n-1}, we get the sequence $(0, 1, 6, 24, 80, 240, \ldots)$ that counts crossing-similarity classes in $\mathcal{M}(n)$ for matchings with at least one crossing. This sequence is entry A001788 of [8] and counts, for example, also 4-cycles in the $(n+1)$-dimensional hypercube.

The situation for nestings is simpler and the number of similarity classes is bigger because nesting sequences are nondecreasing and therefore

$$f_0^1(nes(M)) = f_0^1(nes(N)) \text{ iff } nes(M) = nes(N).$$

By Theorems 1.2 and 2.7, $M \sim_{ne} N$ iff M and N have the same numbers of nestings and the same nesting sequences. For $D \in \mathcal{D}(n)$ we define $ne(D)$ to be the number of ordered pairs t_1, t_2 of distinct tunnels in D such that t_1 covers t_2. The *down sequence* $dos(D)$ of $D = (d_0, d_1, \ldots, d_{2n})$ is $(v_0, v_1, \ldots, v_{2n})$ where v_i is the number of down-steps $d_j - d_{j-1} = (1, -1)$ for $1 \leq j \leq i$. For example, for the Dyck path in Figure 3 we have $ne(D) = 8$ and $dos(D) = (0, 0, 1, 1, 1, 2, 2, 2, 3, 3, 4, 5, 5, 6, 7)$.

Lemma 3.4 *Let $n \in \mathbf{N}$ and $F: \mathcal{M}(n) \to \mathcal{D}(n)$ be the mapping defined above.*

1. *For every $M \in \mathcal{M}(n)$ we have $nes(M) = dos(F(M))$. There is a bijection between the sets $\{nes(M) : M \in \mathcal{M}(n)\}$ and $\mathcal{D}(n)$.*

2. *For every Dyck path $D \in \mathcal{D}(n)$ and every $i \in \mathbf{N}_0$, $0 \le i \le ne(D)$, there is an $M \in F^{-1}(D)$ such that $ne(M) = i$. There is no $M \in F^{-1}(D)$ with $ne(M) > ne(D)$.*

Proof. 1. The first claim follows at once from the definitions. It is also clear that $dos(D)$ is uniquely determined by D and vice versa.

2. We know from the proof of 3 of Lemma 3.2 that $ne(D) = ne(M)$ for the unique noncrossing $M \in F^{-1}(D)$. Now we argue as in the proof of 3 of Lemma 3.2. □

Theorem 3.5 *For $n \in \mathbf{N}$ the set $\mathcal{M}(n)/\sim_{ne}$ of nesting-similarity classes has*

$$2 \cdot 4^{n-1} - \frac{3n-1}{2n+2}\binom{2n}{n}$$

elements.

Proof. By the previous lemma,

$$|\mathcal{M}(n)/\sim_{ne}| = \sum_{D \in \mathcal{D}(n)} (1 + ne(D)) = |\mathcal{D}(n)| + \sum_{D \in \mathcal{D}(n)} ne(D).$$

We claim that this number is equal to the coefficient of x^n in the expansion of the expression

$$C + x^2(2xC' + C)^2 C$$

where $C = C(x) = \sum_{n \ge 0} |\mathcal{D}(n)| x^n = 1 + x + 2x^2 + 5x^3 + \cdots$. It is well known that $C = (1 - \sqrt{1-4x})/2x = \sum_{n \ge 0} \frac{1}{n+1}\binom{2n}{n} x^n$. Using the relations $xC^2 - C + 1 = 0$ and $2xCC' + C^2 = C'$ we simplify the expression to

$$2C(x) + \frac{1/2}{1-4x} - \frac{3/2}{\sqrt{1-4x}}.$$

Using the expansion of $C(x)$, geometric series, and $(1-4x)^{-1/2} = \sum_{n \ge 0} \binom{2n}{n} x^n$ we obtain the formula.

To establish the claim, recall that $\sum_{D\in\mathcal{D}(n)} ne(D)$ counts the triples (D, t_1, t_2) where $D \in \mathcal{D}(n)$ and t_1, t_2 are two distinct tunnels in D such that t_1 covers t_2.Let the segments of D supporting t_i be r_i (up-step) and s_i (down-step). Let the lower endpoints of the segments r_i (s_i) be a_i (b_i) and their upper endpoints be a_i' (b_i'), $i = 1, 2$. The deletion of the interiors of the segments r_1, s_1, r_2, and s_2 splits D in five lattice paths $L_1, \dots, L_5$ where L_1 starts at $(0, 0)$ and ends in a_1, L_2 starts at a_1' and ends at a_2, L_3 starts at a_2' and ends at b_2', L_4 starts at b_2 and ends at b_1', and L_5 starts at b_1 and ends at $(2n, 0)$. Each L_i is nonempty but may be just a single lattice point. The concatenation L_1L_5, where L_5 is appropriately shifted so that a_1 and b_1 are identified in one distinguished point, is a Dyck path and similarly for L_2L_4 with a_2 and b_2 identified and distinguished. L_3 is a Dyck path by itself (after an appropriate shift). We see that the triples (D, t_1, t_2) in question are in a 1-1 correspondence with the triples (E_1, E_2, E_3) where $E_i \in \mathcal{D}(n_i)$, $n_i \in \mathbf{N}_0$, are such that $n_1 + n_2 + n_3 = n - 2$, and moreover E_1 and E_2 have one distinguished lattice point (out of $2n_1 + 1$, respectively $2n_2 + 1$, points). It follows that the number of the triples (E_1, E_2, E_3) is the coefficient of x^{n-2} in $(2xC' + C)^2C$, which proves the claim. $\square$

The values of $|\mathcal{M}(n)/\sim_{ne}|$ form the sequence $(1, 3, 12, 51, 218, 926, \dots)$. Subtracting the Catalan numbers $C_n = |\mathcal{D}(n)|$, we get the sequence (0, 1, 7,37, 176,794, ...) that counts nesting-similarity classes in $\mathcal{M}(n)$ for matchings with at least one nesting. This sequence is entry A006419 of [8] and appears in Welsh and Lehman [11, Table VIb] in enumeration of planar maps. We summarize this identity in the next proposition.

Proposition 3.6 *For $n = 1, 2, \dots$ the formula*

$$2 \cdot 4^{n-1} - \frac{3n+1}{2n+2}\binom{2n}{n}$$

counts the following objects.

1. *The triples (D, t_1, t_2) where D is a Dyck path with semilength n and t_1, t_2 are two distinct tunnels in D such that t_1 covers t_2.*
2. *The nesting-similarity classes in $\{M \in \mathcal{M}(n) :\ ne(M) > 0\}/\sim_{ne}$.*
3. *The vertex-rooted planar maps with two vertices and n faces, which are edge 2-connected and may have loops and multiple edges. See Figure 4 for the case $n = 3$.*

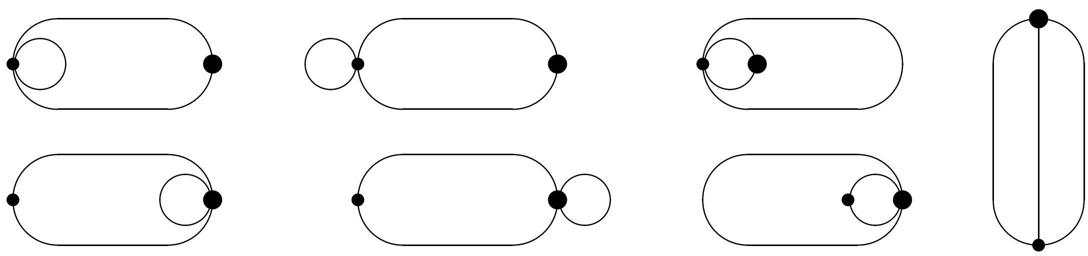

Figure 4: All seven rooted and edge 2-connected planar maps with two vertices and three faces.

Proof. 1 and 2 follow from the proof of Theorem 3.5 and 3 follows by checking the formulas in [11]. Alternatively, it is not too hard to establish bijection between the triples in 1 and the maps in 3. □

The present author proved in [3, Theorem 3.1] that the number of the triples (T, v_1, v_2), where T is a rooted plane tree with n vertices and v_1, v_2 are two (not necessarily distinct) vertices of T such that v_1 lies on the path joining the root of T and v_2, equals

$$\frac{4^{n-1} + \binom{2n-2}{n-1}}{2}.$$

It is straightforward to relate Dyck paths and rooted plane trees and to derive the formula of Theorem 3.5 from this one.

4 Further applications and concluding remarks

Corollary 1.5 presents two matchings M and N such that the distribution of cr on the levels of $\mathcal{T}(M)$ equals the distribution of ne on the levels of $\mathcal{T}(N)$. We show that there are no other substantially different examples.

Proposition 4.1 *Let $M, N \in \mathcal{M}(n)$ be two matchings. We have*

$$cr(\mathcal{T}(M,l)) = ne(\mathcal{T}(N,l)) \text{ for every } l \geq 0$$

if and only if $M = M_n = \{\{1, 2n\}, \{2, 2n-1\}, \ldots, \{n, n+1\}\}$ *and* $N = N_n = \{\{1, 2\}, \{3, 4\}, \ldots, \{2n-1, 2n\}\}$.

Proof. The if part is clear by Theorem 1.2: $cr(M_n) = ne(N_n) = 0$ and

$$cr(\mathcal{T}(M_n, 1)) = ne(\mathcal{T}(N_n, 1)) = \{0, 0, 1, 1, 2, 2, \ldots, n-1, n-1, n\}$$

because

$$\begin{array}{rcl} crs(M_n) & = & (0, 1, 2, \ldots, n-1, n, n-1, \ldots, 2, 1, 0) \\ nes(N_n) & = & (0, 0, 1, 1, 2, 2, \ldots, n-1, n-1, n). \end{array}$$

To show the only if part, we prove that the only matchings $M, N \in \mathcal{M}(n)$ satisfying $cr(M) = ne(N)$ and $f_0^1(crs(M)) = f_0^1(nes(N))$ are M_n and N_n. Since for every $N \in \mathcal{M}(n)$ the sequence $nes(N)$ ends with n, we must have n in $crs(M)$ which means that the middle gap of M must be covered by all edges. Thus all first vertices of the edges in M must precede all second vertices and $crs(M) = (0, 1, 2, \ldots, n-1, n, n-1, \ldots, 2, 1, 0)$. Thus $f_0^1(nes(N)) = \{0, 0, 1, 1, 2, 2, \ldots, n-1, n-1, n\}$ which forces $N = N_n$. Thus $cr(M) = ne(N) = ne(N_n) = 0$ which forces $M = M_n$. $\square$

Therefore we have no other examples of equidistribution of cr and ne on the levels of $\mathcal{T}(M)$ than $M = \emptyset$ and $M = \{\{1, 2\}\}$ because $M_n = N_n$ only for $n = 0, 1$. We call the matchings $M \in \mathcal{M}(n)$ encountered in the proof in which all edges cover the middle gap, equivalently which have $f_0^1(crs(M)) = \{0, 0, 1, 1, 2, 2, \ldots, n-1, n-1, n\}$, *permutational matchings*; they are in 1-1 correspondence with the permutations of $[n]$ and are $n!$ in number.

Because $|\mathcal{M}(n)| = (2n-1)!! = n^n(2/\mathrm{e} + o(1))^n$ and the numbers of crossing-similarity and nesting-similarity classes are only exponential, we have very many examples as in Corollary 1.6 when cr (or ne) has equal distributions on the levels of $\mathcal{T}(M)$ and $\mathcal{T}(N)$ for $M \neq N$. The next corollary follows from the asymptotics of the numbers of similarity classes given in Theorems 3.3 and 3.5.

Corollary 4.2 *Every set of matchings* $X \subset \mathcal{M}(n)$ *contains* $|X|/(2 + o(1))^n$ *mutually crossing-similar matchings and* $|X|/(4 + o(1))^n$ *mutually nesting-similar matchings.*

An explicit example of a big similarity class is provided by permutational matchings in $\mathcal{M}(n)$. They all share the same crossing sequence $(0, 1, 2, \ldots, n-1, n, n-1, \ldots, 2, 1, 0)$ and the same nesting sequence $(0, 0, \ldots, 0, 1, 2, \ldots, n-1, n)$. Hence at least

$$\frac{n!}{\binom{n}{2}+1} = n^n(1/\mathrm{e} + o(1))^n$$

of them are mutually crossing-similar and at least so many of them are mutually nesting-similar.

Crossing and nesting correspond to two of three matchings in $\mathcal{M}(2)$ and the third remaining matching is $\{\{1,2\},\{3,4\}\}$. If two edges of $M \in \mathcal{M}$ form this matching, we say that they form a *camel*. We denote the number of camels in M by $ca(M)$. This statistic behaves on the levels of the subtrees of $\mathcal{T}$ in the same way as cr and ne do.

Corollary 4.3 *Let $M, N \in \mathcal{M}(n)$ be two matchings such that ca has the same distribution on the first two levels of the subtrees $\mathcal{T}(M)$ and $\mathcal{T}(N)$. Then ca has the same distribution on all levels.*

Proof. For $M \in \mathcal{M}(n)$ we have $ca(M) = \binom{n}{2} - (cr(M) + ne(M))$. Thus this result follows by 1 of Theorem 1.1 if we set $A = (\mathbf{Z}, +)$ and $\alpha = \beta = 1$. $\square$

Note that while the number of $M \in \mathcal{M}(n)$ with $cr(M) = 0$ (or with $ne(M) = 0$) is the Catalan number $\frac{1}{n+1}\binom{2n}{n}$, the number of $M \in \mathcal{M}(n)$ with $ca(M) = 0$ is much bigger, namely $n!$ (these are exactly permutational matchings).

It is possible to investigate the general similarity relation $\sim_{A,\alpha,\beta}$ on $\mathcal{M}(n)$ defined, for an abelian group $A = (A, +)$ and two its elements $\alpha, \beta \in A$, by $M \sim_{A,\alpha,\beta} N$ iff $s_{\alpha,\beta}(\mathcal{T}(M,l)) = s_{\alpha,\beta}(\mathcal{T}(N,l))$ for every $l \geq 0$. We consider here only the case $A = (\mathbf{Z}_2, +)$ and define the statistics $cr_2(M), ne_2(M) \in \{0, 1\}$ as parity of the numbers $cr(M), ne(M)$. We define the sequences $crs_2(M)$ and $nes_2(M)$ of M by reducing $crs(M)$ and $nes(M)$ modulo 2. For two matchings $M, N \in \mathcal{M}(n)$ we define $M \sim_{cr,2} N$ iff $cr_2(\mathcal{T}(M,l)) = cr_2(\mathcal{T}(M,l))$ for every $l \geq 0$, and similarly for $M \sim_{ne,2} N$. By Theorems 1.1 and 2.7, $M \sim_{cr,2} N$ iff $cr_2(M) = cr_2(N)$ and $crs_2(M)$ and $crs_2(N)$ are equal as multisets after forgetting the order of terms, and similarly for $\sim_{ne,2}$. (Now $nes_2(M)$ is not nondecreasing and we may have $f_0^1(nes_2(M)) = f_0^1(nes_2(N))$ for $nes_2(M) \neq nes_2(N)$.) We determine the numbers of equivalence classes for $\sim_{cr,2}$ and $\sim_{ne,2}$.

Theorem 4.4 *We have* $|\mathcal{M}(1)/\sim_{cr,2}| = 1$ *and* $|\mathcal{M}(n)/\sim_{cr,2}| = 2$ *for* $n \geq 2$. *The two classes of mod 2 crossing-similarity have* $((2n-1)!!+1)/2$ *and* $((2n-1)!!-1)/2$ *elements. We have* $|\mathcal{M}(1)/\sim_{ne,2}| = 1$, $|\mathcal{M}(2)/\sim_{ne,2}| = 3$, *and* $|\mathcal{M}(n)/\sim_{ne,2}| = 2n$ *for* $n \geq 3$.

Proof. It follows from the definition of $crs(M)$ that $crs_2(M) = (0,\ 1,\ 0,\ 1,\ 0, \ldots,\ 1,\ 0)$ for every matching M. Thus the classes of mod 2 crossing-similarity are determined only by $cr_2(M)$ and, for $n \geq 2$, we have two of them. The fact that

$$|\{M \in \mathcal{M}(n) :\ cr_2(M) = 0\}| - |\{M \in \mathcal{M}(n) :\ cr_2(M) = 1\}| = 1$$

for every $n \geq 1$ was proved by Riordan [6] by generating functions; a simple proof by involution was given by Klazar [4].

To handle nestings modulo 2, recall that $nes(M) = dos(D)$ where $D = F(M)$ and that nesting sequences of the matchings $M \in \mathcal{M}(n)$ are in bijection with the Dyck paths $D \in \mathcal{D}(n)$ (Lemma 3.4). We claim that the n Dyck paths

$$D_1 = udu^{n-1}d^{n-1}, D_2 = u^2du^{n-2}d^{n-1}, \ldots, D_{n-1} = u^{n-1}dud^{n-1}, D_n = u^nd^n$$

(u is the up-step and d is the down-step) realize all possible numbers of 1's and 0's in the sequences $\{dos_2(D) :\ D \in \mathcal{D}(n)\}$ and hence in the sequences $\{nes_2(M) :\ M \in \mathcal{M}(n)\}$. The number of 1's (0's) in $dos_2(D_i)$, $i = 1, 2, \ldots, n$, is $n + \lceil n/2 \rceil - i$ ($1 + i + \lfloor n/2 \rfloor$). It suffices to show that no $dos_2(D)$ has fewer than $\lceil n/2 \rceil$ 1's and fewer than $2 + \lfloor n/2 \rfloor$ 0's. In every D each of the n down-steps contributes to $dos_2(D)$ exactly one 1 (by one of its endpoints) and each of these 1's may belong to at most two downsteps. So we must have at least $\lceil n/2 \rceil$ 1's. The argument for 0's is similar, but now the 0 contributed by the first down-step is never shared (with the next down-steps) and there is one more 0 contributed by the first up-step. So we have at least $1+1+\lfloor n/2 \rfloor$ 0's. Thus, for every $n \geq 1$, $|\{f_0^1(nes_2(M)) :\ M \in \mathcal{M}(n)\}| = n$. If $n \geq 3$, for each D_i there are $M, M' \in F^{-1}(D_i)$ with $ne(M') = ne(M) - 1$ (we take for M the noncrossing matching in $F^{-1}(D_i)$, it has least one nesting, and apply the n-c transformation). Thus, for $n \geq 3$, there are $2n$ classes of mod 2 nesting-similarity. The cases $n = 1, 2$ are easy to treat separately. $\square$

Concluding remarks. Recently, an interesting result for crossings and nestings of higher order was obtained by Chen et al. in [1] where it is proved that for every $k, l, n \in \mathbf{N}$ the number of matchings in $\mathcal{M}(n)$ with no k-crossing and no l-nesting is the same as the number of matchings with no k-nesting and no l-crossing (a similar result is in [1] obtained for set partitions); here k-crossing is a k-tuple of pairwise crossing edges and similarly for k-nesting. Another generalization of crossings and nestings is investigated by Jelínek [2] who is interested in numbers of matchings $M \in \mathcal{M}(n)$ such that M does not contain a fixed permutational matching $N \in \mathcal{M}(3)$ as an ordered submatching.

It may be interesting to try to extend results and methods of the present article to crossings and nestings of higher order. Another research direction may be to apply our method to other structures besides matchings. Finally, one may try to go to higher levels of the description of the enumerative complexity of crossings and nestings — denoting $G: \mathcal{M} \to \mathcal{M}/\sim_{cr}$ the mapping sending M to its equivalence class, when is it the case that $G(\mathcal{T}(M, l)) = G(\mathcal{T}(N, l))$ for every $l \geq 0$; and similarly for $\sim_{ne}$.

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