A simple proof for open cups and caps

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Abstract

Let X be a set of points in general position in the plane. General position means that no three points lie on a line and no two points have the same x-coordinate. $Y \subseteq X$ is a *cup*, resp. *cap*, if the points of Y lie on the graph of a convex, resp. concave function. Denote the points of Y by $p_1, p_2, \ldots, p_m$ according to the increasing x-coordinate. The set Y is *open* in X if there is no point of X above the polygonal line $p_1, p_2, \ldots, p_m$. Valtr [12] showed that for every positive integers k and l there exists a positive integer $g(k,l)$ such that any $g(k,l)$-point set in the plane in general position contains an open k-cup or an open l-cap. This is a generalization of the Erdős-Szekeres theorem on cups and caps. We show a simple proof for this theorem and we also show better recurreces for $g(k,l)$. This theorem implies results on empty polygons in k'-convex sets proved by Károlyi et. al. [5], Kun and Lippner [7] and Valtr [11],[12]. A set of points is k'-convex if it determines no triangle with more than k' points inside.

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1 Definitions and Notations

All sets of points will be throughout this paper in general position in the plane. By *general position* we mean that no three points lie on a line and no two points have the same x-coordinate. Let X be a set of n points and denote its points by p_1, p_2, $\dots p_n$ according to the increasing x-coordinate. Let $Y \subseteq X$ be a set of points q_1, q_2, $\dots q_k$ again ordered by the x-coordinate. For $i = 1, 2, \dots k-1$, let s_i be the slope of the line $q_i q_{i+1}$. The set $Y = \{q_1, \dots, q_k\}$ is a *k-cup* or a *k-cap* if the sequence s_1, s_2, $\dots$, s_k is increasing or decreasing, respectively (see figure 1). In other word if the points lie on the graph of a convex, resp. concave function. The point q_1 is *the left endpoint* of Y and the point q_k is *the right endpoint* of Y.

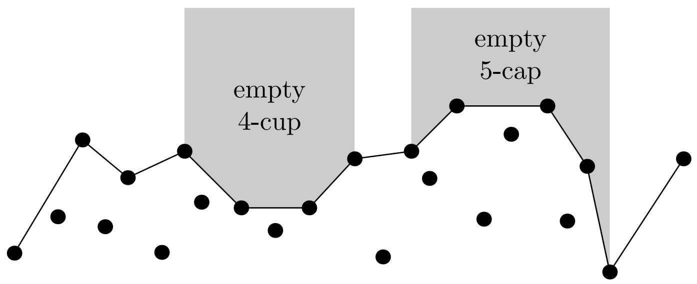


Figure 1: The set of points on the polygonal line is open. There is also an empty 4-cup and an empty 5-cap in the figure.

The set Y is *open* in X if there is no point $p \in X$ with $x(q_1) < x(p) < x(q_k)$ lying above the polygonal line $q_1 q_2 \dots q_k$. *The upper envelope* of Y is the polygonal line $q_{i_1}, q_{i_2}, \dots, q_{i_t}$ where $1 = i_1 < i_2 < \dots < i_t = k$ such that it is the graph of a concave function and there is no point of Y above this line (see figure 2). The point p_L is *the left neighbor* of the point p in set the Y, if $p_L \in Y$ and there is no point $q \in Y$ such that $x(p_L) < x(q) < x(p)$. Similarly is defined *the right neighbor.*

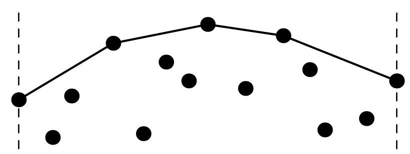


Figure 2: The black polygonal line is the upper envelope of the points in the figure.

2 Introduction

The Erdős-Szekeres theorem [1] says that for every positive integer k there exists a positive integer N such that any N-point set in the plane contains k points that are vertices of a convex polygon. There are several proofs of the theorem using Ramsey theory and a proof using cups and caps. The latter proof gives a much better upper bound on N.

Define $f(k,l)$ to be the smallest positive integer for which X contains a k-cup or an l-cap whenever X has at least $f(k,l)$ poinst. Erdős and Szekeres [1] proved that $f(k,l) = \binom{k+l-4}{k-2} + 1$.

Erdős also asked if for every k there exists N such that any N-point set X in the plane contains k vertices of an empty convex polygon. Empty polygon is a polygon with no point of X in its interior. We say that $Y \subseteq X$ is a k-hole if Y lies in the vertices of an empty convex k-gon. His conjecture holds up to $k = 5$ [3]. In 1983 Horton [4] showed that it is not true for all $k \geq 7$. The question for $k = 6$ was open for a long time. Using a computer Overmars [10] found a configuration of 29 points without empty hexagon and very recently Gerken [2] showed that the conjecture holds also for $k = 6$. See [8] or [9] for a survey.

What is the sufficient condition for the existence of a k-hole? The set X is *l-convex* if and only if every triangle determined by points of X contains at most l points of X in its interior. The l-convex sets were introduced by Valtr [11] and he also showed the following theorem:

Theorem 1 (Valtr). *For every positive integers k and l there exists a positive integer N such that any l-convex N-point set X in the plane contains a k-hole.*

Denote by $n(k,l)$ the smallest positive integer N such that any l-convex N-point set contains a k-hole. In 2001 Károlyi, Pach and Toth [5] proved this theorem for $l = 1$. Later Karolyi, Valtr [6] determined the exact value of $n(k,1)$. The first proof for general l was given by Valtr [11]. He was followed by Kun and Lippner [7] who improved the bound to $n(k,l) \leq (l+2)^{(l+2)^k - 1}$. Finally Valtr [12] again improved the bound to $n(k,l) \leq 2^{\binom{k+l}{l+2} - 1} + 1$. The last Valtr's proof generalizes the result on cups and caps used in the proof of Erdős-Szekeres theorem to open cups and open caps.

Theorem 2 (Valtr). *For every positives integers k and l there exits a positive integer N such that any N-point set in the plane contains an open k-cup or an open l-cap.*

We show a simple proof of theorem 2 in section 3. Theorem 1 for l-convex sets is a corrolary of theorem 2. If we have an $(l-3)$-convex N-point set

X and we want to find a $(k+1)$-hole, we use the projective transformation, which sends the horizontal line l passing through the highest point of the convex hull of X to the infinity. We can assume that there is exactly one point of X on the line l, otherwise we can rotate the point set X a little. We obtain an $(N-1)$-point set $\bar{X}$. We apply theorem 2 on the set $\bar{X}$ and receive either an open k-cup or an open l-cap. In the backward projective transformation the open k-cup corresponds to a $(k+1)$-hole and the open l-cap corresponds to a triangle cointaing at least $(l-2)$-points, but that contradicts the $(l-3)$-convexity of the set X. See Valtr [12] for the details.

We define $g(k,l)$ as the smallest number N such that any N-point set in general position contains an open k-cup or an open l-cap. Valtr [12] showed the following bounds:

$$2^{\binom{\lfloor k/2 \rfloor + \lfloor l/2 \rfloor - 2}{\lfloor k/2 \rfloor - 1}} \leq g(k,l) \leq 2^{\binom{k+l-2}{l-1}-1}.$$

In section 4 we show the recurrences estimating $g(k,l)$ from above (lemma 2). As a corrolary of the recurrences we calculate some upper bounds (lemma 3 and lemma 4), but they do not give us as good bounds as the recurrences themselves. At the end of the section we give a tight upper bound for $l=4$ (lemma 5). In section 5 we show the recurrences estimating $g(k,l)$ from below (lemma 6). We also remark, that this recurrence is not tight and show the idea, how it can be improved. The summary of the previous lemmas is in the following theorem.

Theorem 3 (recurrences). *$g(k,2)=2$, $g(2,l)=2$, $g(k,3)=k$, $g(3,l)=l$, $g(k,4)=2^{k-1}$ and for $k,l \geq 4$ we have*

$$g(k,l) \leq g(k-2,l) \cdot [2g(k,l-1)-3]+2$$
$$g(k,l) \geq g(k-2,l) \cdot (g(k,l-2)-2)+2g(k-1,l).$$

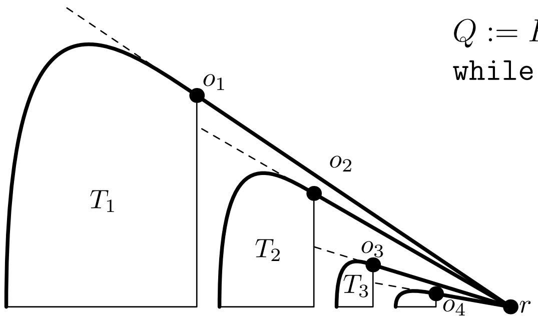
3 The very short proof of Theorem 2

Define $h(k,l,m)$ to be the largest number N such that there is an N-point set in general position which contains neither an open k-cup nor an open l-cap nor an open m-cap ending in the rightmost point. It is easy to see that $h(k,l,2)=1$ and $h(k,l,l)=g(k,l)-1$.

Lemma 1. *$h(k,l,m) \leq h(k-1,l,l) \cdot h(k,l,m-1)+1$ for $k,l \geq 3$.*

Proof. Let P be the set of points in general position maximizing $h(k,l,m)$. That means that P is $h(k,l,m)$-point set which contains neither an open

k-cup nor an open l-cap nor an open m-cap ending in the rightmost point. Denote the rightmost point of P by r. We construct sets T_i by the following algorithm. The construction is ilustrated in the following figure.



$Q := P$; $i := 1$;
`while` $Q \neq \emptyset$ `do`
  Find the upper envelope of Q;
  $o_i :=$ the left neighbor of r
    on the upper envelope of Q;
  $T_i :=$ the points of Q lying to
    the left of o_i including o_i;
  $Q := Q - T_i$; $i := i + 1$;

Any open cap in T_i ending in the point o_i can be extended by the point r and becomes an open cap in P ending in the rightmost point of P. Thus $|T_i| \leq h(k, l, m-1)$.

Let $O = \{o_1, o_2, \dots\}$. The set O is open in P. O contains neither an open k-cup nor an open l-cap, because it would be the open cup or the open cap in P. Moreover O does not contain an open $(k-1)$-cup, because this cup can be extended by the point r. It is because the point r lies above every line determined by two points of O. Hence $|O| \leq h(k-1, l, l)$.

There are at most $|O| \leq h(k-1, l, l)$ sets T_i each containing at most $h(k, l, m-1)$ points and the rightmost point r. That gives us the recurrence. □

Now it is easy to solve the recurrence. We know that $h(k, l, 2) = 1$ and $h(k-1, l, l) = g(k-1, l) - 1$. Denote $g(k-1, l) - 1$ by a. By applying the recurrence from lemma 1 $(l-2)$-times we get $g(k, l) - 1 = h(k, l, l) \leq a(a(a \dots (a \cdot 1 + 1) \dots + 1) + 1) + 1 = a^{l-2} + a^{l-3} + a^{l-4} + \dots + a + 1 = (a^{l-1} - 1)/(a - 1)$. Assuming that $a \geq 2$ which means $k \geq 4$ and $l \geq 3$ we get $g(k, l) - 1 < a^{l-1} - 1 < g(k-1, l)^{l-1} - 1$. Because $g(3, l) = l$ we get $g(k, l) \leq l^{(l-1)^{k-3}}$. That finishes the first proof.

4 The proof of better upper bound

In the first part (lemma 2 and claim 3) we show the recurrences for $g(k, l)$ and in the second part we solve the recurrences (lemma 3 and lemma 4). At the end of the section we show the proof of the tight upper bound for $g(k, 4)$ (lemma 5).

Lemma 2. *$g(k, l) \leq g(k-2, l) \cdot [2g(k, l-1) - 3] + 2$ for $k, l \geq 3$.*

Proof. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of $n = g(k,l) - 1$ points in general position with neither an open k-cup nor an open l-cap.

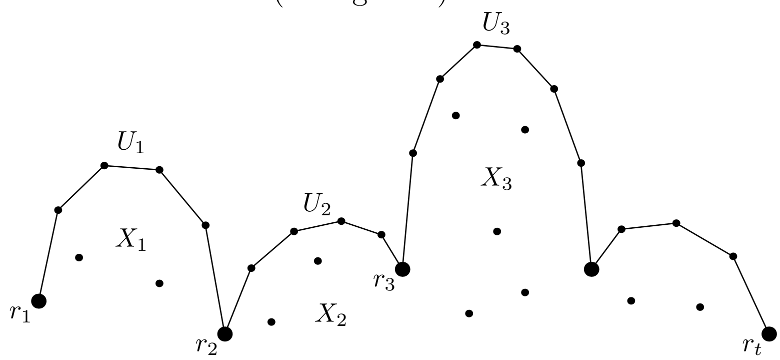
Maximal open cup is the open cup which cannot be extended to a larger cup in X. Let L be the set of left endpoints of maximal open cups with at least 2 points. So for every open cup with the left endpoint $x \notin L$, there is a point in X to the left of x, which extends the open cup. The leftmost point of X is in L, because the two leftmost points of X form an open 2-cup.

Denote the size of L by t and the points of L by $r_1, r_2, \ldots, r_t$. The points of L divide the set X into $t+1$ vertical strips. Denote the sets of points strictly contained in each strip by X_i for $i = 0, 1, 2, \ldots, t$. The leftmost strip is empty, because r_1 is the leftmost point of X.

Claim 1. *Every open cup in X_i can be extended in X by one point to the left and therefore $|X_i| \leq g(k-1,l) - 1$.*

The left endpoint of an open cup in X_i is not in L and thus there is a point in X extending the open cup. Hence there is no open $(k-1)$-cup in X_i and we get $|X_i| \leq g(k-1,l) - 1$.

For $i = 1, 2, \ldots, t-1$ denote the set of points of the upper envelope of $X_i \cup \{r_i, r_{i+1}\}$ by U_i (see the following figure). Let Y be the union of U_i for $i = 1, 2, \ldots, t-1$. The set Y is open in X. That's why if there is an open k-cup resp. an open l-cap in the set Y, then there is the same open k-cup resp. l-cap in the whole set X (see figure 1).

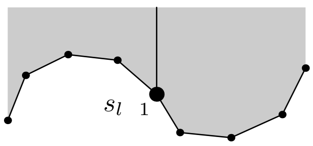


Claim 2. *The set Y does not contain an open $(l-1)$-cap. Thus $|Y| \leq g(k, l-1) - 1$.*

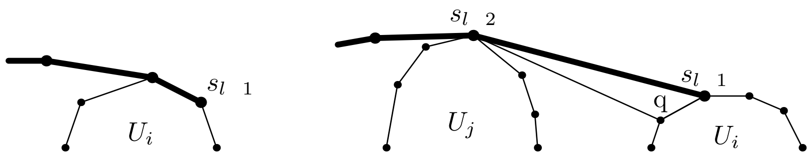
Let us prove it by contradiction. Assume that there is an open $(l-1)$-cap $C = \{s_1, \ldots, s_{l-1}\}$ in Y. By the width of the cap we mean $|x(s_1) - x(s_{l-1})|$. From all the open $(l-1)$-caps in Y choose such a one whose width is the smallest.

Now we show that either there is a narrower open $(l-1)$-cap in Y or the open cap C can be extended in X to the right by one point so we have an open l-cap in X.

Where does the point s_{l-1} lie? If it lies in L then s_{l-1} is also the left endpoint of maximal open cup in X. See the following figure. Either the open cup or the open cap can be extended by one point.



Thus $s_{l-1} \in U_i - L$ for some i (it is an interior point of U_i). If there are at least two points of C in U_i then the cap can be extended by the right neighbor of s_{l-1} in U_i. See the following figure on the left. There must be some right neighbor because $s_{l-1} \notin L$.



In the remaining case s_{l-1} is the only point of C in U_i. The point s_{l-2} lies in U_j for some $j < i$. See previous figure on the right. Denote the left neighbor of s_{l-1} in Y by q. If the triangle $s_{l-2}s_{l-1}q$ is empty then we have the open $(l-1)$-cap $s_1, \ldots, s_{l-2}, q$ which is narrower than C. Otherwise choose $w \in X$ to be the point in the triangle $qs_{l-2}s_{l-1}$ with the largest angle $\angle qs_{l-2}w$. The open l-cap $s_1, \ldots, s_{l-2}, w$ is again narrower than C. This finishes the proof of the claim.

Claim 3. *$g(k,l) \leq g(k-1,l) \cdot g(k,l-1)$ for $k, l \geq 2$.*

By the previous claims there are $t \leq |Y| \leq g(k,l-1) - 1$ vertical strips each containing at most $g(k-1,l) - 1$ points plus one for the point r_i. The leftmost strip is empty. We get $g(k,l)-1 \leq [g(k-1,l)-1+1] \cdot [g(k,l-1)-1]$ and the claim follows.

Using another trick we can get a better recurrence. Similarly as we defined L to be the set of all left endpoints of maximal open cups with at least two points, we can define R to be the set of all right endpoints of maximal open cups with at least two points. For the set R we have similar claims as for the set L because of symmetry.

Denote the points of $R \cup L$ by $P = \{p_1, \ldots, p_{\bar{t}}\}$. The points of P split the plane into $\bar{t}+1$ vertical strips. Denote the set of points strictly contained in each strip by $Z_0, Z_1, \ldots, Z_{\bar{t}}$. Since the leftmost point of X is in L and the rightmost point of X in R, the outer strips are empty. For every set Z_i the first claim hold, because $Z_i \subseteq X_j$ for some j. So every open cup in Z_i can be extended in X by one point to the left. From the symmetric arguments it can also be extended by one point to the right. Thus the set Z_i does not contain an open $(k-2)$-cup and we have $|Z_i| \leq g(k-2,l)-1$.

The number of strips is $\bar{t}+1 = |L|+|R|+1$. The outer strips are empty and the others contain at most $g(k-2,l)-1$ points. By claim 2 the size of L, resp. R is at most $|Y| \leq g(k,l-1)-1$. Altogether we get the recurrence: $g(k,l)-1 \leq [2g(k,l-1)-3]\cdot[g(k-2,l)-1+1]+1$. Don't forget to count the points of P. □

Lemma 3. *$g(k,l) \leq 2^{\binom{k+l-4}{k-2}}$ for k, $l \geq 2$.*

Proof. We prove the formula by induction on k and l. For $k=2$ or $l=2$ we have $g(k,l)=2$ and the formula holds. From the recurrence in claim 3 we get $g(k,l) \leq g(k-1,l)\cdot g(k,l-1) \leq 2^{\binom{k-1+l-4}{k-3}} \cdot 2^{\binom{k+l-1-4}{k-2}} = 2^{\binom{k+l-4}{k-2}}$ □

Lemma 4. *$g(k,l) \leq 2^{\binom{k/2+l-3}{k/2-1}+2^{k/2+l-3}-1}$ for k even and k, $l \geq 2$.*

Proof. We prove the formula by induction on k and l. For $k=2$ or $l=2$ we have $g(k,l)=2$ and the formula holds. For k, $l \geq 3$ apply recurrence from lemma 2 and get $g(k,l) \leq g(k-2,l)\cdot[2g(k,l-1)-3]+2 \leq 2\cdot g(k-2,l)\cdot g(k,l-1)$. Now apply the induction hypothesis and get

$$g(k,l) \leq 2 \cdot 2^{\binom{k/2-1+l-3}{k/2-2}+2^{k/2-1+l-3}-1} \cdot 2^{\binom{k/2+l-1-3}{k/2-1}+2^{k/2+l-1-3}-1}$$

from which the lemma follows. □

Lemma 5. *$g(k,4) \leq 2^{k-1}$ for $k \geq 2$.*

Proof. We prove it by induction on k. For $k=2$, the maximal set with neither an open 2-cup nor an open 4-cap is just one point. So $g(2,4)=2$.

Let $X_{k,4}$ be a maximal set with neither an open k-cup nor an open 4-cap. The upper envelope of $X_{k,4}$ must have three points. If it has more points then we have an open 4-cap. If it has only two points then we can place one new point to the left of $X_{k,4}$ and deep below. This set also contain neither an open k-cup nor an open 4-cap and is larger. That condradict with the maximality of $X_{k,4}$.

Let p be the middle point of the upper envelope of $X_{k,4}$. Denote the set of points to the left of p by L and the set of points to the right of p by R. Every line determined by two points of L goes below p. Otherwise we have an open 4-cap. So every open cup in L can be extended by the point p. Thus

L contains neither an open $(k-1)$-cup nor an open 4-cap. The size of L is at most $2^{k-2}-1$ from the induction hypothesis. Similarly the size of R is at most $2^{k-2}-1$. Altogether with the point p we have $2^{k-1}-1$ points, that is what we wanted to prove. □

The lower bound obtained from lemma 6 is $g(k,4) \geq 2^{k-1}$ and hence the bound is tight.

5 Lower bound

5.1 The recurrence

Lemma 6. $g(k,l) \geq g(k-2,l) \cdot (g(k,l-2)-2) + 2g(k-1,l)$ *for* $k,l \geq 3$.

Proof. Valtr [12] shows the construction proving the recurrence $g(k,l) \geq g(k,l-2) \cdot g(k-2,l)$. This construction can be slightly improved.

The set $Y_{k,l}$ with no open k-cup and no open l-cap can be constructed inductively from the sets $L = Y_{k,l-2}$, $S = Y_{k-2,l}$ as follows.

The points of L divide the plane into $t = |L|+1$ vertical strips. For $i = 1,\ldots,t$ place a tiny copy S_i of S into the strip i in such a way that all lines determined by a pair of points in S_i go below L and all lines determined by a pair of points in L go above S_i. See Valtr [12] for details. The modification is such that instead of the outer sets S_1 and S_t we can place tiny copies of $Y_{k-1,l}$.

□

The lower bound $g(k,l) \geq 2^{\binom{k/2+l/2-2}{k/2-1}}$ for k, l even can be proved by induction from the recurrence $g(k,l) \geq g(k,l-2) \cdot g(k-2,l)$. See Valtr[12].

5.2 Other improvements

Let $X_{k,l}$ be the maximal set of points with neither open k-cup nor open l-cap.

Lemma 7. *Every point* $p \in X_{k,l}$ *is either the left end point of open* $(k-1)$*-cup or the right end point of open* $(l-1)$*-cap.*

Let us note, that p cannot be both the left end point of open $(k-1)$-cup and the right end point of open $(l-1)$-cap, because we would have an open k-cup or open l-cap. See the second figure in the proof of lemma 2.

Proof. Assume that there is a point p for which none of the conditions hold. Then we can double the point p to the points p and p'. Consider the vertical line passing thought p and rotate it very slightly counter clockwise. Denote this line by l. Line l is much steeper than any other line determined by two points in $X_{k,l}$. Now shift p' for very small ϵ along l. Denote the set by $X'_{k,l}$.

The set $X'_{k,l}$ contains neither an open k-cup nor an open l-cap. If there will be such a cup, resp. cap then it has to contain both points p and p', otherwise it correspond to to an open k-cup, resp. open l-cap in $X_{k,l}$. Denote this cup, resp. cap by C. Since p and p' are neigbours in $X'_{k,l}$, they must be neighbours also in C. The line pp' is much steeper than any other line in $X'_{k,l}$ so the points p, p' can be only on the left end of an open cup or on the right end of an open cap. If there is an open k-cup or an open l-cap in $X'_{k,l}$ then we have an open $(k-1)$-cup, resp. open $(l-1)$-cap in original set $X_{k,l}$.

By this construction we got the set $X'_{k,l}$ with neither an open k-cup nor an open l-cap and with more points than $X_{k,l}$. That contradict its maximality. □

There is a symmetric version of this lemma where you change the words left to right and vice-versa. The lower bound on $g(k,l)$ can be further improved by an application of lemma 7 or by its symmetric version.

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