

Distribution of the Size of a Largest Planar Matching and Largest Planar Subgraph in Random Bipartite Graphs

MARCOS KIWI*

Depto. Ing. Matemática and
Ctr. Modelamiento Matemático UMI 2807,
University of Chile
Correo 3, Santiago 170–3, Chile
e-mail: mkiwi@dim.uchile.cl

MARTIN LOEBL†

Dept. of Applied Mathematics and
Institute of Theoretical Computer Science (ITI)
Charles University
Malostranské nám. 25, 118 00 Praha 1
Czech Republic
e-mail: loebl@kam.mff.cuni.cz

Abstract

We address the following question: When a randomly chosen regular bipartite multi-graph is drawn in the plane in the “standard way”, what is the distribution of its maximum size planar matching (set of non-crossing disjoint edges) and maximum size planar subgraph (set of non-crossing edges which may share endpoints)? The problem is a generalization of the Longest Increasing Sequence (LIS) problem (also called Ulam’s problem). We present combinatorial identities which relate the number of r -regular bipartite multi-graphs with maximum planar matching (maximum planar subgraph) of at most d edges to a signed sum of restricted lattice walks in \mathbb{Z}^d , and to the number of pairs of standard Young tableaux of the same shape and with a “descend-type” property. Our results are obtained via generalizations of two combinatorial proofs through which Gessel’s identity can be obtained (an identity that is crucial in the derivation of a bivariate generating function associated to the distribution of LISs, and key to the analytic attack on Ulam’s problem).

Keywords: Gessel’s identity, longest increasing sequence, random bipartite graphs, lattice walks.

1 Introduction

Let U and V henceforth denote two disjoint totally ordered sets (both ordered relations will be referred to by \preceq). Typically, we will consider the case where $|U| = |V| = n$ and denote the elements of U and V by u_1, u_2, \dots, u_n and v_1, v_2, \dots, v_n respectively. Henceforth, we will always

*Gratefully acknowledges the support of MIDEPLAN via ICM-P01–05, and CONICYT via FONDECYT 1010689 and FONDAP in Applied Mathematics.

†Gratefully acknowledges the support of ICM-P01-05. This work was done while visiting the Depto. Ing. Matemática, U. Chile.

assume that the latter enumeration respects the ordered relation in U or V , i.e., $u_1 \preceq u_2 \preceq \dots \preceq u_n$ and $v_1 \preceq v_2 \preceq \dots \preceq v_n$.

Let $G = (U, V; E)$ denote a bipartite multi-graph with color classes U and V . Two distinct edges uv and $u'v'$ of G are said to be *noncrossing* if u and u' are in the same order as v and v' ; in other words, if $u \prec u'$ and $v \prec v'$ or $u' \prec u$ and $v' \prec v$. A matching of G is called *planar* if every distinct pair of its edges is noncrossing. We let $L(G)$ denote the number of edges of a maximum size (largest) planar matching in G (note that $L(G)$ depends on the graph G and on the ordering of its color classes).

For the sake of simplicity we will concentrate solely in the case where $|E| = rn$ and G is r -regular.

When $r = 1$, an r -regular multi-graph with color classes U and V uniquely determines a permutation. A planar matching corresponds thus to an increasing sequence of the permutation, where an increasing sequence of length L of a permutation π of $\{1, \dots, n\}$ is a sequence $1 \leq i_1 < i_2 < \dots < i_L \leq n$ such that $\pi(i_1) < \pi(i_2) < \dots < \pi(i_L)$. The Longest Increasing Sequence (LIS) problem concerns the determination of the asymptotic, on n , behavior of the LIS for a randomly and uniformly chosen permutation π . The LIS problem is also referred to as ‘‘Ulam’s problem’’ (e.g., in [Kin73, BDJ99, Oka00]). Ulam is often credited for raising it in [Ula61] where he mentions (without reference) a ‘‘well-known theorem’’ asserting that given $n^2 + 1$ integers in any order, it is always possible to find among them a monotone subsequence of $n + 1$ (the theorem is due to Erdős and Szekeres [ES35]). Monte Carlo simulations are reported in [BB67], where it is observed that over the range $n \leq 100$, the limit of the LIS of $n^2 + 1$ randomly chosen elements, when normalized by n , approaches 2. Hammersley [Ham72] gave a rigorous proof of the existence of the limit and conjectured it was equal to 2. Later, Logan and Shepp [LS77], based on a result by Schensted [Sch61], proved that $\gamma \geq 2$; finally, Vershik and Kerov [VK77] obtained that $\gamma \leq 2$. In a major recent breakthrough due to Baik, Deift, Johansson [BDJ99] the asymptotic distribution of the LIS has been determined. For a detailed account of these results, history and related work see the surveys of Aldous and Diaconis [AD99] and Stanley [Sta02].

From the previous discussion, it follows that one way of generalizing Ulam’s problem is to study the distribution of the size of the largest planar matching in randomly chosen r -regular bipartite multi-graphs (for a different generalization see [Ste77, BW88]). This line of research, originating in [KL02], turns out to be relevant for the study of several other issues like the Longest Common Subsequence problem (see [KLM05]), interacting particle systems [Sep77], digital boiling [GTW01], and is directly related to topics such as percolation theory [Ale94] and random matrix theory [Joh99].

1.1 Main Results

We establish combinatorial identities which express $g(n; d)$ — the number of r -regular bipartite multi-graphs with planar matchings with at most d edges — in terms of:

- The number of pairs of standard Young tableaux of the same shape and with a ‘‘descend-type’’ property (Theorem 5).
- A signed sum of restricted lattice walks in \mathbb{Z}^d (Theorem 1).

Our arguments can be extended in order to characterize the distribution of the largest size of planar subgraphs of randomly chosen r -regular bipartite multi-graphs (Theorem 4).

1.2 Models of Random Graphs: From k -regular Multi-graphs to Permutations

Most work on random regular graphs is based on the so called random configuration model of Bender and Canfield and Bollobás [Bol85, Ch. II, § 4]. Below we follow this approach, but first we need to adapt the configuration model to the bipartite graph scenario. Given U , V , n and r as above, let $\bar{U} = U \times [r]$ and $\bar{V} = V \times [r]$. An r -configuration of U and V is a one-to-one pairing of \bar{U} and \bar{V} . These rn pairs are called edges of the configuration. Hence, a configuration can be considered a graph, specifically, a perfect matching with color classes \bar{U} and \bar{V} . Moreover, viewing a configuration as such bipartite graph enables us to speak also about its planar matchings (here the total ordering on $\bar{U} = U \times [r]$ and $\bar{V} = V \times [r]$ is the lexicographic one induced by \preceq and \leq).

The natural projection of $\bar{U} = U \times [r]$ and $\bar{V} = V \times [r]$ onto U and V respectively (ignoring the second coordinate) projects each configuration F to a bipartite multi-graph $\pi(F)$ with color classes U and V . Note in particular that $\pi(F)$ may contain multiple edges (arising from sets of two or more edges in F whose end-points correspond to the same pair of vertex in U and V). However, the projection of the uniform distribution over configurations of U and V is not the uniform distribution over all r -regular bipartite multi-graphs on U and V (the probability of obtaining a given multi-graph is proportional to a weight consisting of the product of a factor $1/j!$ for each multiple edge of multiplicity j). Since a configuration F can be considered a graph, it makes perfect sense to speak of the size $L(F)$ of its largest planar matching.

We denote an element $(u, i) \in \bar{U}$ by u^i and adopt an analogous convention for the elements of \bar{V} . We shall further abuse notation and denote by \preceq the total order on \bar{U} given by $u^i \preceq \tilde{u}^j$ if $u \prec \tilde{u}$ or $u = \tilde{u}$ and $i \leq j$. We adopt a similar convention for \bar{V} .

Let $G_r(U, V; d)$ denote the set of all r -regular bipartite multi-graphs on U and V whose largest planar matching is of size at most d . Note that if $|U| = |V| = n$, then the cardinality of $G_r(U, V; d)$ depends on U and V solely through n . Thus, for $|U| = |V| = n$, let $g(n; d) = |G_r(U, V; d)|$.

The first step in our considerations is an identification of $G_r(U, V; d)$ with a subset of configurations of U and V . Specifically, we associate to an r -regular multi-graph $G = (U, V; E)$ the r -configuration \bar{G} of U and V such that $\pi(\bar{G}) = G$ where: If (u, v) is an edge of multiplicity t in G for which there are i edges (u, v') in G such that $v \prec v'$, and j edges (u', v) in G such that $u \prec u'$, then for every $s \in [t]$, the pairing $(u^{i+s}, v^{j+t-s+1})$ belongs to \bar{G} . Note that the number of edges of \bar{G} equals the number of edges of G .

Let $\bar{G}_r(U, V; d)$ be the collection of configurations \bar{G} associated to some $G \in G_r(U, V; d)$. Observe, that $g(n; d) = |\bar{G}_r(U, V; d)|$.

For an edge (\bar{u}, \bar{v}) we say that $M \subseteq \{\bar{u}' \in \bar{U} : \bar{u}' \preceq \bar{u}\} \times \{\bar{v}' \in \bar{V} : \bar{v}' \preceq \bar{v}\}$ is a planar matching that ends with (\bar{u}, \bar{v}) if the edges in M are non-crossing and $(\bar{u}, \bar{v}) \in M$. Since there is a unique edge incident to every node in \bar{G} , say (\bar{u}, \bar{v}) , we speak of a largest planar matching of \bar{G} up to \bar{u} (or \bar{v}) in order to refer to a largest planar matching that ends with edge (\bar{u}, \bar{v}) .

Note that the way in which \bar{G} is derived from G , implies in particular that for $u \in U$ and $i \leq j$, the size of the maximum planar matching in \bar{G} using nodes up to u^i is at least as large as the size of the maximum planar matching using nodes up to u^j . A similar fact holds for elements $v \in V$.

Several of the concepts introduced in this section are illustrated in Figure 1.

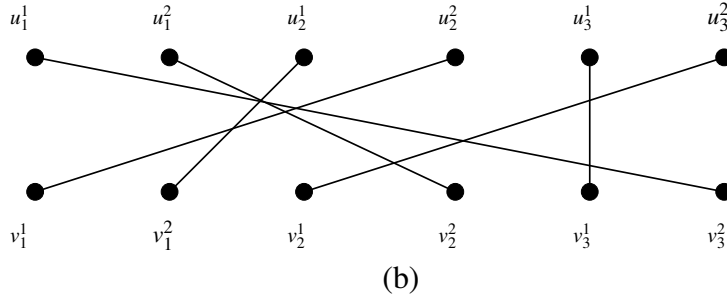
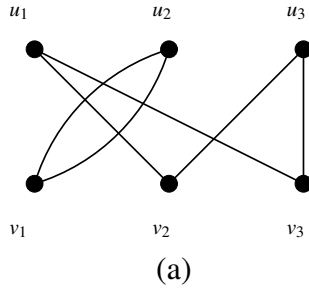


Figure 1: (a) A 2-regular multi-graph G . (b) Configuration \bar{G} associated to G .

1.3 Young tableaux

A (standard) Young tableau of shape $\lambda = (\lambda_1, \dots, \lambda_r)$ where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq 0$, is an arrangement $T = (T_{k,l})$ of $\lambda_1 + \dots + \lambda_r$ distinct integers in an array of left-justified rows, with λ_i elements in row i , such that the entries in each row are in increasing order from left to right, and the entries of each column are increasing from top to bottom (here we follow the usual convention that considers row i to be above row $i + 1$). One says that T has r rows and c columns if $\lambda_r > 0$ and $c = \lambda_1$ respectively. The shape of T will be henceforth denoted $\mathbf{shp}(T)$ and the collection of Young tableau with entries in the set S and with at most d columns will be denoted $T(S; d)$.

The Robinson correspondence (rediscovered independently by Schensted) states that the set of permutations of $[m]$ is in one to one correspondence with the collection of pairs of equal shape tableaux with entries in $[m]$. The correspondence can be constructed through the Robinson–Schensted–Knuth (RSK) algorithm — also referred to as row-insertion or row-bumping algorithm. The algorithm takes a tableau T and a positive integer x , and constructs a new tableau, denoted $T \leftarrow x$. This tableau will have one more box than T , and its entries will be those of T together with one more entry labeled x , but there is some moving around, the details of which are not of direct concern to us, except for the following fact:

Lemma 1 [Bumping Lemma [Ful97, pag. 9]] Consider two successive row-insertions, first row inserting x in a tableau T and then row-inserting x' in the resulting tableau $T \leftarrow x$, given rise to two new boxes B and B' as shown in Figure 2.

- If $x \leq x'$, then B is strictly left of and weakly below B' .
- If $x > x'$, then B' is weakly left of and strictly below B .

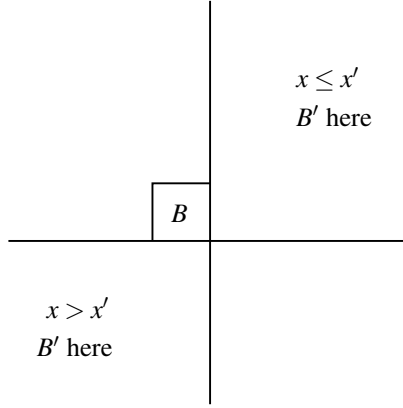


Figure 2: New tableau entries created through row–insertions.

Given a permutation π of $[m]$, the Robinson–Schensted–Knuth (RSK) correspondence constructs $(P(\pi), Q(\pi))$ such that $\mathbf{shp}(P(\pi)) = \mathbf{shp}(Q(\pi))$ by,

- starting with a pair of empty tableaux, repeatedly row–inserting the elements $\pi(1), \dots, \pi(n)$ to create $P(\pi)$, and,
- placing the value i into the box of $Q(\pi)$'s diagram corresponding to the box created during the i –th insertion into $P(\pi)$.

Two remarkable facts about the RSK algorithm which we will exploit are:

Remark 1 [*RSK Correspondence [Ful97, pag. 40]*] The RSK correspondence sets up a one–to–one mapping between permutations of $[m]$ and pairs of tableaux (P, Q) with the same shape.

Remark 2 [*Symmetry Theorem [Ful97, pag. 40]*] If π is a permutation of $[m]$, then $P(\pi^{-1}) = Q(\pi)$ and $Q(\pi^{-1}) = P(\pi)$.

Moreover, it is easy to see that the following holds:

Remark 3 Let π be a permutation of $[m]$. Then, π has no ascending sequence of length greater than d if and only if $P(\pi)$ and $Q(\pi)$ have at most d columns.

The reader interested on an in depth discussion of Young tableaux is referred to [Ful97].

1.4 Walks

We say that $w = w_0 \dots w_m$ is a lattice walk in \mathbb{Z}^d of length m if $\|w_i - w_{i-1}\|_1 = 1$ for all $1 \leq i \leq m$. Moreover, we say that w starts at the origin and ends in \vec{p} if $w_0 = \vec{0}$ and $w_m = \vec{p}$. For the rest of this paper, all walks are to be understood as lattice walks in \mathbb{Z}^d . Let $W(d, m; \vec{p})$ denote the set of all walks of length m from the origin to $\vec{p} \in \mathbb{Z}^d$.

We will often identify the walk $w = w_0 \dots w_m$ with the sequence $d_1 \dots d_m$ such that $w_i - w_{i-1} = \text{sign}(d_i)\vec{e}_{|d_i|}$, where \vec{e}_j denotes the j –th element of the canonical basis of \mathbb{Z}^d . If d_i is negative, then we say that the i –th step is a negative step in direction $|d_i|$, or negative step for short. We adopt a similar convention when d_i is positive.

We say that two walks are equivalent if both subsequences of the positive and the negative steps are the same. For each equivalence class consider the representative for which the positive steps precede the negative steps. Each such representative walk may hence be written as $a_1 a_2 \cdots | b_1 b_2 \cdots$ where the a_i 's and b_j 's are all positive. For an arbitrary collection of walks W , all with the same number of positive and the same number of negative steps, we henceforth denote by W^* the collection of the representative walks in W .

Recall that one can associate to a permutation π of $[d]$ the Toeplitz point $T(\pi) = (1 - \pi(1), \dots, d - \pi(d))$. Note that in a walk from the origin to a Toeplitz point, the number of steps in a positive direction equals the number of steps in a negative direction. In particular, each such walk has an even length.

In cases where we introduce notation for referring to a family of walks from the origin to a given lattice point \vec{p} , such as $W(d, m; \vec{p})$, we sometimes consider instead of \vec{p} a subset of lattice points P . It is to be understood that we are thus making reference to the set of all walks in the family that end at a point in P . A set of lattice points of particular interest to the ensuing discussion is the set of Toeplitz points, henceforth denoted \mathbb{T} .

We now come to a simple but crucial observation: there is a natural identification of $U \times [r]$ with $[rn]$ that respects the total order in each of these sets (\preceq in the former and \leq in the latter). A similar observation holds for $V \times [r]$. Hence, when $m = rn$ the sequences of positive and negative steps in a walk in $W(d, 2m; \mathbb{T})$ can be referred to as:

$$a_{u_1^1} \cdots a_{u_1^r} a_{u_2^1} \cdots a_{u_2^r} \cdots a_{u_n^1} \cdots a_{u_n^r} \quad \text{and} \quad b_{v_1^1} \cdots b_{v_1^r} b_{v_2^1} \cdots b_{v_2^r} \cdots b_{v_n^1} \cdots b_{v_n^r}.$$

Let $W'(d, 2m; T(\pi))$ be the set of all walks in $W^*(d, 2m; T(\pi))$ whose positive steps $a_{u_1^1} \cdots a_{u_n^r}$ and negative steps $b_{v_1^1} \cdots b_{v_n^r}$ satisfy: $a_{u^i} \geq a_{u^{i+1}}$ and $b_{v^i} \geq b_{v^{i+1}}$ for all $u \in U, v \in V$ and $1 \leq i < r$.

2 Counting Planar Matchings and Planar Subgraphs

We are now ready to state the main result of this paper.

Theorem 1

$$g(n; d) = \sum_{\pi} \text{sign}(\pi) |W'(d, 2rn; T(\pi))|.$$

Our proof of Theorem 1 is strongly based on the arguments used in [GWW98] to prove the following result concerning 1-regular bipartite graphs:

Theorem 2 *The signed sum of the number of walks of length $2m$ from the origin to Toeplitz points is $\binom{2m}{m}$ times the number $u_m(d)$ of permutations of length m that have no increasing sequence of length bigger than d .*

This last theorem gives a combinatorial proof of the following well known result:

Theorem 3 [Gessel's Identity] *If $I_\nu(t)$ denotes the Bessel function of imaginary argument, then*

$$\sum_{m \geq 0} \frac{u_m(d)}{(m!)^2} x^{2m} = \det(I_{|r-s|}(2x))_{r,s=1,\dots,d}.$$

We now describe a random process which researchers have studied, either explicitly or implicitly, in several different contexts. Let $X_{i,j}$ be a non-negative random variable associated to the lattice point $(i, j) \in [n]^2$. For $C \subseteq [n]^2$, we refer to $\sum_{(i,j) \in C} X_{i,j}$ as the weight of C . We are interested on the determination of the distribution of the maximum weight of C over all $C = \{(i_1, j_1), (i_2, j_2), \dots\}$ such that i_1, i_2, \dots and j_1, j_2, \dots are strictly increasing.

Johansson [Joh99] considered the case where the $X_{i,j}$ s are independent identically distributed according to a geometric distribution. Sepäläinen [Sep77] and Gravner, Tracy and Widom [GTW01] studied the case where the $X_{i,j}$ s are independent identically distributed Bernoulli random variables (but, in the latter paper, the collections of lattice points $C = \{(i_1, j_1), (i_2, j_2), \dots\}$ were such that i_1, i_2, \dots and j_1, j_2, \dots were weakly and strictly increasing respectively).

The main result of this paper, i.e., Theorem 1, says that if $(X_{i,j})_{(i,j) \in [n]^2}$ is uniformly distributed over all adjacency matrices of r -regular multi-graphs, then the distribution of the maximum weight evaluated at d can be expressed as a signed sum of restricted lattice walks in \mathbb{Z}^d . A natural question is whether a similar result holds if one relaxes the requirement that the sequences i_1, i_2, \dots and j_1, j_2, \dots are strictly increasing. For example, if one allows them to be weakly increasing. This is equivalent to asking for the distribution of the size of a planar subgraph, i.e., the largest set of non-crossing edges which may share endpoints in a uniformly chosen r -regular multigraph. A line of argument similar to the one we will use in the derivation of Theorem 1 yields:

Theorem 4 *Let $\hat{g}(n; d)$ be the number of r -regular bipartite multi-graphs with no larger than d set of non-crossing edges which may share endpoints. Then, $\hat{g}(n; d)$ equals the number of pairs of equal shape Young tableaux in $T([rn]; d)$ satisfying:*

Condition (\hat{T}): *If for each $i \in [n]$ and $1 \leq s < r$, the row containing $r(i-1) + s + 1$ is weakly above the row containing $r(i-1) + s$.*

Moreover,

$$\hat{g}(n; d) = \sum_{\pi} \text{sign}(\pi) \left| \widehat{W}'(d, 2rn; T(\pi)) \right|,$$

where $\widehat{W}'(d, 2rn; T(\pi))$ is the set of all walks in $W^*(d, 2rn; T(\pi))$ whose positive steps $a_{u_1} \cdots a_{u_r}$ and negatives steps $b_{v_1} \cdots b_{v_r}$ satisfy: $a_{u_i} < a_{u_{i+1}}$ and $b_{v_i} < b_{v_{i+1}}$ for all $u \in U$, $v \in V$ and $1 \leq i < r$.

In the rest of the paper we give two independent proofs of Theorem 1.

3 First Proof

Let $m = rn$. Recall that $\overline{G}_r(U, V; d)$ can be thought of as a collection of permutations of $[m]$. Thus, we may think of the RSK correspondence as being defined over $\overline{G}_r(U, V; d)$. In particular, for an r -configuration F of U and V we may write $(P(F), Q(F))$ to denote the pair of Young tableaux associated to the permutation determined by F . Figure 3 shows the result of applying the RSK algorithm to an r -configuration.

We say that a Young tableau in $T([m]; d)$ satisfies

Condition (T): *If for each $i \in [n]$ and $1 \leq s < r$, the row containing $r(i-1) + s$ is strictly above the row containing $r(i-1) + s + 1$.*

1	3
2	5
4	
6	

1	5
2	6
3	
4	

Figure 3: Pair of Young tableaux associated through the RSK algorithm to the 2–configuration of Figure 1.b.

The following result characterizes the image of $\overline{G}_r(U, V; d)$ through the RSK correspondence.

Theorem 5 *The number $g(n; d)$ equals the number of pairs of equal shape tableaux in $T([m]; d)$ satisfying condition (T). Specifically, the RSK correspondence establishes a one–to–one correspondance between $\overline{G}_r(U, V; d)$ and the collection of pairs of equal shape tableaux in $T([m]; d)$ satisfying condition (T).*

Proof: Let $\overline{G} \in \overline{G}_r(U, V; d)$. Remark 3 implies that $P(\overline{G})$ and $Q(\overline{G})$ are tableaux of equal shape that belong to $T([m]; d)$. Corollary 1 implies that, for every $i \in [n]$ the row insertion process through which $P(\overline{G})$ is built is such that the insertion of the values $r(i-1)+1, \dots, ri$ gives rise to a sequence of boxes each of which is strictly below the previous one. This implies that $Q(\overline{G})$ satisfies condition (T).

We still need to show that $P(\overline{G})$ also satisfies condition (T). For $\overline{G} \in \overline{G}_r(U, V; d)$ let the transpose of \overline{G} , denoted \overline{G}^T , be the bipartite graph over color classes U and V such that $u_i v_j$ is an edge of \overline{G}^T if and only if $u_j v_i$ is an edge of \overline{G} . Note that $\overline{G}^T \in \overline{G}_r(U, V; d)$ if and only if $\overline{G} \in \overline{G}_r(U, V; d)$. A direct consequence of Remark 2 is that $(P(\overline{G}^T), Q(\overline{G}^T)) = (Q(\overline{G}), P(\overline{G}))$. Hence, $P(\overline{G})$ must also satisfy condition (T).

Suppose now that (P, Q) is a pair of equal shape tableaux in $T([m]; d)$ both of which satisfy condition (T). Let F be an r –configuration of U and V such that $(P(F), Q(F)) = (P, Q)$ (here we identify u_i^s and v_i^s and view F as a permutation of $[m]$). The existence of F is guaranteed by Remark 1. Remark 3 implies that F ’s largest planar matching is of size at most d . Moreover, since $Q(F)$ satisfies property (T), Lemma 1 implies that the edges of F incident to u_i^s and u_i^{s+1} cross. Similarly, one can conclude that the edges fo F incident to v_i^s and v_i^{s+1} cross. It follows that F belongs to $\overline{G}_r(U, V; d)$. ■

Example 1 *Note that condition (T), as guaranteed by Theorem 5, is reflected in the tableaux shown in Figure 3 (for the tableau in the left; 4, 1 and 3 are strictly above 6, 2 and 5 respectively, while for the tableau in the right; 1, 3 and 5 are strictly above 2, 4 and 6 respectively).*

For a walk $w = a_1 \cdots a_m | b_1 \cdots b_m$ in $W'(d, 2m; T(\pi))$ let $\tilde{w} = \tilde{a}_1 \cdots \tilde{a}_m | \tilde{b}_1 \cdots \tilde{b}_m$ be such that $\tilde{a}_i = a_i$ and $\tilde{b}_i = b_{m-i}$. Denote by $\tilde{W}(d, 2m; T(\pi))$ the collection of all \tilde{w} for which w belongs to $W'(d, 2m; T(\pi))$. Our immediate goal is to establish the following

Theorem 6 *There is a bijection between $\overline{G}_r(U, V; d)$ and the walks in $\widetilde{W}(d, 2m; \vec{0})$ staying in the region $x_1 \geq x_2 \geq \dots \geq x_d$.*

We now discuss how to associate walks to Young tableaux. First we need to introduce additional terminology. We say that a walk $w = a_1 \cdots a_m$ satisfies

Condition (W): If for each $i \in [n]$ and $1 \leq s < r$ it holds that $a_{r(i-1)+s} \geq a_{r(i-1)+s+1}$.

Let φ be the mapping from $T([m]; d)$ to walks in $W(d, m; \mathbb{Z}^d)$ such that $\varphi(T) = a_1 \cdots a_m$ where a_i equals the column in which entry i appears in T . It immediately follows that:

Lemma 2 *The mapping φ is a bijection between tableaux in $T([m]; d)$ satisfying condition (T) and walks of length m starting at the origin, moving only in positive directions, staying in the region $x_1 \geq x_2 \geq \dots \geq x_d$ and satisfying condition (W).*

Proof: If $\varphi(T) = \varphi(T')$ for $T, T' \in T([m]; d)$, then T and T' have the same elements in each of their columns. Since in a Young tableau the entries of each column are increasing from top to bottom, it follows that $T = T'$. We have thus established that φ is an injection.

Assume now that $w = a_1 \cdots a_m$ is a walk of length m starting at the origin, moving only in positive directions, staying in the region $x_1 \geq x_2 \geq \dots \geq x_d$ and satisfying condition (W). Denote by $C(l)$ the set of indices j for which $a_j = l$. Note that since w is a walk in \mathbb{Z}^d , then $C(l)$ is empty for all $l > d$. Let T be the Young tableau whose l -th column entries correspond to $C(l)$ (obviously ordered increasingly from top to bottom). Note that T is indeed a Young tableau since $|C(1)| \geq |C(2)| \geq \dots \geq |C(d)|$ and given that the entries on each row of T are strictly increasing (the latter follows from the fact that w stays in the region $x_1 \geq x_2 \geq \dots \geq x_d$). Observe that T belongs to $T([m]; d)$. We claim that T satisfies condition (T). Indeed, by construction and since w satisfies condition (W), for each $i \in [n]$ it must hold that the indices of the columns of the entries $r(i-1)+1, \dots, ri$ of T is a weakly decreasing sequence. Hence, for every $1 \leq s < r$, the entry $r(i-1)+s+1$ is weakly to the left of $r(i-1)+s$. Since $r(i-1)+s+1 > r(i-1)+s$ and T is a tableau, it must be the case that the entry $r(i-1)+s+1$ is strictly below the entry $r(i-1)+s$. ■

Note that if T and T' belong to $T([m]; d)$ and have the same shape, then $\varphi(T)$ and $\varphi(T')$ are walks that terminate at the same lattice point.

Corollary 1 *There is a bijection between ordered pairs of tableaux of the same shape belonging to $T([m]; d)$ satisfying condition (T), and walks in $\widetilde{W}(d, 2m; \vec{0})$ staying in the region $x_1 \geq x_2 \geq \dots \geq x_d$.*

Proof: By Lemma 2 there is a bijection between ordered pairs of tableaux with the claimed properties and ordered pairs of walks of length m starting at the origin, moving only in positive directions that stay in the region $x_1 \geq x_2 \geq \dots \geq x_d$ and satisfy condition (W). Say such pair of walks are $c_1 \cdots c_m$ and $c'_1 \cdots c'_m$ respectively. Then, $c_1 \cdots c_m | c'_m \cdots c'_1$ is the sought after walk with the desired properties. ■

Figure 4 illustrates the bijection implicit in the proof of Corollary 1. Note that Theorem 6 is an immediate consequence of Theorem 5 and Corollary 1.

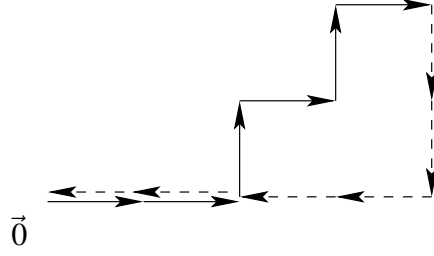


Figure 4: Walk in $\widetilde{W}(d, 2m; \vec{0})$ associated to the pair of Young tableaux of Figure 3 (and thus also to the graph of Figure 1).

Proof: [of Theorem 1] The desired conclusion is an immediate consequence of Theorem 6 and the existence of a parity-reversing involution ρ on the walks w in $\widetilde{W}(d, 2m; \vec{0})$ not staying in the region $x_1 \geq x_2 \geq \dots \geq x_d$. The involution is most easily described if we translate the walks to start at $(d-1, d-2, \dots, 0)$; the walks are then restricted not to lie completely in the region R defined by $x_1 > x_2 > \dots > x_d$. Let N be the subset of the translated walks of $\widetilde{W}(d, 2m; \vec{0})$ not lying completely in R . Let $w = c_1 \dots c_{2m} \in N$ and let t be the smallest index such that the walk given by the initial segment of $c_1 \dots c_t$ of w terminates in a vertex $(p_1, \dots, p_d) \notin R$. Hence, there is exactly one j such that $p_j = p_{j+1}$.

Walk $\rho(w)$ is constructed as follows:

- Leave segment $c_1 \dots c_t$ unchanged.
- For each $i \in [2n]$, define $S(i) = \{s \in [2m] : r(i-1) < s \leq ri\}$, $S_0(i) = \{s \in S(i) : s > t, c_s = j\}$ and $S_1(i) = \{s \in S(i) : s > t, c_s = j+1\}$. For $i \leq n$ (respectively $i > n$), assign the value $j+1$ to the $|S_0(i)|$ first (respectively last) coordinates of $(c_s : s \in S_0(i) \cup S_1(i))$ and the value j to the remaining $|S_1(i)|$ coordinates.

It is easy to see that if w terminates in (q_1, \dots, q_d) , then $\rho(w)$ terminates in $(q_1, \dots, q_{j+1}, q_j, \dots, q_d)$. Hence, ρ reverses the parity of w . Moreover, $\rho \circ \rho$ is the identity. It remains to show that $\rho(w) \in N$. Obviously $\rho(w)$ does not stay in R (as w does not). Hence, it suffices to show the following: if $\rho(w) = a_1 \dots a_m | b_1 \dots b_m$, then for each $i \in [n]$ and $1 \leq s < r$ we have $a_{r(i-1)+s} \geq a_{r(i-1)+s+1}$ and $b_{r(i-1)+s} \leq b_{r(i-1)+s+1}$. This is clearly true for every block $\{r(i-1)+s : 1 \leq s \leq r\}$ completely contained inside w 's unchanged segment (i.e., $1, \dots, t$) and inside w 's modified segment (i.e., $t+1, \dots, 2m$), given that it is true for w and by the definition of ρ . There is still the case to handle where $t \in \{r(i-1)+s : 1 \leq s \leq r\}$. Here, it is true by the following observation: if $t \leq m$ then $c_t = j+1$, otherwise $c_t = j$. ■

4 Second proof

Henceforth let $m = rn$. In this section we introduce two mappings Φ and ϕ . The former is shown to be an injection that, when restricted to $\mathcal{F} = \overline{G}_r(U, V; d)$, takes values in $W'(d, 2m; \vec{0})$. Our first goal is to characterize those walks that belong to $\Phi(\mathcal{F})$. The second mapping ϕ plays

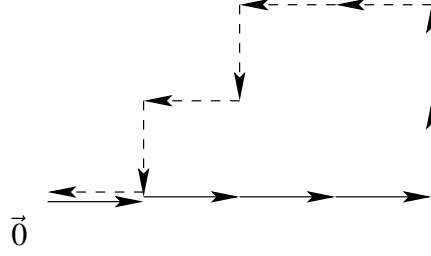


Figure 5: Walk $\Phi(\bar{G}) = 111122|112121$ for the multi-graph G of Figure 1 (direction 1 is to the right and direction 2 is up — negative steps are represented by segmented lines.)

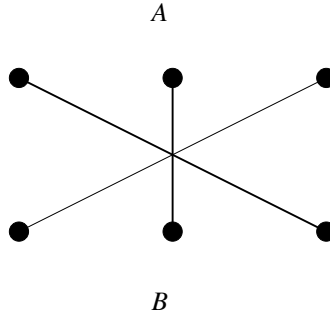


Figure 6: Crossing obtained from left to right ordered sets A and B .

a crucial role in fulfilling this latter objective. Then, relying on the aforementioned characterization we define a parity reversing involution on $W'(d, 2m; \mathbb{T}) \setminus \Phi(\mathcal{F})$. This essentially establishes Theorem 1.

Let Φ be the function that associates to an r -configuration F of U and V the value $\Phi(F) = a_{u_1} \cdots a_{u_n} | b_{v_1} \cdots b_{v_n} \in W^*(d, 2m; \mathbb{Z}^d)$, where

- $a_{\bar{u}}$ equals the largest size of a planar matching of F using nodes up to \bar{u} ,
- $b_{\bar{v}}$ equals the largest size of a planar matching of F using nodes up to \bar{v} .

Note that indeed $\Phi(F) \in W^*(d, 2m; \mathbb{Z}^d)$ when F is an r -configuration of U and V . Figure 5 illustrates the definition of $\Phi(\cdot)$.

The following definition will be instrumental in the introduction of a mapping between walks and configurations.

Definition 1 Let A and B be two linearly ordered sets of equal size. We say that a quasi configuration is obtained from A and B in a crossing way if the first element of A is paired with the last element of B , and so on, until finally the last element of A is paired to the first element of B .

Figure 6 illustrates the concept just introduced. We say that H is a quasi r -configuration of U and V if it can be obtained from a configuration F of U and V by “breaking” (deleting) some of its “edges” (pairings). Note that the same quasi r -configuration G may be obtained by “breaking” different r -configurations.

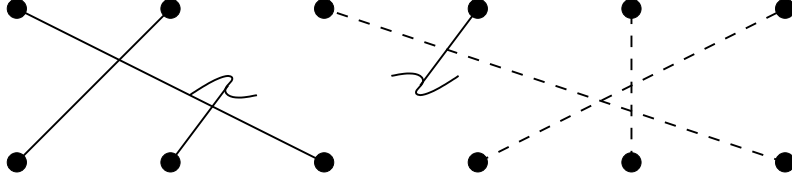


Figure 7: The quasi configuration $\phi(w)$ associated to $w = 112122|122122$. Continuous lines corresponds to the crossing of $A_1(w)$ and $B_1(w)$ and segmented lines to the crossing of $A_2(w)$ and $B_2(w)$.

For $w = a_{u_1}^1 \cdots a_{u_n}^n | b_{v_1}^1 \cdots b_{v_n}^n \in W^*(d, 2m; \mathbb{Z}^d)$, let $A_k(w) = \{\bar{u} : a_{\bar{u}} = k\}$ and $B_k(w) = \{\bar{v} : b_{\bar{v}} = k\}$. We are now ready to introduce a mapping between walks and quasi configurations. Let ϕ be a function that associates to a walk $w \in W^*(d, 2m; \mathbb{Z}^d)$ a quasi r -configuration $\phi(w)$ as follows: for each k , if $|A_k(w)| \geq |B_k(w)|$ then connect the initial segment of $A_k(w)$ of size $|B_k(w)|$ in a crossing way with $B_k(w)$. If $|A_k(w)| \leq |B_k(w)|$, then connect the terminal segment of $B_k(w)$ of size $|A_k(w)|$ in a crossing way with $A_k(w)$. Figure 7 illustrates $\phi(\cdot)$'s definition.

Fact 1 *Let $w \in W^*(d, 2m; \mathbb{Z}^d)$. Then, the edges in $\phi(w)$ incident to two distinct elements of $A_k(w)$ must cross. A similar observation holds for $B_k(w)$.*

Fact 2 *Let $w = \Phi(F)$ for an r -configuration F of U and V and let (\bar{u}, \bar{v}) be a pairing of F . Then, $\bar{u} \in A_k(w)$ if and only if $\bar{v} \in B_k(w)$.*

The following result gives an interpretation in terms of graphs of what it means for a walk starting at the origin to terminate also at the origin.

Lemma 3 *Let $w \in W^*(d, 2m; \mathbb{Z}^d)$. Then, $\phi(w)$ is an r -configuration of U and V if and only if $w \in W^*(d, 2m; \vec{0})$ for some d .*

Proof: A closed walk w passes through the origin if and only if $|A_k(w)| = |B_k(w)|$ for all k . The latter is certainly equivalent to $\phi(w)$ being a configuration. ■

We now prove a technical result.

Lemma 4 *Let $k \in [d]$ be arbitrary. For every r -configuration F of U and V , the set of edges incident to $A_k(\Phi(F))$ equals the set of edges incident to $B_k(\Phi(F))$.*

Proof: Let $\bar{u} \in A_k(\Phi(F))$. There is a unique \bar{v} such that (\bar{u}, \bar{v}) is a pairing of F . By Fact 2, it must hold that $\bar{v} \in B_k(\Phi(F))$. ■

The following result establishes that $\Phi(\cdot)$ is an injection.

Lemma 5 *For every r -configuration F of U and V , it holds that $\phi(\Phi(F)) = F$.*

Proof: By Fact 1 and Lemma 4, $A_k(\Phi(F))$ and $B_k(\Phi(F))$ are equal size sets that must be joined in the crossing way in F . Since a pairing of F is an element of $A_k(\Phi(F)) \times B_k(\Phi(F))$ for some k , it follows that $\phi(\Phi(F)) = F$. ■

Lemma 6 Let \mathcal{F} be a family of r -configurations of U and V . A walk w belongs to $\Phi(\mathcal{F})$ if and only if $\Phi(\phi(w)) = w$ and $\phi(w) \in \mathcal{F}$.

Proof: If $\phi(w) \in \mathcal{F}$, then $w = \Phi(\phi(w))$ belongs to $\Phi(\mathcal{F})$. If $w = \Phi(F)$ for some r -configuration F of U and V , then Lemma 5 implies that $\phi(w) = F$. If in addition $F \in \mathcal{F}$, then one gets that $\phi(w) \in \mathcal{F}$. ■

Two walks in $W^*(d, 2m; \mathbb{Z}^d)$ are certainly equal if their sequence of positive and negative steps agree. The next lemma gives a simpler necessary and sufficient condition for the equality of two walks w and $\Phi(\phi(w))$ when w is a closed walk that goes through the origin. Indeed, it says that one only needs to focus on establishing the equality of the sequence of their positive steps. The result will be useful later in order to establish the equality of two walks w and $\Phi(\phi(w))$.

Lemma 7 Let $w \in W^*(d, 2m; \vec{0})$. Then, $\Phi(\phi(w))$ and w agree in their positive steps if and only if $\Phi(\phi(w)) = w$.

Proof: If $\Phi(\phi(w)) = w$, then $\Phi(\phi(w))$ and w clearly agree in their positive steps. To prove the converse, let $w' = \Phi(\phi(w))$. Assume w' and w agree in their positive steps. First, recall that by Lemma 3, $\phi(w)$ is an r -configuration of U and V . Hence, Lemma 5 implies that $\phi(w') = \phi(\Phi(\phi(w))) = \phi(w)$. Thus, $A_k(w) = A_k(w')$ for every k . Since $\phi(w')$ and $\phi(w)$ are the same configurations, they have the same set of edges. Consider $\bar{v} \in B_k(w)$. There is a unique edge (\bar{u}, \bar{v}) of $\phi(w)$ incident on \bar{v} . By Fact 2, we have that $\bar{u} \in A_k(w) = A_k(w')$. But edge (\bar{u}, \bar{v}) is an edge of $\phi(w')$. Hence, again by Fact 2, we get that $\bar{v} \in B_k(w')$. We have shown that $B_k(w) \subseteq B_k(w')$. The reverse inclusion can be similarly proved. Since k was arbitrary, we conclude that the negative steps of w and w' are the same, and the two walks must thus be equal. ■

For the walk $w = a_{u_1} \cdots a_{u_r} | b_{v_1} \cdots b_{v_n}$, denote by $k(\bar{u})$ and $l(\bar{u})$ the number of occurrences of $a_{\bar{u}}$ and $a_{\bar{u}} - 1$ in $\{\bar{u}' \in \bar{U} : \bar{u}' \preceq \bar{u}\}$ respectively.

Example 2 For the walk 111122|112121 of Figure 5 we have:

\bar{u}	1	2	3	4	5	6
$k(\bar{u})$	1	2	3	4	1	2
$l(\bar{u})$	0	0	0	0	4	4

We say that w satisfies

Condition (C): If for each \bar{u} such that $a_{\bar{u}} > 1$, $l(\bar{u}) > 0$ and the $l(\bar{u})$ -th-to-last appearance of $a_{\bar{u}} - 1$ in the negative steps of w , if it exists, comes before the $k(\bar{u})$ -th-to-last appearance of $a_{\bar{u}}$ in the negative steps of w .

Lemma 8 Let $w = a_{u_1} \cdots a_{u_r} | b_{v_1} \cdots b_{v_n}$ be a walk in $W^*(d, 2m; \mathbb{T})$. Then, $\Phi(\phi(w)) = w$ and $\phi(w)$ is an r -configuration of U and V if and only if w satisfies condition (C).

Proof: Since $\phi(w)$ is an r -configuration of U and V , Lemma 3 implies that w must terminate at the origin. On the other hand, if w satisfies condition (C), then w also needs to terminate at the origin. Hence, $\phi(w)$ would be an r -configuration of U and V . Indeed, let j be the smallest coordinate in which the terminal point of w is positive. Note that $j > 1$ by the definition of

Toeplitz points. Let \bar{u} be maximum so that $a_{\bar{u}} = j$. By the choice of j , there will be fewer than $k(\bar{u})$ appearances of j and at least $l(\bar{u})$ appearances of $j - 1$ among the negative steps. This contradicts the fact that w satisfies condition (C). We thus can assume without loss of generality that w starts and ends at the origin.

Let $\Phi(\phi(w)) = a'_{u_1} \cdots a'_{u_r} | b'_{v_1} \cdots b'_{v_n}$. By Lemma 7, $\Phi(\phi(w))$ and w are distinct if and only if they differ in some positive step. Assume that $a_{\bar{u}} = a'_{\bar{u}}$ for each $\bar{u} \prec \tilde{u}$ and $a_{\bar{u}} \neq a'_{\tilde{u}}$. We claim that $a'_{\tilde{u}} \leq a_{\tilde{u}}$. Indeed, suppose this is not the case. Since $a'_{\tilde{u}}$ equals the size of the largest planar matching of $\phi(w)$ up to \tilde{u} , there is a $\bar{u} \prec \tilde{u}$ such that $a'_{\bar{u}} = a_{\tilde{u}}$ and the edges incident to \bar{u} and \tilde{u} of $\phi(w)$ are non-crossing. We also have $a'_{\bar{u}} = a_{\bar{u}}$ by the choice of \tilde{u} . Hence, \bar{u} and \tilde{u} belong to $A_k(w)$ for some k . By Fact 1 the edges incident to \bar{u} and \tilde{u} must be non-crossing. A contradiction. This establishes our claim.

It follows that $a'_{\tilde{u}} = a_{\tilde{u}}$ if and only if $a'_{\tilde{u}} \geq a_{\tilde{u}}$. We now establish a condition equivalent to $a'_{\tilde{u}} \geq a_{\tilde{u}}$ by considering the following two cases:

- **Case $a_{\tilde{u}} = 1$:** Then, certainly $a'_{\tilde{u}} \geq a_{\tilde{u}}$.
- **Case $a_{\tilde{u}} > 1$:** Then, there is a $\bar{u} \prec \tilde{u}$ such that $a'_{\bar{u}} = a_{\tilde{u}} - 1$ and the edges incident to \bar{u} and \tilde{u} are non-crossing in $\phi(w)$. So we can extend with the edge incident to \tilde{u} the size $a'_{\bar{u}}$ planar matching of $\phi(w)$ up to \bar{u} . Thus, it must be the case that $a'_{\tilde{u}} \geq a_{\tilde{u}}$.

Summarizing $a_{\tilde{u}} = a'_{\tilde{u}}$ if and only if

- $a_{\tilde{u}} = 1$, or
- if $a_{\tilde{u}} > 1$ and there is a $\bar{u} \prec \tilde{u}$ such that $a'_{\bar{u}} = a_{\tilde{u}} - 1$ and the edges \tilde{u} and \bar{u} are non-crossing in $\phi(w)$.

The lemma follows by observing that when $a_{\tilde{u}} > 1$, the fact that w satisfies condition (C) amounts to saying that there is a $\bar{u} \prec \tilde{u}$ such that $a_{\bar{u}} = a_{\tilde{u}} - 1$ and the edges incident to \bar{u} and \tilde{u} are non-crossing in $\phi(w)$. So, all positive steps of $\Phi(\phi(w))$ and w agree if and only if for each \bar{u} such that $a_{\bar{u}} > 1$, $l(\bar{u}) > 0$ and the $l(\bar{u})$ -th-to-last appearance of $a_{\bar{u}} - 1$ in the negative steps of w , if it exists, comes before the $k(\bar{u})$ -th-to-last appearance of $a_{\bar{u}}$ in the negative steps of w . ■

So far in this section we have not directly been concerned with walks $W'(d, 2m; \mathbb{T})$ nor the collection of configurations $\overline{G}_r(U, V; d)$. The next result is the link through which we use all previous results in order to prove Theorem 1.

Lemma 9 *Let $w \in W'(d, 2m; \vec{0})$. If $\Phi(\phi(w)) = w$, then $\phi(w) \in \overline{G}_r(U, V; d)$.*

Proof: Suppose $\Phi(\phi(w)) = w$ and $w = a_{u_1} \cdots a_{u_r} | b_{v_1} \cdots b_{v_n}$ is such that $\phi(w)$ does not belong to $\overline{G}_r(U, V; d)$. Note that since w is a closed walk that goes through the origin, by Lemma 3, we have that $\phi(w)$ is an r -configuration of U and V . Thus, it must be the case that either there is a $u \in U$ such that for some $s < t$ the edges incident to u^s and u^t are non-crossing, or there is a $v \in V$ such that for some $s < t$ the edges incident to v^s and v^t are non-crossing. Without loss of generality assume the former case holds. It follows that, the largest planar matching up to u^s is strictly smaller than the largest planar matching up to u^t , i.e., $a_{u^s} < a_{u^t}$. This contradicts the fact that w belongs to $W'(d, 2m; \vec{0})$. ■

Theorem 7 *The mapping Φ is a bijection between $\overline{G}_r(U, V; d)$ and the collection of walks in $W'(d, 2m; \mathbb{T})$ satisfying condition (C).*

Proof: By Lemma 5 we know that Φ is an injection. We claim it is also onto. Indeed, if w is a walk in $W'(d, 2m; \mathbb{T})$ satisfying condition (C), then Lemmas 3, 6, 8, and 9 imply that $\phi(w)$ belongs to $\overline{G}_r(U, V; d)$ and $\Phi(\phi(w)) = w$. ■

Proof: [of Theorem 1] The desired conclusion is an immediate consequence of Theorem 7 and the existence of a parity-reversing involution ρ on walks w in $W'(d, 2m; \mathbb{T})$ that don't satisfy condition (C). To define ρ , assume $w = a_{u_1^1} \cdots a_{u_n^r} | b_{v_1^1} \cdots b_{v_n^r}$ and let \bar{u} be the smallest index for which w does not satisfy condition (C). Let \bar{v} be such that $b_{\bar{v}}$ is the $l(\bar{u})$ -th-to-last occurrence of $a_{\bar{u}} - 1$ among the negative steps; if $l(\bar{u}) = 0$ then let $\bar{v} = rn + 1$.

Walk $\rho(w)$ is constructed as follows:

- Leave segments $a_{u_1^1} \cdots a_{\bar{u}}$ and $b_{\bar{v}} \cdots b_{v_n^r}$ unchanged.
- For every $i \in [n]$, let $S_0(i) = \left\{ s : a_{u_i^s} = a_{\bar{u}}, \bar{u} \prec u_i^s \right\}$ and $S_1(i) = \left\{ s : a_{u_i^s} = a_{\bar{u}} - 1, \bar{u} \prec u_i^s \right\}$. Assign the value $a_{\bar{u}}$ to the $|S_1(i)|$ first coordinates in $(a_{u_i^s} : s \in S_0(i) \cup S_1(i))$ and the value $a_{\bar{u}} - 1$ to the remaining $|S_0(i)|$ coordinates.

The application of ρ does not change the smallest index not satisfying the sufficient condition of Lemma 8. It follows that $\rho(w)$ also violates condition (C). We claim that $\rho(w) = a'_{u_1^1} \cdots a'_{u_n^r} | b'_{v_1^1} \cdots b'_{v_n^r}$ belongs to $W'(d, 2m; \mathbb{T})$. We need to show that for each $i \in [n]$ and $1 \leq s < r$, we have $a'_{u_i^s} \geq a'_{u_i^{s+1}}$ and $b'_{v_i^s} \geq b'_{v_i^{s+1}}$. This is clearly true for every block $\{u_i^s : s \in [r]\}$ completely contained in the unchanged segments, and also inside the modified segment. The remaining two cases to consider are $\bar{u} = u_i^s$ and/or $\bar{v} = v_j^{s'}$ for some $s < r$ and/or $s' > 1$. Both cases are easy to handle. We leave the details to the reader.

Assume w terminates at $T(\pi)$ for some permutation π of $[d]$. Let τ be a transposition of $a_{\bar{u}}$ and $a_{\bar{u}} - 1$. Finally, we claim that $\rho(w)$ terminates in $T(\pi \circ \tau)$. Indeed, by our choice of \bar{u} , the number of appearances of $a_{\bar{u}}$ in $b_{\bar{v}} \cdots b_{v_n^r}$ is less than $k(\bar{u})$. It must equal to $k(\bar{u}) - 1$, otherwise we could have chosen the index of the $(k(\bar{u}) - 1)$ -th appearance of $a_{\bar{u}}$ for \bar{u} . Hence, in the unchanged segments of the walk w , there is one net positive step in direction $a_{\bar{u}}$ and zero net steps in direction $a_{\bar{u}} - 1$. It follows that in the segment of w that changes, there are $a_{\bar{u}} - \pi(a_{\bar{u}}) - 1$ and $a_{\bar{u}} - 1 - \pi(a_{\bar{u}} - 1)$ net positive steps in directions $a_{\bar{u}}$ and $a_{\bar{u}} - 1$ respectively. Let σ_s denote the s -th coordinate of the terminal point of a walk σ . We get that $\rho(w)_{a_{\bar{u}}} = a_{\bar{u}} - 1 - \pi(a_{\bar{u}} - 1) + 1 = a_{\bar{u}} - \pi(a_{\bar{u}} - 1)$ and similarly $\rho(w)_{a_{\bar{u}} - 1} = a_{\bar{u}} - \pi(a_{\bar{u}}) - 1 = a_{\bar{u}} - 1 - \pi(a_{\bar{u}})$. ■

References

- [AD99] D. Aldous and P. Diaconis. Longest increasing subsequences: From patience sorting to the Baik–Deift–Johansson theorem. *Bull. of the AMS*, 36(4):413–432, 1999.
- [Ale94] K. S. Alexander. The rate of convergence of the mean length of the longest common subsequence. *The Annals of Applied Probability*, 4(4):1074–1082, January 1994.

- [BB67] R. Baer and P. Brock. Natural sorting over permutation spaces. *Mathematics of Computation*, pages 385–410, 1967.
- [BDJ99] J. Baik, P. Deift, and K. Johansson. On the distribution of the length of the longest increasing subsequence of random permutations. *J. Amer. Math. Soc.*, 12:1119–1178, 1999.
- [Bol85] B. Bollobás. *Random Graphs*. Academic Press, 1985.
- [BW88] B. Bollobás and P. Winkler. The longest chain among random points in euclidean space. *Proceedings of the American Mathematical Society*, 103(2):347–353, 1988.
- [ES35] P. Erdős and G. Szekeres. A combinatorial problem in geometry. *Compositio Math.*, 2:463–470, 1935.
- [Ful97] W. Fulton. *Young Tableaux*, volume 35 of *Student Texts*. London Mathematical Society, 1997.
- [GTW01] J. Gravner, C. Tracy, and H. Widom. Limit theorems for height fluctuations in a class of discrete space time growth models. *J. Stat. Phys.*, 102:1085–1132, 2001.
- [GWW98] I. Gessel, J. Weinstein, and H. Wilf. Lattice walks in \mathbb{Z}^d and permutations with no long ascending subsequences. *Electronic J. of Combinatorics*, 5(2):R2, 1998.
- [Ham72] J. M. Hammersley. A few seedlings of research. In *Proc. Sixth Berkeley Sympos. Math. Stat. Prob.*, pages 345–394, Berkeley, Calif., 1972. Univ. of California Press.
- [Joh99] K. Johansson. Shape fluctuations and random matrices. Technical Report math.CO/9903134, arXiv, October 1999.
- [Kin73] J. F. C. Kingman. Subadditive ergodic theory. *The Annals of Prob.*, 1(6):883–909, 1973.
- [KL02] M. Kiwi and M. Loeb. Largest planar matching in random bipartite graphs. *Random Structures and Algorithms*, 21(2):162–181, 2002.
- [KLM05] M. Kiwi, M. Loeb, and J. Matoušek. Expected length of the longest common subsequence for large alphabets. *Advances in Mathematics*, 2005. In press.
- [LS77] B. Logan and L. Shepp. A variational problem on random Young tableaux. *Adv. in Math.*, 26:206–222, 1977.
- [Oko00] A. Okounkov. Random matrices and random permutations. *International Mathematics Research Notices*, pages 1043–1095, 2000.
- [Sch61] C. Schensted. Longest increasing and decreasing subsequences. *Canad. J. Math.*, 13:179–191, 1961.
- [Sep77] T. Seppäläinen. Exact limiting shape for a simplified model of first-passage percolation on the plane. *The Annals of Probability*, 26(3):1232–1250, 1977.

- [Sta02] R. Stanley. Recent progress in algebraic combinatorics. *Bull. of the AMS*, 40(1):55–68, 2002.
- [Ste77] J. M. Steele. Limit properties of random variables associated with a partial ordering of \mathbb{R}^d . *The Annals of Probability*, 5(3):395–403, 1977.
- [Ula61] S. Ulam. Monte Carlo calculations in problems of mathematical physics. In *Modern Mathematics for the Engineers*, pages 261–281. McGraw-Hill, 1961.
- [VK77] A. Vershik and S. Kerov. Asymptotics of the Plancherel measure of the symmetric group and the limiting form of Young tableaux. *Dokl. Akad. Nauk SSSR*, 233:1024–1028, 1977.