Generalised Dualities and Finite Maximal Antichains* (extended abstract)

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Abstract

We fully characterise the situations where the existence of a homomorphism from a digraph G to at least one of a finite set $\mathcal{H}$ of directed graphs is determined by a finite number of forbidden subgraphs. We prove that these situations, called *generalised dualities*, are characterised by the non-existence of a homomorphism to G from a finite set of forests.

Furthermore, we characterise all finite maximal antichains in the partial order of directed graphs ordered by the existence of homomorphism. We show that these antichains correspond exactly to the

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generalised dualities. This solves a problem posed in [13]. Finally, we show that it is NP-hard to decide whether a finite set of digraphs forms a maximal antichain.

1 Introduction and Previous Results

Several classical colouring problems (such as bounding the chromatic number of graphs with given properties) can be treated more generally and sometimes more efficiently in the context of graphs and homomorphisms between them. Recall that, given graphs $G = (V, E)$, $G' = (V', E')$, a *homomorphism* is any mapping $f : V \to V'$ which preserves edges:

$$xy \in E \Rightarrow f(x)f(y) \in E'.$$

This is denoted by $f : G \to G'$. For a recent introduction to the topic of graphs and their homomorphisms, we refer the reader to the book [5].

Let H be a fixed graph (sometimes called a template). For an input graph G, the *H-colouring problem* asks whether there exists a homomorphism $G \to H$. Such a homomorphism is also called an *H-colouring*; the K_k-colouring problem is simply the question whether $\chi(G) \leq k$. Of course, the complexity of the H-colouring problem depends on H. This complexity was determined for undirected graphs in [4]. However, already for directed graphs the problem is unsolved.

The H-colouring problem is also (and perhaps more often) called the *constraint satisfaction problem (CSP(H))*. This is particularly used when the problem is generalised to relational structures and their homomorphisms, as these structures can encode arbitrary constraints. This setting, originally motivated by problems from Artificial Intelligence, leads to the important problem of dichotomy, general heuristic algorithms (consistency check) and, more recently, to an interesting and fruitful algebraic setting (pioneered by Bulatov, Jeavons and Krokhin, cf. [7], [2]).

Further work in the area of CSP complexity led to the following dichotomy conjecture.

Conjecture 1 ([3]). Let H be a finite relational structure. Then CSP(H) is either solvable in polynomial time or NP-complete.

Some particular instances of CSP were studied intensively. This includes the case when the graphs for which there exists an H-colouring are determined by well-described forbidden subgraphs (see [6], [10]) and as a special case, when they are determined by a finite family of forbidden subgraphs. Of course, in these cases we get polynomial instances of CSP.

A pair (F, D) of directed graphs is called a *duality pair* if for every directed graph G, we have $F \to G$ if and only if $G \nrightarrow D$. Here, and from now on, $A \to B$ denotes the fact that there exists a homomorphism from A to B. The duality relationship is denoted by the equation

$$F\!\to\; = \;\nrightarrow\! D$$

where $F\!\to$ denotes the class of graphs admitting a homomorphism from F and $\nrightarrow\! D$ the class of graphs not admitting a homomorphism to D. The dualities in the category of directed graphs are characterised in [8], [12]:

Theorem 2 ([8], [12])**.** *Given a directed graph F, there exists a directed graph D_F such that (F, D_F) is a duality pair if and only if F is homomorphically equivalent to an orientation of a tree. For a Δ-tree F, such a Δ-structure D_F is unique up to homomorphism equivalence.*

We say that A and B are homomorphically equivalent if both $A \to B$ and $B \to A$. The unique D such that (F, D) is a duality pair is called *the dual* of the Δ-tree F. We use the notation $D = D(F)$.

Here we generalise the notion of a duality pair: for two finite sets of graphs $\mathcal{F}$, $\mathcal{D}$, we say that $(\mathcal{F}, \mathcal{D})$ is a *generalised duality* if for any graph G, there exists $F \in \mathcal{F}$ such that $F \to G$ if and only if $G \to D$ for no $D \in \mathcal{D}$; briefly

$$\bigcup_{F \in \mathcal{F}} F\!\to\; = \bigcap_{D \in \mathcal{D}} \nrightarrow\! D.$$

The special case $|\mathcal{D}| = 1$ is characterised by the following theorem proved in [12].

Theorem 3 ([12])**.** *Let $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ be a finite nonempty set of Δ-structures. The pair $(\mathcal{F}, \{D\})$ is a generalised duality if and only if $D = \prod_{i=1}^{m} D_i$ and (F_i, D_i) is a duality pair for $i = 1, 2, \ldots, m$.*

When $p = |\mathcal{D}| = 1$, the generalised duality $(\mathcal{F}, \mathcal{D})$ is called a *finitary homomorphism duality* in [12]. The theorem states that the finitary dual is the product of the duals of the Δ-trees $F_1, \ldots, F_m$; this product will be denoted by $D(F_1, \ldots, F_m)$ or $D(M)$ if $M = \{F_1, \ldots, F_m\}$.

We can also consider the case $\mathcal{F} = \emptyset$. Then $\mathcal{D} = \{\mathbf{1}\}$, where $\mathbf{1}$ is a single vertex with a loop.

The relation $\to$ induces a partial order $\mathcal{C}$ on the classes of homomorphic equivalence of graphs. This order is called the *homomorphism order*. The homomorphism order is actually a distributive lattice, with the disjoint union (the sum) of graphs being the supremum and the categorical product being the infimum. The standard order-theoretic terminology is applied here.

Particular properties of the homomorphism order, that were studied, are density (solved for undirected graphs by Welzl [15] and for directed graphs by Nešetřil and Tardif [12]) and the description of finite maximal antichains.

Earlier results characterise all maximal antichains of size 1 and 2 in the homomorphism order of directed graphs.

Theorem 4 ([14]). *The only maximal antichains of size 1 in the homomorphism order of directed graphs are directed paths of length 0, 1, and 2 and a single vertex with a loop.*

Theorem 5 ([13]). *The maximal antichains of size 2 in the homomorphism order of directed graphs are precisely the pairs $\{T, D_T\}$, where T is a tree different from P_0, P_1 and P_2, and D_T is its dual.*

2 Generalised Dualities

In this section, we characterise all generalised dualities. We restrict ourselves to the case $|\mathcal{F}| \geq 2$, as the other cases are described in the previous section. First, we present a construction of generalised dualities from a family of forests.

2.1 The Construction

Let $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ be an arbitrary fixed nonempty finite set of core (cf. [5]) Δ-forests that are pairwise incomparable.

Consistently with the above notation, let $\mathcal{F}_c = \{C_1, \ldots, C_n\}$ be the set of all distinct connected components of the graphs in $\mathcal{F}$; each of these components is a core Δ-tree.

A subset $M \subseteq \mathcal{F}_c$ is a *quasitransversal* if it satisfies

(T1) M is an antichain, i.e. for every $C \neq C' \in M$ we have $C \parallel C'$, and

(T2) M *supports* $\mathcal{F}$, i.e. for every $F \in \mathcal{F}$ there exists $C \in M$ such that $C \to F$.

For two quasitransversals M, M' we define $M \preceq M'$ if and only if for every $C' \in M'$ there exists $C \in M$ such that $C \to C'$. Note that this order is different from the homomorphism order of forests corresponding to the quasitransversals. On the other hand, we have:

Lemma 6. *Let M, M' be two quasitransversals. Then $D(M) \to D(M')$ if and only if $M \preceq M'$*

Lemma 7. *The relation $\preceq$ is a partial order on the set of all quasitransversals.*

A quasitransversal M is a *transversal* if

(T3) M is a maximal quasitransversal in $\preceq$.

Set $\mathcal{D} = \mathcal{D}(\mathcal{F}) = \{D(M) : M \text{ is a transversal}\}$.
We have:

Theorem 8. *The pair $(\mathcal{F}, \mathcal{D})$ is a generalised duality.*

Before outlining the proof, we give three of examples.

Example. First, let $\mathcal{F} = \{T_1, T_2, \dots, T_n\}$ be a set of pairwise incomparable trees and D_1, D_2, $\dots$, D_n their respective duals. By (T2), every transversal contains all these trees. Therefore there exists only one transversal $M = \{T_1, T_2, \dots, T_n\}$ and $\mathcal{D} = \{D_1 \times D_2 \times \dots \times D_n\}$. This situation shows how the finitary duality is a special case of the generalised duality.

Now, let T_1, T_2, T_3 and T_4 be pairwise incomparable trees with duals D_1, D_2, D_3, D_4. Let $\mathcal{F} = \{T_1 + T_2, T_1 + T_3, T_4\}$. Then we have two transversals $\{T_1, T_4\}$ and $\{T_2, T_3, T_4\}$; and $\mathcal{D} = \{D_1 \times D_4, D_2 \times D_3 \times D_4\}$.

Finally, let $T_1 \to T_3$ and $\mathcal{F} = \{T_1 + T_2, T_3 + T_4\}$. The transversals are $\{T_1\}$, $\{T_2, T_3\}$ and $\{T_2, T_4\}$. Hence $\mathcal{D} = \{D_1, D_2 \times D_3, D_2 \times D_4\}$.

Proof of Theorem 8. Let X be a Δ-structure such that $X \to D$ for some $D \in \mathcal{D}$. We want to prove that $F_i \nrightarrow X$ for $i = 1, \dots, m$. For contradiction, assume that $F_i \to X$ for some i. Let M be the transversal for which $D(M) = D$. By (T2), there exists $C \in M$ such that $C \to F_i \to X$, therefore $X \nrightarrow D(C)$. That is a contradiction with the assumption that $X \to D \to D(C)$.

Now, let X be a Δ-structure such that $F_i \nrightarrow X$ for $i = 1, \dots, m$. We want to prove that there exists $D \in \mathcal{D}$ such that $X \to D$. Let C_{j_i} be a component of F_i such that $C_{j_i} \nrightarrow X$ for $i = 1, \dots, m$. Let $M' = \min_{\to}\{C_{j_i} : i = 1, \dots, m\}$; by $\min_{\to} S$ we mean the set of all elements of S that are minimal with respect to the homomorphism order $\to$. Because M' is a quasitransversal, there exists a transversal M such that $M' \preceq M$. We have that $C \nrightarrow X$ for each $C \in M$, and therefore $X \to D(M) \in \mathcal{D}$. $\square$

2.2 The Characterisation

We will now prove that all generalised dualities are of the above form.

Theorem 9. *If $(\mathcal{F}, \mathcal{D})$ is a generalised duality, then all elements of $\mathcal{F}$ are forests and $\mathcal{D} = \mathcal{D}(\mathcal{F})$; in particular, $\mathcal{D}$ is uniquely determined by $\mathcal{F}$.*

Proof. The proof consists of five steps. Suppose that $\mathcal{F} = \{A_1, A_2, \ldots, A_m\}$ and $\mathcal{D} = \{D_1, D_2, \ldots, D_p\}$. Consistently with the above notation, let $\mathcal{F}_c = \{C_1, C_2, \ldots, C_n\}$ be the set of all distinct connected components of the structures in $\mathcal{F}$. Quasitransversals and transversals are defined in the same way as above; we note that neither for the definition nor for proving Lemma 7 do we need the fact that the elements of $\mathcal{F}_c$ are trees.

For a quasitransversal M, let $\overline{M} = \{C' \in \mathcal{F}_c : C \in M \Rightarrow C \nrightarrow C'\}$.

Fact 1. *If $M \subseteq \mathcal{F}_c$ is a transversal, then there exists a unique Δ-structure $D \in \mathcal{D}$ that satisfies*

1. *$C \nrightarrow D$ for every $C \in M$,*
2. *$C' \to D$ for every $C' \in \overline{M}$.*

Proof. If $\overline{M} = \emptyset$, let $D \in \mathcal{D}$ be arbitrary. Otherwise set $S = \sum_{C' \in \overline{M}} C'$. Because $(\mathcal{F}, \mathcal{D})$ is a generalised duality, either there exists $F \in \mathcal{F}$ such that $F \to S$ or there exists $D \in \mathcal{D}$ such that $S \to D$. If $F \to S$, by (T2) there exists $C \in M$ satisfying $C \to F \to S$, and since C is connected, $C \to C'$ for some $C' \in \overline{M}$, which is a contradiction with the definition of $\overline{M}$. Therefore there exists $D \in \mathcal{D}$ that satisfies $S \to D$. It can be checked that such D satisfies both (1) and (2) and that such a graph must be unique. $\square$

For a transversal M, the unique $D \in \mathcal{D}$ satisfying the conditions (1) and (2) above is denoted by $d(M)$.

Fact 2. $\mathcal{D} = \{d(M) : M \text{ is a transversal}\}$.

Proof. Let $D \in \mathcal{D}$. We want to show that $D = d(M)$ for a transversal M. Let $M' = \min_{\to}\{C' \in \mathcal{F}_c : C' \nrightarrow D\}$ be the set of all $\mathcal{C}(\Delta)$-minimal components that are not homomorphic to D. The set M' is a quasitransversal: if some $F \in \mathcal{F}$ is not supported by M', then all its components are homomorphic to D, and also $F \to D$, a contradiction.

Let M be a transversal such that $M' \preceq M$. To prove that $D = d(M)$, it suffices (by the uniqueness part of Fact 1) to check conditions (1) and (2). □

Fact 3. *For two distinct transversals M_1, M_2, we have (a) $\overline{M_1} \cap M_2 \neq \emptyset$, (b) $d(M_1) \neq d(M_2)$.*

Proof.

(a) By (T3), $M_1 \npreceq M_2$, so there exists $C_2 \in M_2$ such that $C_1 \nrightarrow C_2$ for any $C_1 \in M_1$. Obviously $C_2 \in \overline{M_1} \setminus \overline{M_2} \subseteq \overline{M_1}$. Since we chose $C_2 \in M_2$, we have $C_2 \in \overline{M_1} \cap M_2$.

(b) Let $C_2 \in \overline{M_1} \cap M_2$, as above. Then $C_2 \to d(M_1)$ and $C_2 \nrightarrow d(M_2)$. □

Fact 4. *If M is a transversal, then the pair $(M, \{d(M)\})$ is a finitary homomorphism duality, and consequently $d(M) = D(M)$.*

Proof. We can prove that for a Δ-structure G, the following statements are equivalent:

(1) $G \in \bigcap_{C \in M}(C \nrightarrow)$
(2) $C \nrightarrow G$ for any $C \in M$
(3) $C \nrightarrow G + \sum_{\check{C} \in \overline{M}} \check{C}$ for any $C \in M$
(4) $G + \sum_{\check{C} \in \overline{M}} \check{C} \to d(M)$
(5) $G \to d(M)$
(6) $G \in (\to d(M))$

The equivalence (1) $\Leftrightarrow$ (6) is precisely the definition of finitary duality. □

Fact 5. *Each component $C \in \mathcal{F}_c$ is a tree.*

For the proof, we use the following

Theorem 10 ([12]). *Let A and C be relational structures such that $A < C$, and C is a connected structure that is not a tree. Then there exists a structure X such that $A < X < C$.*

Proof of Fact 5. Using Theorem 10, it can be proved that if some component $C \in \mathcal{F}_c$ is not a tree, then there exists a digraph X such that $X < C$ and X is homomorphic to exactly the same elements of $\mathcal{F}_c$ as C, and moreover for any $C' \in \mathcal{F}_c$, $C' \neq C$, we have $C' \to C$ if and only if $C' \to X$. Then if G is created by replacing C with X in some $F \in \mathcal{F}$, the graph G violates the definition of generalised duality (no $F \in \mathcal{F}$ is homomorphic to G and G is homomorphic to no $D \in \mathcal{D}$). □

We finish the proof of Theorem 9. All elements of $\mathcal{F}$ are forests by virtue of Fact 4, Fact 5 and Theorem 3. The set $\mathcal{D}$ is uniquely determined as a consequence of Fact 2 and due to Fact 4 and Theorem 3 it is determined by the transversal construction. □

3 Finite Maximal Antichains

First, we discuss when a generalised duality forms a maximal antichain; precisely, for what families $\mathcal{F}$ of incomparable forests is $\mathcal{Q} = \mathcal{F} \cup \mathcal{D}(\mathcal{F})$ a maximal antichain in the homomorphism order of Δ-structures.

Obviously, if a generalised duality forms an antichain, then it is maximal. It is also evident that $F \nrightarrow D$ for any $F \in \mathcal{F}$, $D \in \mathcal{D}$. So, a generalised duality does not form an antichain if and only if there exist $D \in \mathcal{D}$ and $F \in \mathcal{F}$ such that $D \to F$.

Let $P_1 = (\{1,2\}, \{(1,2)\})$ be the Δ-structure consisting of a single edge (the path of length 1). If $P_1 \in \mathcal{F}_c$, then obviously $\mathcal{F} = \{P_1\}$ and $\mathcal{D} = \{\mathbf{0}\}$. So for the rest, we can assume that $P_1 \notin \mathcal{F}_c$.

Let $P_2 = (\{1,2,3\}, \{(1,2),(2,3)\}$ be the directed path of length 2.

Lemma 11. *Let $\mathcal{F}$ be a set of pairwise incomparable core Δ-forests. Then $\mathcal{F} \cup \mathcal{D}(\mathcal{F})$ is not an antichain if and only if $\mathcal{F} = \{\mathbf{0}\}$, $\mathcal{F} = \{P_1\}$, or $\mathcal{F} = \{P_2\}$.*

Proof. We have just observed that if $\mathcal{F} \cup \mathcal{D}(\mathcal{F})$ is not an antichain, there exist $D \in \mathcal{D}$ and $F \in \mathcal{F}$ such that $D \to F$. Fix such F and D.

Let A be a Δ-structure. If there exists a Δ-tree T such that $A \to T$, we say that A is *balanced*. It is easy to see that A is balanced if and only if it is homomorphic to a Δ-forest.

Since F is a Δ-forest, we have that D is balanced. Moreover, by Theorem 9, $D = D(M)$ for a transversal $M \subseteq \mathcal{F}_c$.

Let Z_s be the orientation of a cycle of length $2s+1$ such that Z_s does not contain a directed path of length 3. Since Z_s is not balanced, $Z_s \nrightarrow D = D(M)$; therefore (by the definition of finitary duality) for every positive integer s there exists $C \in M$ such that $C \to Z_s$. Hence there exists some $C \in Z_s$ such that $|V(C)| > 2s+1$. Since any proper subgraph of Z_s is homomorphic to P_2, we get that $C \to P_2$. This finishes the proof as the other implication is evident. $\square$

We have now observed that only three generalised dualities that are not antichains exist: $(\emptyset, \{\mathbf{1}\})$, $(\{P_1\}, \{\mathbf{0}\})$ and $(\{P_2\}, \{P_1\})$. Let us now consider the question when a maximal antichain is not a generalised duality.

Observe that a finite maximal antichain $\mathcal{Q}$ is formed from a generalised duality if and only if there exist disjoint sets $\mathcal{F}$, $\mathcal{D}$ such that $\mathcal{Q} = \mathcal{F} \cup \mathcal{D}$ and for an arbitrary Δ-structure X there exists $F \in \mathcal{F}$ such that $F \to X$ or there exists $D \in \mathcal{D}$ such that $X \to D$.

Lemma 12. *Let $\mathcal{Q}$ be a finite maximal antichain in the homomorphism order of digraphs. Then the following are equivalent:*

1. *$\mathcal{Q}$ is not formed from a generalised duality, i.e. whenever $\mathcal{Q} = \mathcal{F} \cup \mathcal{D}$, the pair $(\mathcal{F}, \mathcal{D})$ is not a generalised duality,*
2. *$\mathcal{Q}$ is one of the sets $\{\mathbf{0}\}$, $\{P_1\}$, or $\{P_2\}$.*

Proof. All the sets $\{\mathbf{0}\}$, $\{P_1\}$, $\{P_2\}$ are obviously finite maximal antichains not formed from a generalised duality.

The other implication is proved by splitting $\mathcal{Q}$ into $\mathcal{F}$ and $\mathcal{D}$ in a suitable way, which allows us to show that all elements of $\mathcal{F}$ are balanced. But since $(\mathcal{F}, \mathcal{D})$ is not a generalised duality, we can use an argument similar to the above proof to show that $\mathcal{F}$ is one of the sets $\{\mathbf{0}\}$, $\{P_1\}$, or $\{P_2\}$. $\square$

Thus we come to the astonishing correspondence between generalised dualities and maximal antichains. This solves a problem posed in [13], where maximal antichains of size 2 were characterised.

Theorem 13. *The correspondence*

$$(\mathcal{F}, \mathcal{D}) \;\mapsto\; Q = \mathcal{F} \cup \{D \in \mathcal{D} : D \nrightarrow F \textit{ for any } F \in \mathcal{F}\}$$

is a one-to-one correspondence between generalised dualities and finite maximal antichains in the homomorphism order of directed graphs.

Proof. Follows immediately from Theorems 4 and Lemmas 11 and 12. $\square$

4 Extensions

4.1 MAC Decidability

We are interested in the following decision problem, called the *MAC decision problem*: given a finite nonempty set $\mathcal{Q}$ of Δ-structures, decide whether $\mathcal{Q}$ is a maximal antichain. The results of the previous section allow us to state the following result.

Theorem 14. *The MAC decision problem is decidable. Moreover, it is NP-hard.*

Another consequence of Theorem 13 is the following.

Theorem 15. *Let $\mathcal{Q}$ be a finite maximal antichain in $\mathcal{C}$. An element of $\mathcal{Q}$ that is comparable with an input structure A can be found in polynomial time.*

4.2 Duality Decidability

Using a recent result of [9], we can deduce that it is decidable whether for a set $\mathcal{H}$ of Δ-structures there exists a set $\mathcal{F}$ of Δ-structures such that $(\mathcal{F}, \mathcal{H})$ is a generalised duality.

It is easy to see that $\mathcal{H}$ is the right-hand side of a generalised duality if and only if each structure in $\mathcal{H}$ is a finitary dual and they are pairwise incomparable. The former is decidable (and even in NP) due to [9], the latter is obviously in NP. It also follows from [9] that in general, the problem is NP-complete.

4.3 GCSP Dichotomy

As an analogy to CSP, we define *GCSP*, the *generalised constraint satisfaction problem*, as the following: given a finite set $\mathcal{H}$ of Δ-structures, decide for an input Δ-structure G whether there exists $H \in \mathcal{H}$ such that $G \to H$.

Note that if $(\mathcal{F}, \mathcal{D})$ is a generalised duality, then $\mathrm{GCSP}(\mathcal{D})$ is polynomially solvable.

As in Conjecture 1, one could ask whether there is a dichotomy for GCSP. However, this problem is not very captivating, as the positive answer to the dichotomy conjecture for CSP would imply a positive answer here as well:

Theorem 16. *Let $\mathcal{H}$ be a finite nonempty set of pairwise incomparable Δ-structures.*

1. *If CSP(H) is tractable for all $H \in \mathcal{H}$, then GCSP($\mathcal{H}$) is tractable.*
2. *If CSP(H) is NP-complete for some $H \in \mathcal{H}$, then GCSP($\mathcal{H}$) is NP-complete.*

Thus from the complexity (and dichotomy) point of view, generalised CSP is equivalent to CSP. But their first-order definability is another matter: it is both interesting and more involved.

4.4 First-order Definable GCSP

We remark that GCSP($\mathcal{H}$) is first-order definable if and only if there exists a set $\mathcal{F}$ such that $(\mathcal{F}, \mathcal{H})$ is a generalised duality. This result is an extension of a similar theorem for CSP contained in [1], and its proof follows the same way.

5 Summary and Concluding Remarks

In Sect. 2, we characterised all the generalised dualities $(\mathcal{F}, \mathcal{D})$ in the category of directed graphs: the set $\mathcal{D}$ such that $(\mathcal{F}, \mathcal{D})$ is a generalised duality exists if and only if $\mathcal{F}$ is a finite family of forests; if this is the case, $\mathcal{D}$ is determined uniquely (up to homomorphic equivalence).

In Sect. 3 we described all finite maximal antichains in the homomorphism orderof directed graphs. They all appear to be formed from generalised dualities by taking all maximal elements of $\mathcal{F} \cup \mathcal{D}$.

We mention here that the result on generalised dualities extends to the fully general setting of relational structures with relations of arbitrary arity. Similarly, Theorem 13 can be generalised for relational structures with one relation of arbitrary arity. The maximal antichains which are not of the form $\mathcal{F} \cup \mathcal{D}$ for a generalised duality $(\mathcal{F}, \mathcal{D})$ are $\{\mathbf{0}\}$, $\{P_1\}$ and $\mathcal{S}$, where $\mathcal{S}$ is the set of all core trees with two edges. Both these results will appear in the full version of this paper.

Let us note that the characterisation of finite maximal antichains is hard and interesting for infinite graphs. It has been proved in [11] that for every countable infinite graph G, G not equivalent to K_1, K_2, K_ω, there exists a graph H incomparable with G. There are also infinitely many maximal antichains, however, as pointed out in [11], all maximal antichains seem to contain a finite graph.

We believe that the interplay of order theoretic notions (such as maximal antichain) and descriptive complexity notions (such as generalised duality and first order definability) leads to further insight into the structure of CSP. For example, the duality theorems present a rich supply of non-trivial CSP problems, for which polynomial algorithms exist.

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