A note on k-walks in bridgeless graphs

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Abstract

We show that every bridgeless graph of maximum degree Δ has a spanning $\lceil(\Delta+1)/2\rceil$-walk. The bound is optimal.

1 Introduction

Following Jackson and Wormald [6], we define a *k-walk* in a graph G to be a closed spanning walk visiting each vertex at most k times, where $k \geq 1$ is an integer. Being an interesting variation on the notion of a Hamilton cycle, this concept has received considerable attention (see, e.g., [2, 3, 5]).

Our aim in this note is to determine the least possible $k = k(\Delta)$ such that every graph of maximum degree Δ admits a k-walk. For general graphs, this problem is trivial since a tree of maximum degree Δ has a Δ-walk [6], and it clearly does not admit any k-walk with $k < \Delta$. The situation changes if we restrict ourselves to *bridgeless* (i.e., 2-edge-connected) graphs. We prove the following result:

Theorem 1. *Every bridgeless graph of maximum degree Δ admits a $\lceil(\Delta+1)/2\rceil$-walk.*

Theorem 1 follows directly from a more general statement (Theorem 5) which we prove in Section 2. In Section 3, we complement this result by showing that the bound in Theorem 1 is best possible.

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[5]Supported by the NSFC (60373012 and 10471078), SRSDP (20040422004) and PDSF (2004036402) of China.

2 The upper bound

All the graphs we consider are finite and loopless, multiple edges are allowed. Throughout this section, G is a graph. Its vertex set and the edge set are denoted by $V(G)$ and $E(G)$, respectively. If W is a walk in G, we let $p_W(x)$ denote the number of times a vertex $x \in V(G)$ is visited by W. An *edge-cut* in G is an inclusionwise minimal set of edges whose removal disconnects G.

Let v be a vertex of G and e_1, e_2 be two distinct edges incident with v. Let v_i be the endvertex of e_i distinct from v. We recall the operation of *splitting* e_1 *and* e_2 *off* v. The resulting graph $G(v, e_1, e_2)$ is defined to be G with an added vertex v^* and the edges e_1, e_2 replaced with e_1^*, e_2^*, where e_i^* has ends v^* and v_i. The following assertion is an easy consequence of Fleischner's Splitting Lemma [4] (see also [9, Theorem A.5.2]):

Lemma 2. *Let v be a vertex of degree at least 4 in a bridgeless graph G. There exist edges e_1, e_2 incident with v such that the graph $G(v, e_1, e_2)$ is bridgeless.*

Lemma 3. *Let v be a vertex of a graph G, let e_1, e_2 be two edges incident with v, and $H = G(v, e_1, e_2)$. If W is a spanning closed walk in H such that $p_W(v^*) \leq 2$ (where v^* is defined as above), then G admits a closed walk $\tilde{W}$ such that*

(i) for all $z \in V(G) \setminus \{v\}$, $p_{\tilde{W}}(z) \leq p_W(z)$, and

(ii) $1 \leq p_{\tilde{W}}(v) \leq p_W(v) + 1$.

Proof. Enumerate the vertices visited by W as

$$W = x_0 x_1 \dots x_\ell,$$

where $x_0 = x_\ell$. Any operations on the indices of the vertices in W are performed modulo ℓ. A *subwalk* of W is a walk of the form

$$[x_i, x_j] = x_i x_{i+1} \dots x_{j-1} x_j.$$

We write $[x_i, x_j]^-$ for the reverse subwalk $x_i x_{i-1} \dots x_{j+1} x_j$.

If $p_W(v^*) = 1$, then we may set $\tilde{W} = W$. Thus, it may be assumed that $p_W(v^*) = 2$. Let the two occurences of v^* in W be x_i and x_j, where $i < j$. We use the symbols v_1, v_2 as introduced in the definition of splitting.

Suppose first that both neighbors of x_i on W coincide with v_1, i.e., $[x_{i-1}, x_{i+1}] = v_1 v^* v_1$. Then we may set

$$\tilde{W} = [x_0, x_{i-1}]\, [x_{i+2}, x_{j-1}]\, v\, [x_{j+1}, x_\ell]$$

(see Figure 1a). Note that we may indeed concatenate the subwalks $[x_0, x_{i-1}]$ and $[x_{i+2}, x_{j-1}]$ since x_{i+2} is a neighbor of $x_{i-1} = x_{i+1}$. It is easy to check that $\tilde{W}$ satisfies the conditions (i)–(ii). By symmetry, we may assume that the neighbors of x_i on W are v_1 and v_2, and the same holds for x_j. We now distinguish two cases.

Case 1: $[x_{i-1}, x_{i+1}] = [x_{j-1}, x_{j+1}] = v_1 v^* v_2$. We set

$$\tilde{W} = [x_0, x_{i-1}]\, [x_{j-2}, x_{i+2}]^- \, [x_{j+1}, x_\ell]$$

(see Figure 1b). Note that the conditions (i)–(ii) are satisfied. By symmetry, this case also covers the possibility that $[x_{i-1}, x_{i+1}]$ and $[x_{j-1}, x_{j+1}]$ equal $v_2 v^* v_1$.

Case 2: $[x_{i-1}, x_{i+1}] = [x_{j-1}, x_{j+1}]^- = v_1 v^* v_2$. Since W is spanning, there is k such that $x_k = v$. We may assume that $i < k < j$ since the other possibility ($k < i$ or $k > j$) is symmetric. The walk

$$\tilde{W} = [x_0, x_{i-1}]\, v\, [x_{k-1}, x_{i+2}]^- \, [x_{j-1}, x_{k+1}]^- \, v\, [x_{j+1}, x_\ell]$$

(see Figure 1c for an illustration) meets the requirements.

Since we have covered, up to symmetry, all the possibilities, the proof is complete. $\square$

Spanning closed walks correspond to edge weight functions in the following straightforward way. Let w be a function assigning to each edge $e \in E(G)$ a non-negative integer $w(e)$. For any set $X \subset E(G)$, we define

$$w(X) = \sum_{e \in X} w(e).$$

The function w is an *Eulerian weight* if for each edge-cut C in G, the value $w(C)$ is positive and even. Note that if w is an Eulerian weight, then each vertex v must be incident with an edge of nonzero weight, since the set

$$\partial v = \{e : e \text{ is incident with } v\}$$

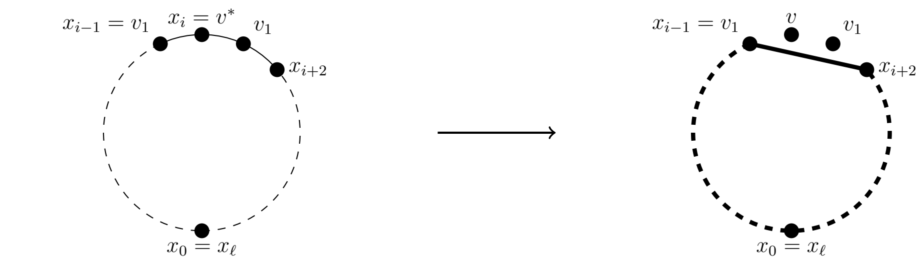
contains an edge-cut.

Lemma 4. *Let G be a graph and $k \geq 1$ a positive integer. The graph G has a k-walk if and only if it admits an Eulerian weight w such that for each $v \in V(G)$,*

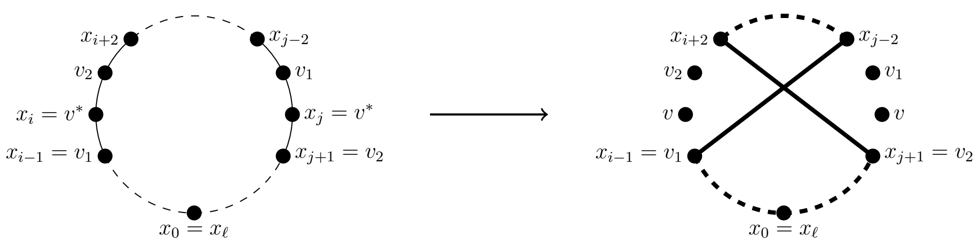
$$w(\partial v) \leq 2k. \tag{1}$$

Proof. If G has a k-walk W, then the function assigning each edge the number of times it is traversed by W (in any direction) is clearly an Eulerian weight satisfying (1). Conversely, let w be such an Eulerian weight. Replacing each edge e by $w(e)$ parallel edges (or deleting it if $w(e) = 0$), we obtain a (connected) Eulerian graph of maximum degree at most $2k$. Any Euler trail in the new graph determines a k-walk in G. $\square$

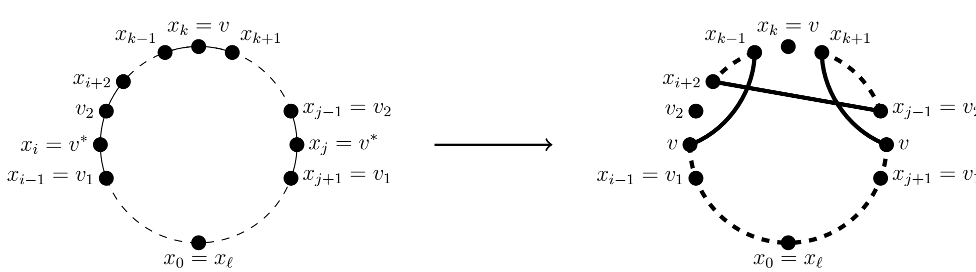
We now proceed to prove the main result of this paper.



(a) W contains a subwalk $v_1 v^* v_1$.



(b) W contains two subwalks $v_1 v^* v_2$.



(c) W contains subwalks $v_1 v^* v_2$ and $v_2 v^* v_1$.

Figure 1: The possibilities considered in the proof of Lemma 3. Dashed lines represent walks, edges are drawn solid. In each case, thick lines give the resulting walk $\tilde{W}$ in G.

Theorem 5. *Every bridgeless graph admits a closed spanning walk W such that for each vertex x,*

$$p_W(x) \leq \left\lceil \frac{\deg(x)+1}{2} \right\rceil. \tag{2}$$

Proof. By induction. We first establish the assertion for graphs with maximum degree $\Delta \leq 3$. Then, we prove that if $\Delta(G) \geq 4$, the assertion holds for G whenever it holds for all bridgeless graphs that are smaller than G in a certain sense.

Assume first that $\Delta(G) \leq 3$. Since the minimum degree is at least 2 and the claim is clearly true if G is a circuit, we may assume that G is a subdivision of a cubic bridgeless graph H. By the well-known Petersen theorem (see, e.g., [1, Corollary 2.2.2]), H has a 1-factor F. Let $w : E(G) \to \{1, 2\}$ be a function whose value $w(e)$ is 2 if the edge of H corresponding to e belongs to F, and 1 otherwise. It is easy to see that w is an Eulerian weight in G. By Lemma 4, G admits a 2-walk.

Next, assume that $\Delta(G) \geq 4$ and (2) holds for all graphs G' such that either $\Delta(G') < \Delta(G)$, or $\Delta(G') = \Delta(G)$ and G' has fewer vertices of maximum degree. We show that the assertion holds for G.

Let v be any vertex of degree $\Delta(G)$. Lemma 2 ensures that there are two edges e_1, e_2 such that $G(v, e_1, e_2)$ is bridgeless. Since the resulting graph has fewer vertices of degree $\Delta(G)$, the induction hypothesis implies that $G(v, e_1, e_2)$ admits a closed spanning walk W_0 satisfying (2). Using Lemma 3, we find a closed spanning walk $\tilde{W}_0$ in G such that for each vertex $x \in V(G) \backslash \{v\}$, $p_{\tilde{W}_0}(x) \leq p_{W_0}(x)$, and

$$1 \leq p_{\tilde{W}_0}(v) \leq p_{W_0}(v) + 1.$$

Clearly, the closed walk $\tilde{W}_0$ in G is spanning, satisfies (2) at all vertices $x \neq v$, and

$$p_{\tilde{W}_0}(v) \leq p_{W_0}(v) + 1 \leq \left\lceil \frac{(\deg(v)-2)+1}{2} \right\rceil + 1 = \left\lceil \frac{\deg(v)+1}{2} \right\rceil.$$

It follows that $W = \tilde{W}_0$ satisfies (2) at all vertices of G. □

3 The lower bound

Theorem 6. *For every even $\Delta \geq 4$, there is a 2-connected graph G with $\Delta(G) = \Delta$ and no $(\Delta/2)$-walk.*

Proof. Let $k = \Delta - 1$. For $i \in \{1, \ldots, 9\}$, take a copy H_i of the complete bipartite graph $K_{2,k}$, with the degree k vertices denoted by a_i and b_i.

The graph G is obtained from the disjoint union of the graphs $H_1, \ldots, H_9$ by adding new vertices a and b, together with edges

$$\{aa_i : i \in \{1, 4, 7\}\} \cup \{bb_i : i \in \{3, 6, 9\}\} \cup \{b_i a_{i+1} : i \in \{1, 2, 4, 5, 7, 8\}\}$$

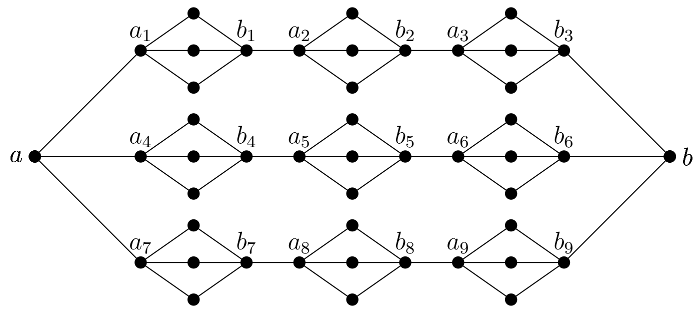


Figure 2: A 2-connected graph with maximum degree 4 and no 2-walk.

(see Figure 2 for an illustration with $k = 3$). Note that the maximum degree of G is $k + 1 = \Delta$.

We now show that G has no $(\Delta/2)$-walk. Assume the contrary. By Lemma 4, there is an Eulerian weight w satisfying

$$w(\partial v) \leq \Delta \tag{3}$$

for each vertex v.

Since $w(\partial a)$ is even, there is an edge incident with a that receives an even value. We may assume that $w(aa_1)$ is even. Since each pair of edges from the set

$$C = \{aa_1, b_1a_2, b_2a_3, b_3b\}$$

forms an edge-cut, at most one edge $e \in C$ has $w(e) = 0$. Consequently, for some $i \in \{1, 2, 3\}$, both edges in C that are incident with either a_i or b_i are assigned a positive even value by w. Let C_i be the set consisting of these two edges. We have

$$w(E(H_i)) = w(\partial a_i) + w(\partial b_i) - w(C_i) \leq 2\Delta - 4 \tag{4}$$

by (3).

For each vertex d of degree 2 in H_i, ∂d is an edge-cut, whence $w(\partial d) \geq 2$. It follows that

$$w(E(H_i)) \geq 2k = 2\Delta - 2,$$

contradicting (4). It follows that G does not admit any $(\Delta/2)$-walk. □

Recall that a *trail* in a graph is a walk using each edge at most once. By a well-known result of Jaeger [7, 8], every 4-edge-connected graph G admits a spanning closed trail. It is easy to see that if the maximum degree of G is Δ, then such a trail gives rise to a $\lceil\Delta/2\rceil$-walk in G. For even Δ, this improves on the bound of Theorem 1 by one. Since the tightness example constructed in the proof of Theorem 6 makes a heavy use of edge-cuts of size 2, one may wonder whether such an improvement is possible even for 3-edge-connected graphs G. We leave this as an open problem:

Problem 7. *Does every 3-edge-connected graph of maximum degree Δ admit a $\lceil \Delta/2 \rceil$-walk?*

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