COMPLETE CONGRUENCES ON TOPOLOGIES AND DOWN-SET LATTICES

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Dedicated to Bernhard Banaschewski

ABSTRACT. From the work of Simmons about nuclei in frames it follows that a topological space X is scattered if and only if each congruence Θ on the frame of open sets is induced by a unique subspace A so that $\Theta = \{(U, V) \mid U \cap A = V \cap A\}$, and that the same holds without the uniqueness requirement iff X is weakly scattered (corrupt). We prove a seemingly similar but substantially different result about quasidiscrete topologies (in which arbitrary intersections of open sets are open): each complete congruence on such a topology is induced by a subspace if and only if the corresponding poset is (order) scattered, i.e. contains no dense chain. More questions concerning relations between frame, complete, spatial, induced and open congruences are discussed as well.

1. INTRODUCTION: THE ROLE OF DOWN-SET LATTICES IN DUALITIES

Each subset A of a partially ordered set (*poset*) X with order relation $\leq$ generates a *down-set* (*lower set, decreasing set, initial interval*)

$$\downarrow A = \{x \in X \mid x \leq a \text{ for some } a \in A\}$$

and an *up-set* (*upper set, increasing set, final interval*)

$$\uparrow A = \{x \in X \mid a \leq x \text{ for some } a \in A\}.$$

As usual, $\downarrow a$ denotes the *principal ideal* $\downarrow\{a\}$, and $\uparrow a$ the *principal filter* $\uparrow\{a\}$. Since the pioneering work of Alexandroff [1] one knows that the lattices $\mathfrak{U}(X)$ of all up-sets and, by dualization, the lattices $\mathfrak{D}(X)$ of

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all down-sets are precisely the *quasidiscrete topologies*, i.e. those T_0 topologies which are closed under arbitrary intersections. Moreover, the open sets in the *lower Alexandroff topology* $\mathfrak{D}(X)$ are precisely the closed sets with respect to the *upper Alexandroff topology* $\mathfrak{U}(X)$. In other words, the down-sets are just the complements of the up-sets. Sending each poset to the corresponding quasidiscrete space with the upper Alexandroff topology and keeping fixed the underlying maps of the morphisms yields a concrete isomorphism between the category of posets with isotone (i.e. order preserving) maps and the category of quasidiscrete spaces with continuous maps. The inverse isomorphism is established by restriction of the *specialization functor*, which sends any topological space X to $X_\leq$, the underlying set equipped with the *specialization order*, defined by

$$x \leq y \;\Leftrightarrow\; x \in \overline{\{y\}} \;\Leftrightarrow\; \text{for all open } U,\; x \in U \;\text{ implies }\; y \in U\,.$$

This quasiorder is antisymmetric, hence a partial order, iff X is a T_0-space. Thus, the specialization functor forgets the topological structure and adds an order structure instead.

The (up- or) down-set lattices are very special complete lattices, namely those in which arbitrary joins are unions and arbitrary meets are intersections. Abstractly, they are the *superalgebraic* lattices, i.e. those which are join generated by their *completely join prime* or *supercompact* elements (the principal ideals). Being completely distributive and join generated by completely join irreducible elements, such lattices may be seen as the most obvious generalization of finite distributive lattices. Indeed, the Birkhoff duality between finite posets and finite distributive lattices [11] naturally extends to a duality between arbitrary posets with isotone (order preserving) functions on the one side and superalgebraic lattices with complete homomorphisms, preserving arbitrary joins and meets, on the other [8, 20, 36]. One duality functor sends each poset to its down-set lattice and any isotone function to its preimage map. A duality functor in the opposite direction is obtained by associating with any superalgebraic lattice the subposet of all supercompact elements and with any complete homomorphism between superalgebraic lattices the restriction of its lower adjoint to the supercompact elements. Thus, the superalgebraic lattices may be regarded as the pointfree counterparts of Alexandroff spaces, in the same vein as spatial frames correspond to sober spaces via the open set functor and the spectrum functor in the opposite direction [36, 43].

Completely distributive lattices and their relationships to down-set lattices have been studied quite extensively a long time ago, mainly by

Raney [54, 55]. Down-set lattices play a role in diverse areas of order theory and topology, for example, in the theory of standard completions for ordered sets [3, 17, 27], in connection with continuous and algebraic lattices [36], and also in the more general context of $\mathcal{Z}$-continuous and $\mathcal{Z}$-algebraic posets [10, 20, 24, 25]. For further categorical aspects of the down-set functor, see [8]. A further perspective to duality aspects of down-set lattices was opened by Banaschewski [4], who charcterized them as the bounded distributive topological lattices with a Boolean (compact and zero-dimensional) topology.

Renewed interest in down-set lattices comes from their role in obtaining relational semantics and developing the correspondence theory of algebraic logics generalizing modal logics [33]. In this setting the object of study are partially ordered algebras and more specially often distributive lattice-ordered algebras. The objective is to associate, in a natural way, with each of these abstract algebras, A, a relational structure, $S(A)$, so that A may be represented as a concrete algebra over the powerset of the set underlying $S(A)$ that is constructed by means of first order definable subcollections and operations. In doing this one has two routes. One is to obtain from the algebra A its topo-relational dual space, A_* (through, e.g., extended Priestley duality), and then to forget the topology thus leaving a relational structure $S(A)$ by which A may be represented concretely over its powerset. Another route from A to $S(A)$ is to first complete the algebra A to obtain an algebra A^σ that lies within an extended Birkhoff duality between down-set lattices with additional 'complete' operations and certain relational structures. The two assignment routes may be seen as:

$$\begin{array}{ccc} A & \mapsto & A^\sigma \\ \updownarrow & & \updownarrow \\ A_* & \mapsto & S(A) \end{array}$$

Here the lower horizontal assignment is given by the forgetful functor; the upper assignment is given by canonical extension [32, 44] which is a candidate for the algebraic or pointfree version of the left adjoint to the forgetful functor on the lower horizontal level. The extended Birkhoff duality that effects the correspondence between A^σ and $S(A)$ is, in the distributive setting, none other than the simple duality between down-set lattices and partially ordered sets mentioned above. By using the upper route then relational semantics, and even topological duality can be studied by a combination of canonical extension theory and discrete duality [16, 33, 37] and [34, 35], respectively. In this approach, in most available cases, dualities are correspondence results overlayed with obvious necessary conditions.

To illustrate, in the case of a Boolean algebra A, the structure $X = S(A)$ is the poset of all prime filters of A (ordered by inclusion), and A_* is that poset equipped with Stone's hull-kernel topology, while A^σ is the powerset $\mathfrak{P}(X)$. Here, the topological duality $A \leftrightarrow A_*$ is Stone's duality for Boolean algebras (see [59] for Stone's original approach and [43] for a modern treatment), while the discrete duality $X \leftrightarrow \mathfrak{P}(X)$ on the right is the one between sets and complete atomic Boolean algebras. By virtue of that duality, complete congruences of $\mathfrak{P}(X)$ correspond to subsets of X. This lifts to a duality result by 'overlaying' the property that a subset of X must be closed in the Boolean topology in order to restrict correctly to a quotient of the abstract Boolean algebra A sitting inside A^σ (as the Boolean subalgebra of clopen sets). As a well-known and useful consequence, the congruences of a Boolean algebra are in one-to-one correspondence with the closed subsets of the dual space.

For bounded distributive lattices, $X = S(A)$ is the prime filter spectrum of the (bounded) distributive lattice A (see [60] and [43]), and A_* is that poset equipped with the Priestley or patch topology [51], while A^σ is (up to isomorphism) the up-set lattice $\mathfrak{U}(X)$. In this case, we have the Priestley duality $A \leftrightarrow A_*$ on the left and the aforementioned Alexandroff duality $X \leftrightarrow \mathfrak{U}(X)$ between posets and superalgebraic lattices on the right. Again, an important aspect of the left hand duality is that congruences of the abstract algebra A are in one-to-one correspondence to the closed subsets of the dual space A_*. This result however is not the simple overlay of topology on the corresponding result for the discrete structures: the complete congruences of a down-set lattice are *not* in general in one-to-one correspondence with the subsets of the dual poset. The duality result is of course understandable in this setting, e.g., by lifting the congruences to the free Boolean extension and then applying the Boolean duality result. Nevertheless it is a natural question to ask which down-set lattices have the property that their complete congruences are in one-to-one correspondence to the subsets of the underlying poset, by associating with each subset A the *induced congruence* $\Theta_A = \{(U, V) \in \mathfrak{D}(X)^2 \mid U \cap A = V \cap A\}$. The main result of this paper is the answer to the above question: This is the case if and only if the underlying poset is *(order) scattered*, i.e., it does not contain a densely ordered chain.

Our approach will be through the extensive theory of pointfree topology. We consider down-set lattices as special frames and give various abstract characterizations of these, as well as of their complete quotients, the completely distributive or *supercontinuous* lattices. Here, the majority of the results and tools needed are already available, but for selfcontainment and a streamlined exposition, we include most of the necessary proofs, since the pertinent literature is rather dispersed and some arguments would become rather circuitous.

Our main interest lies in the understanding of complete quotients and congruences (compatible with arbitrary joins and meets) of superalgebraic frames. The problem treated here is parallel to a well-known problem in pointfree topology: In the adjunction between topological spaces and frames, subspaces of a space give rise to surjective frame homorphisms on the frame side. Thus, in pointfree topology, frame quotients (and thus frame congruences) are considered as 'pointfree' or generalized subspaces. The question is then which spaces have the property that all generalized subspaces (i.e. frame quotients of their topology) are '*bona fide* subspaces' (i.e. isomorphic to the topology of a subspace). This problem was solved by Simmons [57] (see also Niefield and Rosenthal [50, 56]): the spaces in question are exactly the *corrupt* or *weakly scattered* ones (in which all nonempty closed sets have weakly isolated points; see Section 3). We see how our result may be regarded as the corresponding fact for Alexandroff T_0 spaces: such a space (resp. the associated poset) is *order-scattered* if and only if its generalized subspaces are '*bona fide* subspaces'. This sounds quite similar to the above statement about frame congruences, but *topological scatteredness*, though loosely related to order-scatteredness [49], is something different, and the proof requires other techniques.

We take the occasion to explore the import of complete congruences for frames in general, give a characterization of the complete congruences among the frame congruences for an arbitrary open set frame, and show that it is precisely the T_1-topologies for which only the *open* subspaces induce complete congruences (while for Alexandroff spaces, *all* subsets do). Moreover, it turns out that *topological scatteredness* of an Alexandroff space is a much stronger property than *order scatteredness*: the posets whose upper Alexandroff spaces are scattered are the Noetherian ones (in which every directed subset has a greatest element).

After this introductory section we shall give, in the second section, the promised characterizations of down-set lattices and their complete quotients in the setting of frames. Crucial will be the observation that the points of a poset are *separated* (by the complements of a principal ideal and a principal filter) iff the MacNeille completion is supercontinuous (cf. [9, 21]), while the points are *principally separated* (by a principal ideal and a complementary principal filter) iff the MacNeille completion is superalgebraic, hence isomorphic to a down-set lattice (see [21]). Using this, we prove, in frame-theoretic guise, the quotient version of our main result, namely, that all the complete quotients of a superalgebraic lattice are superalgebraic if and only if its subposet of supercompact elements is scattered. In the third section, we revisit the duality between frames and spaces and identify all the pertinent types of frame congruences including general, spatial, induced, open and complete congruences. We explore their mutual relationships and arrive, in the last section, at our main theorem.

As we hope that this paper also will be of interest outside the point-free topology community, we introduce all the specialized tools and definitions and include proofs whenever this does not lead us too far afield. For readers interested in a choice-free development, we point out that many of our deductions work in set theory without the Axiom of Choice. In a few cases, we use the weaker *Principle of Dependent Choices* is used are marked by an asterisk. This rather 'obvious' (but in ZF unprovable) principle says that, given a relation R on a nonempty set X such that for each $x \in X$ there is a y with $x \, R \, y$, there exists a sequence (x_n) with $x_n R \, x_{n+1}$ for all n.

Convention. Statements where the Principle of Dependent Choices is used are marked by an asterisk.

2. Superalgebraic and supercontinuous lattices

Our first goal is to provide the tools that will allow us to understand up- or down-set lattices and their complete quotients in a 'pointfree framework'. Although, by the usual definition of the specialization order, the up-set lattices are regarded as the lattices of open sets in the associated Alexandroff topologies (*A-topologies* for short), one may take the down-set lattices as well, because they are just the open set lattices with respect to the dual order. In the present context, we prefer to work with down-set lattices, because the underlying poset X sits, right side up, in its down-set lattice $\mathfrak{D}(X)$.

A *frame* (resp. *co-frame*) is a complete lattice L satisfying the 'infinite' distributive law

$$a \wedge \bigvee B = \bigvee \{a \wedge b \mid b \in B\} \quad \text{(resp.} \quad a \vee \bigwedge B = \bigwedge \{a \vee b \mid b \in B\} \text{)}$$

for all $a \in L$ and $B \subseteq L$. A *frame homomorphism* preserves all joins and all finite meets, while a *complete homomorphism* preserves all joins and all meets. 'Classical' frames are the lattices $\mathfrak{O}(X)$ of open sets of topological spaces X. Their isomorphic copies are called *spatial frames*. Abstractly, they are characterized by the condition that every element is a meet of ($\wedge$-)primes. If $f : X \to Y$ is a continuous mapping between topological spaces then

$$\mathfrak{O}(f) : \mathfrak{O}(Y) \to \mathfrak{O}(X),\ U \mapsto f^{-1}[U]$$

is a frame homomorphism, and in case Y is an Alexandroff space, $\mathfrak{O}(f)$ is even a complete homomorphism. For more details about frames see, e.g., [43] or [52].

In order to provide the desired abstract characterizations of A-topologies, let us recall the relevant definitions. An element c of a poset X is *compact* if the set $X \setminus {\uparrow} c$ is closed under directed joins, and $\vee$-*prime* if $X \setminus {\uparrow} c$ is closed under finite joins; if $X \setminus {\uparrow} c$ is even a principal ideal ${\downarrow} d$ then c is *supercompact* in X (and d is supercompact in the dual of X). We denote by $\mathsf{sc}X$ the subposet of all supercompact elements. In a complete lattice, these are precisely the $\bigvee$-*primes*, satisfying $c \in {\downarrow} A$ whenever $c \leq \bigvee A$ (while compact elements are characterized by the condition that $c \leq \bigvee A$ implies $x \leq \bigvee F$ for some finite $F \subseteq A$). A complete lattice L is *(super-)algebraic* if each of its elements is a join of (super-)compact elements (see, e.g., Banaschewski and Niefield [7]). A poset X is *separated* (or a *Raney poset* [47]) if for $a \not\leq b$ in X there are $c, d \in X$ such that $a \not\leq d$, $c \not\leq b$, and ${\uparrow} c \cup {\downarrow} d = X$; and X is *principally separated* if, in addition, c and d may be chosen so that $c \not\leq d$ (whence $X \setminus {\uparrow} c = {\downarrow} d$; see [21, 22]). Finally, a poset is *weakly atomic* [14] if each non-trivial interval contains a covering pair $u \prec v$ (so that $u < v$ but no x satisfies $u < x < v$). Parts of our first characterization theorem have been known for a long time; concerning (a)$\Leftrightarrow$(e), see Büchi [12] or Raney [54]; the equivalence (d)$\Leftrightarrow$(e) (d)$\Leftrightarrow$(e) was first observed by Bruns [13]. See also [15] and [22].

Theorem 2.1. *For a complete lattice L, the following conditions are equivalent:*

(a) *L is superalgebraic.*

(b) *L is algebraic, and its dual is a spatial frame.*

(c) *L is principally separated.*

(d) *L is a weakly atomic frame and a co-frame.*

(e) *L is isomorphic to a down-set frame (A-topology).*

(f) *The join map $\bigvee : \mathfrak{O}(\mathsf{sc}L) \to L$ is an isomorphism.*

Proof. (a) $\Rightarrow$ (b). Supercompact elements are $\vee$-prime and compact.

(b) $\Rightarrow$ (a). It suffices to show that any compact element $c \in L$ is a join of supercompact elements. By spatiality of the dual lattice, c is the join of a minimal finite set F of $\vee$-primes. For any such $p \in F$ and any set A with $p \leq \bigvee A$, we find a finite $E \subseteq A$ such that $c \leq \bigvee(E \cup (F \setminus \{p\}))$, and as $p \not\leq b$ for $b \in F \setminus \{p\}$, there is an $a \in E \subseteq A$ with $p \leq a$. Hence, each $p \in F$ is supercompact.

(a) $\Rightarrow$ (f) $\Rightarrow$ (e) is straightforward.

(e) $\Rightarrow$ (d). Clearly, any down-set system $\mathfrak{D}(X)$ is a frame and a co-frame; moreover, it is weakly atomic, since for $A \subset B$ in $\mathfrak{D}(X)$ and for $x \in B \setminus A$, the down-set $V = A \cup {\downarrow} x$ covers the down-set $U = V \setminus \{x\}$ in the interval $[A, B]$.

(d) $\Rightarrow$ (c). If $a \not\leq b$, i.e. $a \wedge b < a$, choose u, v with $a \wedge b \leq u \prec v \leq a$. By the co-frame property, the minimum $c = \min\{x \,|\, v \leq x \vee u\}$ exists, is $\bigvee$-irreducible (indeed, $x < c$ implies $x \leq u$, while $c \not\leq u$) and consequently supercompact, because L is a frame. For $d = \max(L \setminus {\uparrow} c)$ we get $a \not\leq d$ (since $c \leq a$), $c \not\leq b$ (otherwise $v \leq a \wedge (b \vee u) \leq u$), and ${\uparrow} c \cup {\downarrow} d = L$.

(c) $\Rightarrow$ (a). Given $a \not\leq b$ in L, choose c, d with $a \not\leq d$, $c \not\leq b$ and ${\uparrow} c \cup {\downarrow} d = L$. Then c is supercompact with $c \leq a$ but $c \not\leq b$. Thus, each element is a join of supercompact ones. $\square$

Of course, the conditions (c) – (e) are self-dual. Thus, in contrast to algebraic lattices, the duals of superalgebraic lattices are again superalgebraic.

The Axiom of Choice allows to omit the word 'spatial' in (b), because then (d) is easily derived for any algebraic frame, with the help of Zorn's Lemma (cf. [14]).

Now, the well-known duality between posets and superalgebraic lattices (see [20] or [36] for details, and [18] for a more general framework) may be stated as follows (for corresponding notions, also applicable in the non-distributive setting, see [18, 20, 24] and [30]):

Proposition 2.2. *Assigning to each isotone map $f : X \to Y$ the map*

$$\mathfrak{D}(f) : \mathfrak{D}(Y) \to \mathfrak{D}(X), \ B \mapsto f^{-1}[B],$$

one obtains a duality between the category of posets with isotone maps (resp. the isomorphic category of quasidiscrete spaces) and the category of superalgebraic lattices with complete homomorphisms (preserving arbitrary joins and meets). Moreover, f is an order embedding iff $\mathfrak{D}(f)$ is onto.

Concerning the last fact, note that if f is an embedding then $U = \mathfrak{D}(f)(\downarrow f[U])$ for each $U \in \mathfrak{D}(X)$; conversely, if $\mathfrak{D}(f)$ is onto then each principal ideal $\downarrow y$ of X is the preimage of some $U \in \mathfrak{D}(Y)$, whence $f(x) \le f(y) \in U$ implies $x \in f^{-1}[U] = \downarrow y$, i.e. $x \le y$.

Next, to the characterization of *complete homomorphic* (rather than *isomorphic*) images of A-topologies. Recall that a complete lattice is *continuous* (in the sense of Scott [36]) if each $a \in L$ is the join of its *way-below elements* $c \ll a$, belonging to all directed down-sets whose join is above a. Replacing directed with arbitrary down-sets in that definition, one obtains the *supercontinuous* lattices. Thus, a complete lattice L is supercontinuous iff each element $a \in L$ is the join of all $c \lhd a$, the latter meaning that c belongs to every down-set whose join is above a. We call $\lhd$ the *superway-below relation* (elsewhere referred to as the *well-below relation* or *long-way-below relation*, and also denoted by $\lll$). By definition, c is supercompact (resp. compact) iff $c \lhd c$ (resp. $c \ll c$). The major part of the resulting theory of supercontinuous and superalgebraic lattices is entirely parallel to that of continuous and algebraic lattices (see Gierz *et al.* [36]), being based on the non-conditional versions of the way-below relation and of compact elements. In fact, both mainstreams are special instances of the general theory of $\mathcal{Z}$*-algebraic* and $\mathcal{Z}$*-continuous* posets and lattices (see [10, 24, 25]). For example, the superway-below relation of supercontinuous lattices (cf. [55]) has the same interpolation property as the way-below relation of continuous lattices (see [36]):

Proposition 2.3. *The superway-below relation $\lhd$ of a supercontinuous lattice is idempotent, i.e. transitive and interpolative:*

$$a \lhd b \;\Leftrightarrow\; a \lhd c \lhd b \;\; \textit{for some } \; c.$$

A lattice L is superalgebraic iff it is supercontinuous and for $a \lhd b$ in L, there is a supercompact c with $a \le c \le b$ (hence $a \lhd c \lhd c \lhd b$).

Corollary 2.4. *Every down-set frame is supercontinuous. Its supercompact elements are just the principal ideals, and one has for any two down-sets A, B:*

$$A \lhd B \;\Leftrightarrow\; A \subseteq \downarrow x \subseteq B \;\; \textit{for some } \; x \in X.$$

Next, a remark about morphisms between supercontinuous lattices (see [10, 24] and [36]).

Proposition 2.5. *A map between supercontinuous lattices is a complete homomorphism iff it has a left (=lower) adjoint preserving the superway-below relation.*

From Raney's pioneering work [55] we know that the supercontinuous lattices are just the *completely distributive* ones. These may be characterized, without using choice functions, by the identity

$$\bigwedge\{\bigvee A : A \in \mathcal{X}\} = \bigvee\{\bigwedge B : B \in \mathcal{X}^{\#}\},$$

where $\mathcal{X}$ is any system of subsets and $\mathcal{X}^{\#}$ consists of all subsets of the union $\bigcup \mathcal{X}$ that intersect each member of $\mathcal{X}$. We add here some alternative characterizations of supercontinuity, essentially due to Raney [54, 55].

Theorem 2.6. *For a complete lattice L, the following conditions are equivalent:*

(a) *L is supercontinuous.*
(b) *L is completely distributive.*
(c) *L is separated.*
(d) *L satisfies $\bigwedge\{\bigvee A : A \in \mathcal{Y}\} = \bigvee \bigcap \mathcal{Y}$ for all $\mathcal{Y} \subseteq \mathfrak{D}(L)$.*
(e) *L is the complete homomorphic image of a down-set frame (A-topology).*
(f) *The join map $\bigvee : \mathfrak{D}(L) \to L$ preserves arbitrary meets (and joins).*

Proof. (a) ⇒ (c). For $a \not\leq b$, there is a $c \not\leq b$ that belongs to every down-set having a join above a. Since the down-set $A = L \setminus {\uparrow} c$ does not contain c, we must have $a \not\leq d := \bigvee A$. Furthermore, if $x \notin {\downarrow} c$ then $x \in A$ and so $x \in {\uparrow} d$.

(b) is equivalent to (d) by the identity $\bigcap\{{\downarrow} A : A \in \mathcal{X}\} = {\downarrow}\{\bigwedge B : B \in \mathcal{X}^{\#}\}$.

(c) ⇒ (d). Let $\mathcal{Y}$ be any collection of down-sets. For $a := \bigwedge\{\bigvee A : A \in \mathcal{Y}\}$ and $b := \bigvee \bigcap \mathcal{Y}$, it is clear that $b \leq a$. Assuming $a \not\leq b$, we find $c, d \in L$ with $a \not\leq d$, $c \not\leq b$, and $L = {\uparrow} c \cup {\downarrow} d$. Then $\bigvee A \not\leq d$ for all $A \in \mathcal{Y}$, and so $A \not\subseteq {\downarrow} d$, whence $A \cap {\uparrow} c \neq \emptyset$. As each $A \in \mathcal{Y}$ is a down-set, it follows that $c \in \bigcap \mathcal{Y}$, in contrast to $c \not\leq \bigvee \bigcap \mathcal{Y}$. Thus, $\bigvee(\bigcap \mathcal{Y}) = \bigwedge \bigvee[\mathcal{Y}]$.

(d) ⇒ (f) ⇒ (e). Being left adjoint to the principal ideal map $a \mapsto {\downarrow} a$ from L into $\mathfrak{D}(L)$, the join map preserves joins, and by hypothesis (d), it also preserves meets.

(e) ⇒ (a) follows from the fact that superalgebraic lattices are supercontinuous and that supercontinuity, respectively, complete distributivity is preserved by complete homomorphisms (see [10] for more general results on *$\mathcal{Z}$-continuous posets*). □

Again, (c) is obviously a self-dual statement, so supercontinuity is a self-dual property, too. Furthermore, combining Theorem 2.1 with 2.6, we conclude:

Corollary 2.7. *The supercontinuous lattices are precisely the complete homomorphic images of superalgebraic lattices.*

By the isomorphism $\mathfrak{P}(X) \simeq \{0,1\}^X$, it is clear from Theorem 2.1 that

a complete lattice is superalgebraic iff it is completely embeddable in a discrete cube $\{0,1\}^X$.

In full analogy to that embedding theorem, one can prove that

* *a complete lattice is supercontinuous iff it is completely embeddable in a real cube* $[0,1]^X$.

However, the known proofs (see, e.g. [19, 36]) require Dependent Choices, similarly as in the case of Urysohn's Lemma for normal spaces.

The coincidence between superalgebraicity and principal separation extends to the non-complete case as follows (see [9] and [21]). The *cut operator* of a poset X associates with any subset A the *cut generated by* A, the intersection of all principal ideals containing A. Its range is $\mathfrak{N}(X)$, the *completion by cuts*, *Dedekind-MacNeille completion* or *normal completion* [11, 46]. See [5] for categorical aspects and [21] for a different characterization of that completion as a reflector. The latter reference also contains most of the following facts, establishing the "completion-invariance" of (principal) separation:

Theorem 2.8. (1) *A poset X is principally separated iff the completion $\mathfrak{N}(X)$ is superalgebraic iff the cut operator induces an isomorphism between $\mathfrak{D}(\mathsf{sc}X)$ and $\mathfrak{N}(X)$.*

(2) *A poset X is separated iff the completion $\mathfrak{N}(X)$ is supercontinuous iff the cut operator induces a complete homomorphism from $\mathfrak{D}(X)$ onto $\mathfrak{N}(X)$.*

Proof. (1) If X is principally separated and $P = \mathsf{sc}X$ its subposet of supercompact elements, the restricted cut operator $\Gamma : \mathfrak{D}(P) \to \mathfrak{N}(X)$, $A \mapsto \bigcap\{\downarrow y \,|\, A \subseteq \downarrow y\}$ has the left adjoint

$$\mathrm{L} : \mathfrak{N}(X) \to \mathfrak{D}(P),\ C \mapsto P \cap C.$$

Indeed, if $C \subseteq \Gamma(A)$ then for $c \in P \cap C$ there is a $d \in X$ with $X \backslash {\uparrow} c = {\downarrow} d$, so $c \notin A$ would imply $A \subseteq {\downarrow} d$ and $c \in C \subseteq \Gamma(A) \subseteq {\downarrow} d$, a contradiction; thus, $\mathrm{L}(C) \subseteq A$, which in turn entails $C \subseteq \Gamma(A)$, since $a \in C$ implies $a = \bigvee\{c \in P \,|\, c \leq a\} \subseteq \Gamma(A)$. Consequently, Γ preserves meets. Furthermore, Γ is onto, since ${\downarrow} x = \Gamma(P \cap {\downarrow} x)$ by join-density of P, and each cut is an intersection of principal ideals. But Γ is also an embedding: for $A, B \in \mathfrak{D}(P)$ with $A \not\subseteq B$ and $c \in A \setminus B$, we get $\uparrow c \cap B = \emptyset$, hence $B \subseteq \downarrow d$ and $c \notin \downarrow d$ for $d = \max(X \setminus \uparrow c)$; thus, $A \not\subseteq \Gamma(B)$. We see that $\Gamma : \mathfrak{D}(P) \to \mathfrak{N}(X)$ is an isomorphism, whence $\mathfrak{N}(X)$ is superalgebraic, by Theorem 2.1.

On the other hand, if we assume that $\mathfrak{N}(X)$ is superalgebraic, then for $a \not\leq b$ in X, i.e. $\downarrow a \not\subseteq \downarrow b$ in $\mathfrak{N}(X)$, there is a supercompact cut $C \subseteq \downarrow a$ with $C \not\subseteq \downarrow b$. The equation $C = \bigvee_{\mathfrak{N}(X)} \{\downarrow c : c \in C\}$ forces C to coincide with one $\downarrow c$, whence $c \leq a$ but $c \not\leq b$; now, there is a cut D with $x \notin D \Leftrightarrow \downarrow x \not\subseteq D \Leftrightarrow C \subseteq \downarrow x$, i.e. $D = X \setminus \uparrow c$, and as D is a $\bigwedge$-prime cut, it must be a principal ideal $\downarrow d$. Thus, X is principally separated.

(2) If X is separated then the restricted cut operator $\Gamma : \mathfrak{D}(X) \to \mathfrak{N}(X)$ is onto and preserves joins, being left adjoint to the inclusion map $\mathfrak{N}(X) \hookrightarrow \mathfrak{D}(X)$. But Γ also preserves meets, as can be shown by a similar argument as above. Hence, $\Gamma : \mathfrak{D}(X) \to \mathfrak{N}(X)$ is a complete homomorphism, and in particular, $\mathfrak{N}(X)$ is supercontinuous by Theorem 2.6.

On the other hand, if we assume that $\mathfrak{N}(X)$ is supercontinuous and that $a \not\leq b$ in X, then we have $\downarrow a \not\subseteq \downarrow b$ in $\mathfrak{N}(X)$, so there is a cut $C \lhd \downarrow a$ with $C \not\subseteq \downarrow b$. Pick $c \in C$ with $c \not\leq b$. Then $\downarrow c \lhd \downarrow a$, hence $\downarrow a \not\subseteq \bigvee_{\mathfrak{N}(X)} \{\downarrow x : c \not\leq x\} = \Gamma(X \setminus \uparrow c)$, and we find a $d \in X$ with $a \not\leq d$ and $X \setminus \uparrow c \subseteq \downarrow d$, i.e. $\uparrow c \cup \downarrow d = X$. Thus, X is separated. □

Recall that a poset is said to be *(order) scattered* if the chain $\mathbb{Q}$ of rationals is not embeddable in it. At least for chains, that notion has a long history dating back to the pioneering work of Hausdorff [38]. Recall also that a nontrivial poset is *dense* if it has no covering pairs. An easy application of Dependent Choices yields:

Proposition 2.9.* *The following conditions are equivalent for a poset X:*

(a) *X is order scattered.*
(b) *X has no dense subchain.*
(c) *Every subposet of X is weakly atomic.*

For deeper results about scattered ordered sets, we refer to [28].

Example 2.10. Like every chain, the real unit interval $\mathbb{I} = [0, 1]$ (with the usual order) is separated, hence supercontinuous (being complete). Here, the superway-below relation $\lhd$ (which coincides with the way-below relation on the half-open interval $]0, 1]$) is just the strict order

$<$. As a strict order is irreflexive, it is clear that there are no supercompact elements in $\mathbb{I}$; that is, $\mathbb{I}$ is extremely non-superalgebraic. But, by Theorem 2.6, it is a quotient of its (superalgebraic!) down-set frame under the complete homomorphism $\bigvee : \mathfrak{D}(\mathbb{I}) \to \mathbb{I}$. Up to isomorphism, $\mathfrak{D}(\mathbb{I})$ is obtained by doubling each element of $\mathbb{I}$. Though being superalgebraic, hence weakly atomic, the complete chain $\mathfrak{D}(\mathbb{I})$ is clearly not scattered.

The previous example is generic among examples of supercontinuous lattices which are not superalgebraic, in that every supercontinuous but not superalgebraic lattice cannot be weakly atomic and must therefore contain a copy of the real unit interval. We are now ready for the first version of our main result, characterizing scattered posets by a common property of their subposets, and by another one of the complete quotients of their down-set lattices.

Theorem 2.11.* *The following conditions are equivalent for a poset X:*

(a) *X is order scattered.*

(b) *Every separated subposet of X is principally separated.*

(c) *Every complete quotient of $\mathfrak{D}(X)$ is superalgebraic (weakly atomic).*

Proof. (a)$\Rightarrow$(c). Let h be a complete homomorphism from $\mathfrak{D}(X)$ onto a complete lattice L. By Proposition 2.6, L is supercontinuous (thus, by Theorem 2.1, it is superalgebraic iff weakly atomic). Assume L is not superalgebraic. Then there exist $a \lhd b$ in L such that the interval $[a, b]$ contains no supercompact elements (Proposition 2.3). Therefore, $\lhd$ is not only iinterpolative but also irreflexive on $[a, b]$, whence $a \leq u \lhd v \leq b$ entails $u < v$. By Proposition 2.5, the left adjoint g of h is an embedding that preserves joins and the superway-below relation. We define now a relation $\sqsubset$ on X by

$$x \sqsubset y \;\Leftrightarrow\; {\downarrow} x \subseteq g(u) \subset g(v) \subseteq {\downarrow} y \text{ for some } u, v \in [a, b] \text{ with } u \lhd v.$$

This relation $\sqsubset$ is not empty, since by the interpolation property of $\lhd$, we find a_1, b_1 with $a \lhd a_1 \lhd b_1 \lhd b$, so that Proposition 2.3 and the embedding property of g yield x_1, y_1 with

$$g(a) \subseteq {\downarrow} x_1 \subseteq g(a_1) \subset g(b_1) \subseteq {\downarrow} y_1 \subseteq g(b), \text{ hence } x_1 \sqsubset y_1.$$

A similar argument shows that the relation $\sqsubset$ interpolates, i.e. $x \sqsubset y$ implies $x \sqsubset z \sqsubset y$ for some z: given u, v with ${\downarrow} x \subseteq g(u) \subset g(v) \subseteq {\downarrow} y$, choose elements u_1, v_1 with $u \lhd u_1 \lhd v_1 \lhd v$ and a z satisfying the inclusions

$$ {\downarrow} x \subseteq g(u) \subset g(u_1) \subseteq {\downarrow} z \subseteq g(v_1) \subset g(v) \subseteq {\downarrow} y, \text{ hence } x \sqsubset z \sqsubset y.$$

Now, since $x \sqsubset y$ implies $\downarrow x \subset \downarrow y$, i.e. $x < y$, the usual recursive interpolation procedure (using Dependent Choices again) yields a dense subchain of X. Thus, X is not order scattered, in contrast to (a).

(c) $\Rightarrow$ (b). Let Y be a separated subposet of X. The relativization map

$$\mathrm{R} : \mathfrak{D}(X) \to \mathfrak{D}(Y),\ A \mapsto A \cap Y$$

is a surjective complete homomorphism. Furthermore, since Y is separated, the restricted cut operator

$$\Gamma : \mathfrak{D}(Y) \to \mathfrak{N}(Y),\ A \mapsto \bigcap \{\downarrow y : A \subseteq \downarrow y\}$$

is a surjective complete homomorphism, too (see Theorem 2.8), and so is the composite map $\Gamma \circ \mathrm{R} : \mathfrak{D}(X) \to \mathfrak{N}(Y)$. Hence, $\mathfrak{N}(Y)$ is superalgebraic, and again by Theorem 2.8, it follows that Y is principally separated.

(b) $\Rightarrow$ (a). If X is not order scattered then it contains a subposet isomorphic to $\mathbb{Q}$, which is separated (like every chain) but not principally separated (being dense). □

Our final remark about supercontinuity is:

Proposition 2.12. * *Every supercontinuous frame is spatial.*

Proof. Given $a \not\leq b$ in a supercontinuous frame, use the approximation and interpolation property of the superway-below relation in order to find, via Dependent Choices, a sequence of elements (c_n) such that $c_{n+1} \lhd c_n \lhd a$ but not $c_n \leq b$. Then it is easy to see that the element $p = \bigvee \{x \mid c_n \not\leq x \text{ for all } n\}$ is $\wedge$-irreducible, hence prime, and above b but not above a. □

By similar arguments, one proves that *even every continuous frame is spatial* (see [36]), the basic ingredient for the Hofmann-Lawson duality of continuous frames. However, the spatiality of continuous frames requires additional choice principles like the Prime Ideal Theorem, though not the full strength of the Axiom of Choice (cf. [43]).

3. Congruences on topologies and frames

Every frame (indeed, every complete lattice) L gives rise to a specific topological space, its *spectrum* $X = \mathfrak{S}(L)$, whose points are the ($\wedge$-)prime elements of L and whose *closed* sets are the 'shadows' $\uparrow a \cap X$ $(a \in L)$ (cf. [36]). Alternately, points may be obtained as the characters, i.e. frame homomorphisms into the two-element chain 2, with each $a \in L$ yielding an open set $\{x : x(a) = 1\}$ (see, e.g, [43, 52]). In this setting, frame homomorphisms $h : L \to M$ give rise to continuous maps $\mathfrak{S}(h) : \mathfrak{S}(M) \to \mathfrak{S}(L)$, defined by $x \mapsto x \circ h$. The spectrum functor $\mathfrak{S}$ together with the open set functor $\mathfrak{O}$ in the opposite direction yields the categorical (dual) adjunction at the heart of point free topology, relating frames with topological spaces. The dual adjunction becomes a duality when restricted to sober spaces and spatial frames, respectively.

A space X is *weakly sober* if each $\wedge$-irreducible closed set is the closure of a point, or, equivalently, if each frame homomorphism from $\mathfrak{O}(X)$ into 2 is of the form λ_x, where x is a point and $\lambda_x(U) = 1 \Leftrightarrow x \in U$. One can show easily (see [53]) that if X is weakly sober then for every frame homomorphism $h : \mathfrak{O}(X) \to \mathfrak{O}(Y)$ there is a continuous $f : Y \to X$ such that $h = \mathfrak{O}(f)$, and that in *sober* spaces (that is, in weakly sober T_0-spaces), the map f stipulated above is unique. For more about sobriety see, e.g., [40] and [43].

Note that a frame L is spatial if and only if distinct elements of L give rise to distinct open sets of the space $\mathfrak{S}(L)$. Although all countable frames are spatial (being necessarily weakly atomic and therefore meet generated by their completely meet-irreducible, hence prime elements), there exist non-spatial frames in abundance (prominent examples are all atomless complete Boolean algebras). We know that all down-set frames are spatial, and that they are captured within the class of all frames as the superalgebraic ones, so that there is a way of recapturing the points of a quasidiscrete space faithfully. This is through the supercompact elements, or in a format more related to the description of the spectrum by characters, as the *complete* (rather than frame) homomorphisms into 2. However, the duality functor from superalgebraic lattices to quasidiscrete spaces is *not* induced by the spectrum functor from spatial frames to sober spaces: $\vee$-prime elements need not be supercompact, and quasidiscrete spaces need not be sober.

As mentioned in the introduction, our interest here is in understanding complete congruences (compatible with arbitrary joins and meets) of superalgebraic frames. But, in order to set them in context, we consider several related types of congruences on general frames and give some partial insight into how they relate. In the category of frames, the *frame congruences*, i.e., those corresponding to images under frame homomorphisms, are of course the most central ones. It is not hard to see that, via the spectrum functor, surjective frame homomorphisms yield embeddings. Thus it is natural that surjective frame homomorphisms and thus frame congruences are identified as *generalized subspaces* in pointfree topology. On the other hand, if X is a topological space and A is a subspace of X then the inclusion map $e : A \hookrightarrow X$ induces a surjective frame homomorphism

$$\mathfrak{O}(e) : \mathfrak{O}(X) \to \mathfrak{O}(A),\ U \mapsto U \cap A$$

whose kernel is the *induced congruence* Δ_A given by

$$U \Delta_A V \;\Leftrightarrow\; U \cap A = V \cap A.$$

A first obvious question concerning such induced congruences is: which spaces have the property that the inducing subspaces are unique? The answer has been given in [53]:

Proposition 3.1. *A space X is T_D if and only if each frame congruence on $\mathfrak{O}(X)$ is induced by at most one subspace of X.*

Here T_D is the separation axiom between T_0 and T_1 requiring that each point x has an open neighborhood U such that $U \setminus \{x\}$ is open [2, 61] (for related material on T_D-spaces, see [13, 18, 29, 39, 53]).

Proof. Let X be T_D and suppose $A, B \subseteq X$ but $A \nsubseteq B$. Choose $x \in A \setminus B$ and an open U such that $x \in U$ and $V = U \setminus \{x\}$ is open. Then $U \cap A \neq V \cap A$ while $U \cap B = V \cap B$; hence $\Delta_A \neq \Delta_B$.

On the other hand, if for each $x \in X$ the congruence Δ_X is distinct from $\Delta_{X\setminus\{x\}}$, there exist open sets $U \neq V$ in $\mathfrak{O}(X)$ such that $U \setminus \{x\} = V \setminus \{x\}$. That is possible only if one of them, say U, contains x, and $V = U \setminus \{x\}$. □

The next question is: which congruences have a chance to be induced? A congruence on a frame L is called *spatial* provided the corresponding quotient frame is spatial (that is, isomorphic to a topology). On open set frames $\mathfrak{O}(X)$, certainly each induced congruence Δ_A is spatial (as the quotient will be, up to isomorphism, $\mathfrak{O}(A)$), but not all spatial congruences have to be induced. The precise criterion is:

Proposition 3.2. *A space X is weakly sober if and only if every spatial congruence on $\mathfrak{O}(X)$ is induced. Thus, in weakly sober spaces the spatial congruences are exactly the induced ones.*

Proof. Suppose X is weakly sober and Θ is a spatial congruence on $\mathfrak{O}(X)$, i.e. there is an isomorphism $i : \mathfrak{O}(X)/_\Theta \to \mathfrak{O}(Y)$. Then $g = i \circ p_\Theta : \mathfrak{O}(X) \to \mathfrak{O}(Y)$ (with $p_\Theta : \mathfrak{O}(X) \to \mathfrak{O}(X)/_\Theta$ the canonical epimorphism) is a surjective frame homomorphism, and by weak sobriety, there is a continuous $f : Y \to X$ with $g = \mathfrak{O}(f)$. Now, Θ is the kernel of p_Θ, hence of g, and consequently, the image $A = f[Y]$ is a subspace of X with $\Theta = \Delta_A$; indeed,

$$U \Theta V \Leftrightarrow g(U) = g(V) \Leftrightarrow f^{-1}[U] = f^{-1}[V]$$
$$\Leftrightarrow f[f^{-1}[U]] = f[f^{-1}[V]) \Leftrightarrow A \cap U = A \cap V.$$

Conversely, assume each spatial congruence on $\mathfrak{O}(X)$ is of the form Δ_A, and let W be a $\wedge$-irreducible member of $\mathfrak{O}(X)$. Then

$$\Theta = \{(U, V) \in \mathfrak{O}(X) \times \mathfrak{O}(X) : U \subseteq W \Leftrightarrow V \subseteq W\}$$

is a spatial congruence, since the quotient is a two-element frame (consisting of the two congruence classes $\mathfrak{O}(W)$ and $\mathfrak{O}(X) \setminus \mathfrak{O}(W)$). Thus, our hypothesis yields an $A \subseteq X$ with $\Theta = \Delta_A$. In particular, for $V \in \mathfrak{O}(X)$ with $V \not\subseteq W$, we have $X \Theta V$ (since $X \not\subseteq W$) and so $A = A \cap X = A \cap V$. In other words, $V \not\subseteq W$ implies $A \subseteq V$. We apply this to the open sets $X \setminus \overline{x}$ (which contain W if and only if x is not in W). Assuming $W \neq X \setminus \overline{x}$ for all x, we obtain $A \subseteq \bigcap\{X \setminus \overline{x} : x \notin W\} = W$, hence $A \cap X = A \cap W$ and so $X \Theta W$, i.e. $X = W$, impossible as W is prime. By way of contradiction, we see that W is the complement of a point closure, and X is weakly sober. □

Corollary 3.3. *A space is sober if and only if each spatial congruence on $\mathfrak{O}(X)$ is induced by a unique subspace.*

Thus, for a weakly sober space X all quotients of the open set frame that *do* correspond to spaces are induced by actual subspaces of X, which are unique in the sober case. The much stronger property of X that *all* generalized subspaces (that is, all quotients of its open set frame) are induced by (unique) actual subspaces of X is much rarer. Identifying the spaces with that property is of course a natural task, and it has been settled by Simmons [57] (in the language of nuclei rather than congruences; see Theorem 3.4 below). The crucial property is *(topological) scatteredness*, requiring an isolated point in each nonempty (closed) subspace. That topological property should not be confused with the order-scatteredness introduced earlier, but there exist interesting relations between the two notions (see, e.g., Mislove [48, 49]).

Niefield and Rosenthal [50, 56] solved the analogous pointfree problem and gave several characterizations of those frames L for which all quotients are spatial. One of these conditions says that every element $a \in L \setminus \{1\}$ has an *essential prime*, that is, a prime element p such that $a = b \wedge p$ for some $b > a$. We omit the easy proof of that fact and refer to [56], Ch. 4.4, for details. In an open set frame $\mathfrak{O}(X)$, each complement $X \setminus \overline{\{x\}}$ of a point closure is prime, and it is an essential prime for an open set V, i.e. $V = U \cap (X \setminus \overline{\{x\}}) \subset U$ for some open U, iff x is a *weakly isolated point* of the proper closed subspace $A = X \backslash V$, meaning that $x \in A \cap U \subseteq \overline{\{x\}}$ for some $U \in \mathfrak{O}(X)$. (Niefield and Rosenthal used slightly different definitions of essential primes and of weakly isolated points, which however amount to the same results.) A space is said to be *weakly scattered* [50] if each nonempty closed subspace has a weakly isolated point (Simmons called such spaces *corrupt*). Every weakly scattered space is weakly sober, since for irreducible closed A, the condition $x \in A \cap U \subseteq \overline{\{x\}}$ entails $\overline{\{x\}} \subseteq A \subseteq \overline{\{x\}} \cup (X \setminus U)$, and then $A \not\subseteq X \backslash U$ forces $A = \overline{\{x\}}$. On the other hand, as already observed by Simmons [57], *a space is scattered iff it is weakly scattered and* T_D. Now, combining the previous remarks with Propositions 3.1 and 3.2, we arrive at Simmons' Theorem, translated from nuclei to congruences (which are just the kernels of nuclei):

Theorem 3.4. *A space X is weakly scattered (resp. scattered) if and only if each frame congruence on $\mathfrak{O}(X)$ is induced by a subspace (resp. a unique subspace).*

4. Complete congruences on frames and superalgebraic lattices

In the duality between superalgebraic lattices and posets (further exploited in correspondence theory [33] and extended Priestley duality [34]) complete congruences and induced congruences stand in the same relationship as do frame congruences and induced congruences in pointfree topology. Before getting into that in detail though we explore the import of complete congruences for frames more generally.

Our first remark concerns spatiality. By Corollary 2.7 and Proposition 2.12, we may note:

Proposition 4.1. *Every complete congruence on a supercontinuous lattice and, in particular, on a down-set lattice, is spatial (but not necessarily induced).*

Under the well-known one-to-one correspondence between frame congruences and *nuclei*, i.e. closure operators on frames preserving finite meets [43, 52], the so-called *open nuclei*

$$u_a : L \rightarrow L, x \mapsto a \wedge x \ (a \in L)$$

correspond to the *open congruences*

$$\Delta_a = \{(x, y) \in L \times L \mid x \wedge a = y \wedge a\},$$

which evidently are not only frame but complete congruences. In particular, we see that *if a congruence on an open set frame is induced by an open subspace then it is complete.*

In order to get a reversible statement, we need a slightly more complicated property. Recall that in any topological space we can define the specialization order $\leq$ with $x \leq y \Leftrightarrow x \in \overline{\{y\}}$, and that each open set is an up-set, while the converse is true only for Alexandroff spaces. A subset A of a space X is said to be *quasiopen* [41] if $\uparrow V \in \mathfrak{O}(X)$ for all $V \in \mathfrak{O}(A)$.

Proposition 4.2. *A subset A of a space X is quasiopen if and only if the congruence induced by A is complete.*

Proof. Suppose $A \subseteq X$ is quasiopen and consider the quotient map

$$h : \mathfrak{O}(X) \to \mathfrak{O}(A),\ U \mapsto U \cap A$$

and define $g : \mathfrak{O}(A) \to \mathfrak{O}(X)$ by $g(V) = \uparrow V$. Then $g(V) \subseteq U \Leftrightarrow V \subseteq U \Leftrightarrow V \subseteq h(U)$. That is, g is a left adjoint to h and thus h preserves all (joins and) meets, i.e. Δ_A is complete.

For the converse, suppose the quotient map $h : \mathfrak{O}(X) \to \mathfrak{O}(A)$ is complete and g is its left adjoint. Then, for any $V \in \mathfrak{O}(A)$, we have $\uparrow V = \bigcap\{U \in \mathfrak{O}(X) \,|\, V \subseteq U\} = \bigcap\{U \in \mathfrak{O}(X) \,|\, V \subseteq h(U) = U \cap A\} = g(V) \in \mathfrak{O}(X)$. □

Corollary 4.3. *Openness of $\uparrow A$ is necessary and openness of A is sufficient for completeness of the induced congruence Δ_A.*

Corollary 4.4. *A T_0 space is quasidiscrete if and only if each induced congruence is complete.*

Example 4.5. Even in chains with the upper (Scott) topology (cf. [36]), openness of $\uparrow A$ alone is necessary but not sufficient for completeness of Δ_A: for example, in the real line $\mathbb{R}$, the integers form a subchain $\mathbb{Z}$ for which $\uparrow \mathbb{Z} = \mathbb{R}$ is trivially open, whereas for no nonempty proper open subset $V =]r, \infty[$, the set $\uparrow (\mathbb{Z} \cap V) = [s, \infty[$ (with $s = \min\{z \in \mathbb{Z} : r \leq s\}$) is open, and therefore, $\Delta_{\mathbb{Z}}$ cannot be complete.

At the other extreme, on T_1-topologies, only the *open* subspaces induce complete congruences. Indeed, a space X is T_1 iff $A = \uparrow A$ for all $A \subseteq X$, and we conclude:

Proposition 4.6. *A space X is T_1 if and only if the open subsets are precisely those which induce complete congruences on $\mathfrak{O}(X)$.*

Proof. It only remains to show that the latter condition entails the validity of the T_1-axiom. First, it implies T_D, by Proposition 3.1 and the fact that different open sets always induce distinct congruences. Now, for $x \in X$ and $V \in \mathfrak{O}(X)$, the set $\uparrow(V \setminus \{x\})$ is either equal to V (if $x \notin V$ or there exists a $y \in V$ with $y < x$), or equal to $V \setminus \overline{x}$ (if x is minimal in V, whence $V \setminus \overline{x} = V \setminus \downarrow x = V \setminus \{x\}$). In any case, $\uparrow(V \setminus \{x\})$ is open, and by Proposition 4.2, $X \setminus \{x\}$ induces a complete congruence; thus, by hypothesis, $X \setminus \{x\}$ is open. □

To guarantee that the complete congruences on $\mathfrak{O}(X)$ are precisely those which are induced by open subsets, a weaker separation axiom than T_1 is sufficient, namely, the spatial counterpart of *subfitness* of frames, introduced by Isbell [42], and requiring that

$$\text{for } a \not\leq b \text{ there is a } c \text{ with } a \vee c = 1 \text{ but } b \vee c \neq 1.$$

This property has been referred to as *conjunctivity* by Simmons [57, 58] and others, being the dual of the *disjunction property* introduced by Wallman already in the thirties [62] and studied further by Banaschewski and Harting [6] (see also [23]). A characteristic property of subfit frames L is that every *co-dense* frame homomorphism $h : L \to M$ (satisfying $h(x) = 1$ only if $x = 1$) is one-to-one (cf. [6] and [42]). Although subfitness is not hereditary in general, it is inherited by *open* sublocales. In other words:

Lemma 4.7. *Every principal ideal of a subfit frame is subfit.*

Proof. Let L be a subfit frame and $s \in L$. Given $a, b \leq s$ with $a \not\leq b$, choose a c such that $a \vee c = 1 \neq b \vee c$. Then s, the top element of the principal ideal $\downarrow s$, satisfies

$$a \vee (c \wedge s) = (a \wedge s) \vee (c \wedge s) = (a \vee c) \wedge s = s,$$

while $b \vee (c \wedge s) = s$ would lead to the contradiction

$$b \vee c = b \vee (c \wedge s) \vee c = s \vee c = a \vee (c \wedge s) \vee c = a \vee c = 1. \quad \square$$

As remarked by Isbell [42] and Simmons [57], an open set frame $\mathfrak{O}(X)$ is subfit (conjunctive) if and only if the underlying space X fulfils the following condition:

(C) For all $U \in \mathfrak{O}(X)$ and $x \in U$, there is a $y \in \overline{\{x\}}$ with $\overline{\{y\}} \subseteq U$.

Simmons also noted the 'equation' $T_1 = C + T_D$.

Proposition 4.8. *If X is a space satisfying condition C then the complete congruences on $\mathfrak{O}(X)$ are exactly those which are induced by open subspaces.*

More generally, any complete congruence resp. homomorphism on a subfit frame is open.

Proof. Let $h : L \to M$ be complete and onto, and put $s = \bigwedge\{x \in L \mid h(x) = 1\}$. Then $h(s) = 1$. Consider the onto homomorphism

$$\widehat{s} : L \to {\downarrow}s, \ x \mapsto x \wedge s$$

and define $k : {\downarrow}s \to M$ by setting $k(x) = h(x)$. Then we have

$$k(\widehat{s}(x)) = h(x \wedge s) = h(x) \wedge h(s) = h(x)$$

and hence k is an onto homomorphism. Now let $k(x) = 1$, $x \leq s$. Then $h(x) = 1$ so that $x = s$. Thus, $k(x) = 1$ only for $x = s$ and, by subfitness of ${\downarrow}s$, the homomorphism k is one-to-one and hence an isomorphism. It follows that the kernel congruence of h is the open congruence induced by s: indeed, $h(x) = h(y) \Leftrightarrow k(\widehat{s}(x)) = k(\widehat{s}(y)) \Leftrightarrow x \wedge s = y \wedge s$. □

It remains open whether condition C (subfitness) is not only sufficient but also necessary in order that all complete congruences be open.

Putting together Corollary 4.3 and Proposition 4.8 we have:

Corollary 4.9. *A quasidiscrete topology is subfit if and only if it is discrete, that is, the corresponding order is the identity relation. Thus, subfit superalgebraic frames are already complete atomic Boolean algebras.*

We have seen that on T_D-topologies all induced congruences are distinct. Among these, T_1-topologies are characterized by the property that each complete congruence is induced by precisely one subset and this subset is open. Also among the T_D-spaces we have the quasidiscrete spaces, but for these *all* the induced congruences are complete.

There remains the question: under what circumstances are the complete congruences on a topology $\mathfrak{O}(X)$ exactly the induced ones? This is the 'discrete' version of the question answered in Theorem 3.4. We know from Corollary 4.4 that X necessarily has to be quasidiscrete, but that is not enough: the lower Alexandroff topology $\mathfrak{D}(\mathbb{I})$ of the real unit interval $\mathbb{I}$ (see Example 2.10) has non-induced complete congruences, like the kernel of the join map from $\mathfrak{D}(\mathbb{I})$ onto $\mathbb{I}$. This congruence cannot be induced by any subspace, since it identifies any closed interval $[0, a]$ with the half-open interval $[0, a[$; but $[0, a]$ is not congruent to $[0, a[$ modulo Δ_A whenever $a \in A$, and the empty set induces the trivial congruence.

The general situation is clarified by the next easy observation.

Proposition 4.10. *A complete congruence on a quasidiscrete topology is induced if and only if the corresponding quotient is superalgebraic (isomorphic to a quasidiscrete topology).*

Proof. For any induced congruence Δ_A on a quasidiscrete topology $\mathfrak{O}(X)$, the associated quotient is isomorphic to the quasidiscrete subspace $\mathfrak{O}(A)$, hence superalgebraic. Conversely, if A is a subspace such that $\mathfrak{O}(X)/_\Theta$ is superalgebraic, choose an isomorphism i from $\mathfrak{O}(X)/_\Theta$ onto a quasidiscrete topology $\mathfrak{O}(Y)$ and compose it with the complete homomorphism p_Θ associated with Θ; then Θ is the kernel congruence of the complete homomorphism $i \circ p_\Theta$ (cf. the proof of Proposition 3.2, and also [26] for a more general result.) □

This brings us back to our Theorem 2.11, characterizing those posets for which the complete quotients of the corresponding Alexandroff topology are exactly the induced quotients. Recall that a poset (or the associated quasidiscrete space) is *order-scattered* provided the poset does not contain a densely ordered subchain. Now, combining the previous proposition with Theorem 2.11, we arrive at:

Theorem 4.11.* *A poset is order-scattered if and only if the induced congruences of the associated Alexandroff topology (or down-set lattice) are exactly the complete ones.*

Finally, let us establish various characterizations of those quasidiscrete spaces for which not only every complete congruence, but even every frame congruence is induced. We already know that these are the (weakly) scattered ones (Theorem 3.4). In order to get more handy order-theoretical criteria, we recall some conditions that will be relevant for our purposes. A poset is *well-founded* (*co-well-founded*) if each nonempty subset has a minimal (maximal) element, and *Noetherian* if each directed subset has a greatest element, or equivalently, if each directed down-set ("ideal" [36]) is finitely generated, hence principal. A poset admitting no properly ascending sequences satisfies the *Ascending Chain Condition*, abbreviated *ACC*. The following implications obviously hold in set theory without choice:

$$\textit{co-well-founded} \;\Rightarrow\; \textit{Noetherian} \;\Rightarrow\; \textit{ACC}$$

The reverse implications are true as well, provided Dependent Choices are assumed.

Theorem 4.12. * *For a quasidiscrete space X and the corresponding poset $X_\leq$, the following conditions are equivalent:*

- (a) *X is topologically scattered.*
- (b) *$X_\leq$ is co-well-founded.*
- (c) *Each frame congruence on $\mathfrak{O}(X)$ is induced (by a unique subset).*
- (d) *Each frame congruence on $\mathfrak{O}(X)$ is complete.*
- (e) *Each spatial congruence on $\mathfrak{O}(X)$ is induced.*
- (f) *Each spatial congruence on $\mathfrak{O}(X)$ is complete.*
- (g) *X is sober.*
- (h) *$X_\leq$ is Noetherian.*
- (i) *$X_\leq$ satisfies the ACC.*

Proof. (a) $\Leftrightarrow$(b) . Both conditions say that each nonempty subset A contains a point x with $\uparrow x \cap A = \{x\}$ (here $\uparrow x$ is the least neighborhood of x).

For the equivalence of (a) and (c) , see Theorem 3.4.

The implications (c) $\Rightarrow$(d) $\Rightarrow$(f) and (c) $\Rightarrow$(e) $\Rightarrow$(f) are clear.

(f) $\Rightarrow$(h) . Given a directed subset D of $X_\leq$, let Θ be the kernel of the two-valued frame homomorphism $h : \mathfrak{O}(X) \to 2$ with $h(U) = 1 \Leftrightarrow U \cap D \neq \emptyset$. Then Θ is a spatial congruence (having two classes). By hypothesis (f) , Θ is complete, so there is a smallest up-set U with $U \cap D \neq \emptyset$, and then, for $d \in U \cap D$, we get $U = \uparrow d$, so d must be the greatest element of D.

For (e) $\Leftrightarrow$(g) see Corollary 3.3.

(g) $\Leftrightarrow$(h) is an immediate consequence of the known (and simple) fact that the irreducible closed sets of any quasidiscrete space are the directed down-sets of the corresponding poset.

(h) $\Rightarrow$(i) is obvious, and (i) $\Rightarrow$(a) is the aforementioned straightforward application of the Principle of Dependent Choices. ☐

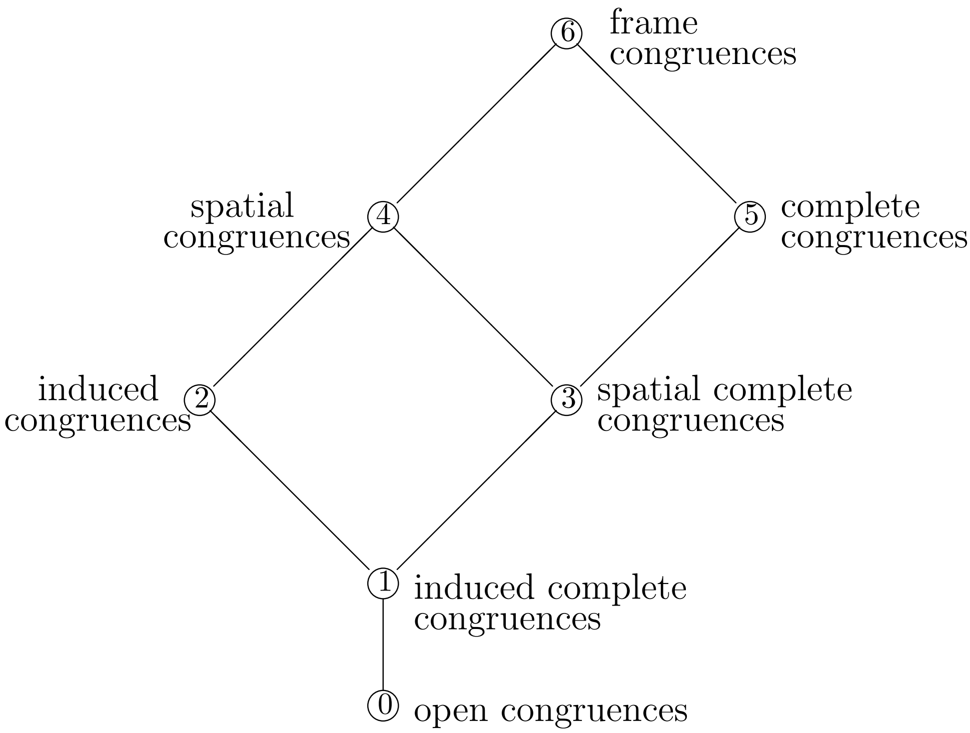
The previous proof shows that the following equivalences and implications hold true without any choice principles:

$$(a) \Leftrightarrow (b) \Leftrightarrow (c) \Rightarrow (d) \Rightarrow (e) \Leftrightarrow (f) \Leftrightarrow (g) \Leftrightarrow (h) \Rightarrow (i) .$$

Example 4.13. The chain ω of natural numbers is well-founded but not co-well-founded, hence not scattered as a quasidiscrete space, although all complete congruences are induced, being at the same time the complete congruences for the up-set lattice of the dual ω^d (which is co-well-founded). Thus, topological scatteredness is sufficient but not necessary for the coincidence of complete and induced congruences.

In the diagram below, we sketch the congruence types discussed in this paper and the hierarchy between them. At each of the nodes (1) – (6), an example is given that has the indicated property and exactly those properties that are reachable *upwards* from the node. This will demonstrate that no further implications than the previously established ones are valid for the studied properties of congruences on topologies.

Diagram: congruences on a spatial frame $L = \mathfrak{O}(X)$



(1) $3 = \{0, 1, 2\}$ is the Sierpinski topology on $2 = \{0, 1\}$ (with the only nontrivial open set $1 = \{0\}$). The congruence on 3 identifying $0 = \emptyset$ with $1 = \{0\}$ is spatial, complete and induced by $\{1\}$, but not by any *open* subspace.

(2) Induced and spatial but not complete is the congruence $\Delta_{\{\omega\}}$ on the Scott topology of the ordinal $\omega{+}1$, which consists of all up-sets of $\omega{+}1$ except $\{\omega\}$.

(3) A spatial but not induced complete congruence on $\mathfrak{D}(\mathbb{I})$ is the kernel of the join mapping $\bigvee : \mathfrak{D}(\mathbb{I}) \to \mathbb{I}$ (see Example 2.10).

(4) The upper Alexandroff topology on ω is the trace of the Scott topology on $\omega{+}1$ and dually isomorphic to $\omega{+}1$; as in (1), the spatial congruence that identifies all nonempty open subsets is not complete – but here it is not induced either.

(5) It seems hard to find examples of complete but not spatial congruences on topologies. The quotients of topologies under complete homomorphisms are precisely the so-called *quasitopologies* which may be characterized by an infinite distributive law (see [22, 25]). The only known example of a quasitopology without points was constructed by Kříž and Pultr [45]. Once one has found a non-spatial quasitopology L, one has a complete homomorphism from $\mathfrak{D}(L)$ onto L, and its kernel is then a complete but not spatial congruence on $\mathfrak{D}(L)$.

(6) Frame congruences that are neither spatial nor complete are obtained as follows: take any T_1-space X without isolated points (like the reals). The *Boolean nucleus* w_0 (see [43]) sends any open set to the interior of its closure; the range of that nucleus is the complete Boolean frame $\mathfrak{RO}(X)$ of all regular open sets, and the kernel is a frame congruence but not spatial (because $\mathfrak{RO}(X)$ is not atomic). Moreover, that congruence cannot be complete, since each of the open sets $X \setminus \{x\}$ is congruent to X, while their intersection (= meet) is empty.

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