DUALITIES IN FULL HOMOMORPHISMS

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ABSTRACT. In this paper we study dualities of graphs and, more generally, relational structures with respect to full homomorphisms, that is, mappings that are both edge- and non-edge-preserving. The research was motivated, a.o., by results from logic (concerning first order definability) and Constraint Satisfaction Problems. We prove that for any finite set of objects $\mathcal{B}$ (finite relational structures) there is a finite duality with $\mathcal{B}$ to the left. It appears that surprising richness of these dualities leads to interesting problems of Ramsey type; they are which are explicitly analyzed in the simplest case of graphs.

INTRODUCTION

We will illustrate the motivation and the type of results to be presented by the simple example of finite binary relations (i.e. directed graphs). Given such relations $G = (X, R)$ and $G' = (X', R')$ a mapping $f : X \to X'$ is said to be a *homomorphism* $G \to G'$ if

$$(x, y) \in R \Rightarrow (f(x), f(y)) \in R'.$$

Homomorphisms capture many combinatorial properties of graphs and relations, see [7]. Of particular interest is the following class defined for a fixed relation B:

$$\{G \mid \text{there is an } f : G \to B\}.$$

In the particular case when B is the complete graph (symmetric, without loops) with k vertices this is the class of all k-colorable graphs; consequently, more generally, for a relation B we speak of the class of all B-colorable relations, or the *B-color class*. Considering more general objects (n-ary relations, relational systems) we obtain this way

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representations of the so called Constraint Satisfaction Problem. This is why we denote the above class by

$$\mathbf{CSP}(B).$$

$\mathbf{CSP}(B)$ can be represented in a complementary way by *forbidding* (instead of *requiring*) homomorphisms, namely as

$$\mathbf{Forb}(\mathcal{A}) = \{G \mid \text{there is no } f : A \to G \text{ with } A \in \mathcal{A}\}$$

(it suffices to take $\mathcal{A} = \{A \mid \text{there is no } f : A \to B\}$). We are interested in the cases when such an $\mathcal{A}$ can be chosen finite. This cannot be done for every B (consider for example the class of all 3-colorable graphs; the set of minimal forbidden relations is then infinite and coincides with so called 4-critical graphs), but it is not quite a rare phenomenon. If we have such a finite $\mathcal{A}$ we speak of a *finite duality*

$$\mathbf{Forb}(\mathcal{A}) = \mathbf{CSP}(B).$$

We also say that B *has finite duality.* Finite dualities were defined in [13]. They are being intensively studied from the logical point of view, and also in the optimization (mostly CSP) context. The following has been recently proved (as a combination of results of [2, 12]):

Theorem. *Let B be a finite binary relation. Then the following statements are equivalent.*

(i) *The class $\mathbf{CSP}(B)$ is first order definable;*
(ii) *B has finite duality; explicitly, there exists a finite set $\mathcal{A}$ such that $\mathbf{Forb}(\mathcal{A}) = \mathbf{CSP}(B)$;*
(iii) *$\mathbf{Forb}(\mathcal{A}) = \mathbf{CSP}(B)$ for a finite set $\mathcal{A}$ of finite oriented trees.*

In fact similar theorems hold for more general finite relational structures. Thus, finite dualities for finite relational structures are well characterized, and it can be shown that they abound.

In a sharp contrast with that, there are no finite dualities for (general) finite algebras. It has been recently shown [9] that there are no such dualities at all. Namely, one has

Theorem. *For every finite set $\mathcal{A}$ of finite algebras of a given type $(n_i)_{i\in T}$ and every other finite algebra B there exists a finite algebra A such that $A \in \mathbf{Forb}(\mathcal{A})$ and $A \notin \mathbf{CSP}(B)$.*

(This concerns the standard homomorphisms $f : (X, (\alpha_i)_{i\in T}) \to (X', (\alpha'_i)_{i\in T})$ satisfying

$$(*) \qquad x = \alpha_i(x_1, \dots, x_{n_i}) \Rightarrow f(x) = \alpha'_i(f(x_1), \dots, f(x_{n_i})).\ )$$

This is a striking difference. The aim of this paper is to study the situation of the relations and relational systems with a type of homomorphisms that are structurally closer to the homomorphisms of algebras then to the standard ones, but, surprisingly, admit plenty of finite dualities. Homomorphisms of algebras automatically satisfy more than $(*)$: for instance if f is one-one, this requirement is equivalent to

$$(**) \qquad x = \alpha_i(x_1, \ldots, x_{n_i}) \Leftrightarrow f(x) = \alpha'_i(f(x_1), \ldots, f(x_{n_i}))).\ )$$

Now given finite relations $G = (X, R)$ and $G' = (X', R')$ a mapping $f : X \to X'$ is said to be a *full homomorphism* $G \to G'$ (see [7]) if

$$(x, y) \in R \Leftrightarrow (f(x), f(y)) \in R'$$

(similarly for n-ary relations and relational systems, see 1.2 below). The category of all relations and all their full homomorphisms is much more restrictive than the category of all homomorphisms but on the other hand it is more sensitive to the scheme of dualities as seen from the following theorem (a special case of our main result on general relational systems proved in Section 3).

Theorem. *For every relation B there exists a finite duality*

$$\mathbf{Forb}_{\mathsf{full}}(\mathcal{A}) = \mathbf{CSP}_{\mathsf{full}}(B).$$

(The classes $\mathbf{Forb}_{\mathsf{full}}(\mathcal{A})$ and $\mathbf{CSP}_{\mathsf{full}}(B)$ are defined in a complete analogy with the classes above, only with full homomorphisms instead of the general ones.)

The paper is organized as follows. In Section 1 we review the basic definitions. We treat our problems in a fairly general categorical setting; this also explains our detailed exposition in this introduction. In Section 2 we consider the dualities still in the abstract way, and in Section 3 we prove our main result (3.3). In Sections 4 an 5 we deal with the binary relations and then with the even more special classes of undirected graphs; in particular we have a procedure that produces (albeit not very effectively) finite "left hand sides" to the $\mathbf{CSP}(B)$'s, or even the $\mathbf{CSP}$'s of finite systems $B_1, \ldots, B_k$.

1. Preliminaries

1.1. We will be concerned with very special categories of a combinatorial nature. In particular, we will typically assume the following properties.

(bi-LocFin) The category is *bi-locally finite*, that is, for any object A there are (up to isomorphism) only finitely many monomorphisms $B \to A$ and only finitely many epimorphisms $A \to B$.

(wFac) The category has a *weak (epi-mono) factorization*, that is, every morphism f can be written as $f = m \cdot e$ with m a monomorphism and e an epimorphism.

(Ch) The category has *choice*, that is, every epimorphism is a retraction.

Only basic facts and notions from category theory (monomorphisms, epimorphisms, retractions and coretractions, products) are assumed; see, for instance, the opening chapters of [10].

1.2. An *n-ary relation* on a set X is a subset $R \subseteq X^n$, and a mapping $f : X \to Y$ is a *homomorphism* with respect to R, S if

$$(x_1, \dots, x_n) \in R \quad \Rightarrow \quad (f(x_1), \dots, f(x_n)) \in S.$$

The mappings with the (much) stronger property

$$(x_1, \dots, x_n) \in R \quad \Leftrightarrow \quad (f(x_1), \dots, f(x_n)) \in S$$

will be called *full homomorphisms*.

A (finite) *type* is a finite collection $\Delta = (n_t)_{t \in T}$ of natural numbers, and a *relational structure* of type Δ on X is a collection $R = (R_t)_{t \in T}$ where the R_t are n_t-ary relation on X; (X, R) is then referred to as a *relational object*. A *(full) homomorphism* $f : (X, R = (R_t)_{t \in T}) \to (Y, S = (S_t)_{t \in T})$ is a mapping that is a (full) homomorphism with respect to R_t, S_t for each $t \in T$.

The category of all relational objects of type Δ and full homomorphisms will be denoted by

$$\mathbf{Rel}_{\mathsf{full}}(\Delta).$$

The category of undirected graphs (resp. connected undirected graphs) with full homomorphisms will be viewed as a full subcategory of $\mathbf{Rel}_{\mathrm{full}}((2))$; that is, the set of edges is represented as a symmetric antireflexive binary relation. It will be denoted by

$$\mathbf{Graph}_{\mathsf{full}} \quad \text{resp.} \quad \mathbf{ConnGraph}_{\mathsf{full}}.$$

Note that the mentioned categories satisfy all the properties from 1.1.

1.3. With a category $\mathcal{C}$ we will associate the preordered class $\widetilde{\mathcal{C}} = (\widetilde{C}, \to)$ of the objects from $\mathcal{C}$ with the preorder

$$A \to B \quad \equiv_{\mathrm{df}} \quad \exists f : A \to B \text{ in } \mathcal{C}.$$

Thus, for a set $\mathcal{A}$ of objects of $\mathcal{C}$,

$\uparrow\!\mathcal{A} \equiv_{\mathrm{df}} \{C \in \mathcal{C} \mid \exists A{\in}\mathcal{A} \quad A \to C\}, \quad \downarrow\!\mathcal{A} \equiv_{\mathrm{df}} \{C \in \mathcal{C} \mid \exists A{\in}\mathcal{A} \quad C \to A\}.$

We will write

$$A \sim B \quad \text{if} \quad A \to B \text{ and } B \to A$$

and speak of $\sim$-*equivalence classes* or simply of *equivalence classes.*

The fact that there is no $f : A \to B$ will be indicated by

$$A \nrightarrow B.$$

1.4. An object A of a category $\mathcal{C}$ is said to be *reduced*, or a *core*, if each $f : A \to A$ is an isomorphism.

Lemma. *Let $\mathcal{C}$ satisfy (bi-LocFin), (wFac), and (Ch). Then*

1. the sets $\mathcal{C}(A, B)$ of morphisms $A \to B$ are (up to isomorphism) finite, and

2. an object A in $\mathcal{C}$ is reduced iff there is no proper (that is, non-isomorphic) retraction out of A.

Proof. 1 is trivial.

2: If A is reduced and $r : A \to B$ is a retraction, with $r \cdot m : B \to B$ identical, then we have that $m \cdot r : A \to A$ is an isomorphism and hence also r.

Now suppose that $f : A \to A$ is not an isomorphism. If when factored as $f = me$, m monic and e epic, e is not an isomorphism then we have found a proper retraction out of A. So suppose that e is an isomorphism, so that f is a monomorphism. By 1 there are integers $n, k > 0$ such that f^{n+k} is equivalent to f^n, say $f^n h = f^{n+k}$ for an isomorphism h. Since f^n is a monomorphism, $f^k = 1$. But then f is both the left factor of an epimorphism and the right factor of a monomorphism, and hence is itself both. And in a category with (Ch), that implies that f is an isomorphism. $\square$

1.5. Proposition. *If a category $\mathcal{C}$ satisfies (bi-LocFin), (wFac), and (Ch) then each $\sim$-equivalence class contains (up to isomorphism) exactly one core object.*

Proof. If two reduced objects A and B are equivalent then then they are, trivially, isomorphic.

Now let A be any object. Consider the class $\mathcal{M}$ of all the coretractions $m : A_m \to A$ and (pre)order it by $m \prec n$ iff there is an f such that $m = nf$. By (bi-LocFin), $\mathcal{M}$ is, up to isomorphism, finite and hence there is an $m \in \mathcal{M}$ minimal in $\prec$. Then A_m cannot admit a proper retraction $A_m \to B$, for such a B would be smaller in $\prec$, and hence it is reduced by 1.4. $\square$

1.6. Proposition. *Let $\mathcal{C}$ satisfy (bi-LocFin), (wFac) and (Ch). Then*

1. if A is reduced then every $A \to B$ is a monomorphism, and

2. for every A and every property $\mathcal{P}$ satisfied by A there exists an $A_0 \to A$ minimal in $\to$ such that it still satisfies $\mathcal{P}$.

Proof. 1. Set, by (wFac),

$$f = (A \xrightarrow{\;e\;} C \xrightarrow{\;m\;} B).$$

By (Ch) e is a retraction an by 1.4 it is an isomorphism.

2. By 1 and (bi-LocFin) we have, in $\to$, under each object only finitely many $\sim$-classes. Hence we have minimal objects with any property $\mathcal{Q}(B)$ that is satified by some object (here: $\mathcal{Q}(B) \equiv$“$B \to A$ and $\mathcal{P}(B)$”). □

1.6.1. Remark. Note that in the categories from 1.2, monomorphisms are precisely the embeddings of induced objects. Thus, searching for objects smaller then a given one can be restricted to its subobjects.

2. Dualities and Ramsey lists

2.1. Let $\mathcal{A}$ be a subclass of obj $\mathcal{C}$, the class of objects of $\mathcal{C}$. Write

$$\begin{array}{lll} X \to \mathcal{A} & \text{for} & \exists A \in \mathcal{A},\ X \to A, \\ \mathcal{A} \to X & \text{for} & \exists A \in \mathcal{A},\ A \to X, \\ X \not\to \mathcal{A} & \text{for} & \forall A \in \mathcal{A},\ X \not\to A, \\ \mathcal{A} \not\to X & \text{for} & \forall A \in \mathcal{A},\ A \not\to X. \end{array}$$

Set

$$\begin{aligned} &\mathbf{Forb}(\mathcal{A}) = \{X \mid \mathcal{A} \not\to X\}, \quad \mathbf{CSP}(\mathcal{B}) = \{X \mid X \to \mathcal{B}\} \\ &\text{and} \quad \mathcal{N}(\mathcal{A}) = \{X \mid X \not\to \mathcal{A}\}. \end{aligned}$$

Note. $\mathbf{Forb}(\mathcal{A})$ resp. $\mathbf{CSP}(\mathcal{B})$ are, of course, understood in the category discussed. Thus, if we are in the categories of full homomorphisms, they designate the classes $\mathbf{Forb}_{\mathsf{full}}(\mathcal{A})$ resp. $\mathbf{CSP}_{\mathsf{full}}(\mathcal{B})$ from the Introduction.

A *finite duality* in $\mathcal{C}$ is a couple $\mathcal{A}, \mathcal{B}$ of <u>finite</u> subsets of obj $\mathcal{C}$ such that

$$\mathcal{A} \not\to X \quad \text{iff} \quad X \to \mathcal{B}, \quad \text{that is,} \quad \mathbf{Forb}(\mathcal{A}) = \mathbf{CSP}(\mathcal{B}).$$

2.2. Proposition. *We have*

$$\mathcal{N}(\mathcal{B}) \not\to X \quad \textit{iff} \quad X \to \mathcal{B}$$

and

$$\mathcal{A} \not\to X \quad \textit{iff} \quad X \to \mathbf{Forb}(\mathcal{A}).$$

In other words,

$$\mathbf{Forb}(\mathcal{N}(\mathcal{B})) = \mathbf{CSP}(\mathcal{B}) \quad \textit{and} \quad \mathbf{Forb}(\mathcal{A}) = \mathbf{CSP}(\mathbf{Forb}(\mathcal{A})).$$

Proof. The desired condition $\mathcal{A} \not\to X$ iff $X \to \mathcal{B}$ coincides in the general setting of a preordered class $(P, \leq)$ with the equality

$$P \setminus \uparrow\mathcal{A} = \downarrow\mathcal{B}.$$

Now we have $\mathbf{Forb}(\mathcal{A}) = P \setminus \mathcal{A}$ and $\mathcal{N}(\mathcal{B}) = P \setminus \downarrow\mathcal{B}$. Thus $P \setminus (\uparrow \mathcal{N}(\mathcal{B})) = P \setminus (P \setminus \downarrow\mathcal{B}) = \downarrow\mathcal{B}$, and $P \setminus \uparrow\mathcal{A} = \downarrow(P \setminus \uparrow\mathcal{A}) = \downarrow\mathbf{Forb}(\mathcal{A})$. □

2.3. An object A will be called *critical* with respect to a class of objects $\mathcal{B}$ if

- it is reduced,
- $A \not\to \mathcal{B}$, and
- if $A' \to A \not\to A'$ then $A' \to \mathcal{B}$.

Thus, since we can restrict ourselves to reduced objects, by 1.6 the third condition amounts to requiring that every proper subobject A' of A minorizing an element of $\mathcal{B}$.

Set

$$\mathcal{N}_0(\mathcal{B}) = \{X \in \mathcal{N}(\mathcal{B}) \mid X \text{ critical w.r.t. } \mathcal{B}\}.$$

We have

2.3.1. Proposition. *If $\mathcal{C}$ is a category satisfying* (bi-LocFin), (wFa) *and* (Ch)*, then*

$$\mathcal{N}_0(\mathcal{B}) \not\to X \quad \textit{iff} \quad X \to \mathcal{B}.$$

Proof. Use 2.2 and 1.6.1: there is an $A \in \mathcal{N}(\mathcal{B})$ with $A \to X$ iff there is such an A in $\mathcal{N}_0(B)$. □

2.4. The Propositions in 2.2 and 2.3.1 are not necessarily finite dualities, since neither $\mathbf{Forb}(\mathcal{A})$ nor $\mathcal{N}(\mathcal{A})$ nor $\mathcal{N}_0(A)$ is necessarily finite just because $\mathcal{A}$ is finite. However, we will see that in the categories we are interested in, a finite $\mathcal{B}$ can always be extended to a finite duality $\mathbf{Forb}(\mathcal{A}) = \mathbf{CSP}(\mathcal{B})$. This leads to the following definition.

2.5. A collection of reduced objects $\mathcal{A} = \{A_1, \ldots, A_n\}$ is said to be a *Ramsey list*, or, briefly, to be *Ramsey*, if there is a finite $\mathcal{F} \subseteq \text{obj } \mathcal{C}$ such that for each core X that is not isomorphic with an object from $\mathcal{F}$, some of the A_i is isomorphic to a subobject of X. (The reader can consult [11] and [5] for general background of Ramsey theory.)

2.5.1. Proposition. *Let $\mathcal{C}$ satisfy (bi-LocFin), (wFac) and (Ch). Then a finite $\mathcal{A}$ is Ramsey iff* $\mathbf{Forb}(A)$ *is a finitely generated downset, that is iff there is a finite duality*

$$\mathcal{A} \not\to X \quad \textit{iff} \quad X \to \mathcal{B}.$$

Proof. If there is such a duality then it suffices to take for $\mathcal{F}$ the set of all subobjects of the elements of $\mathcal{B}$.

On the other hand, if $\mathcal{A}$ is Ramsey then $\mathcal{A} \not\to X$ iff

$$X \to \mathbf{Forb}(\mathcal{A}) = X \to \{X \mid \mathcal{A} \not\to X\} = X \to \{X \mid \mathcal{A} \not\to X \text{ and } X \in \mathcal{F}\}. \quad \square$$

3. The category of relational systems

3.1. Convention. In this section we will deal with the finite dualities in $\mathbf{Rel}_{\mathsf{full}}(\Delta)$. Just to avoid too many indices we will present the proof in 4.3 as if for one n-ary relation. If one reads n_t for n and R_t for every relation constituting the relational system, and if one does everything simultaneously, one obtains correctly the general result.

3.2. If $B = (X, R)$ is an object of $\mathbf{Rel}_{\mathsf{full}}(\Delta)$ write $X = X_B$, $R = R_B$.

Proposition. *Let $\mathcal{B}$ be a finite set of objects of $\mathbf{Rel}_{\mathsf{full}}(\Delta)$. Let $\Delta = (n_t)_{t\in T}$ and let $m > \max_t n_t$. Then, with possibly finitely many exceptions, every A critical with respect to $\mathcal{B}$ can be embedded into an object of $\mathbf{Rel}_{\mathsf{full}}(\Delta)$ carried by X^m where*

$$X = X_B \cup \{\omega\}$$

for some $B \in \mathcal{B}$ and $\omega \notin X_B$.

Proof. Consider an A critical with respect to $\mathcal{B}$. For every $a \in A$ there is a $B_a \in \mathcal{B}$ such that $A \setminus \{a\} \to B_a$. If A is sufficiently large, there are distinct $a_1, \ldots, a_m$ such that the B_{a_i} coincide. Denote $B = B_{a_i}$ the common value.

Since A is reduced, it suffices to find a full homomorphism from A into an object as stated.

Recall the convention 3.1. For every $i = 1, \ldots, m$ there is a full homomorphism

$$f_i : A \setminus \{a_i\} \to B.$$

Set

$$X_{B_i^+} = X \ (= X_B \cup \{\omega\}) \quad \text{and} \quad X_{A_i^+} = X_A$$

and define

$$f_i^+ : X_{A_i^+} \to X_{B_i^+}$$

by setting $f_i^+(x) = f_i(x)$ if $x \neq a_i$, and $f_i^+(a_i) = \omega$.

Now put

$(y_1,\dots,y_n)\in R_{B_i^+}$ iff either $(y_1,\dots,y_n)\in R_S$ or at least one of the y_j's is ω.

Further define the relation for A_i^+ by

$$(x_1,\dots,x_n)\in R_{A_i^+} \quad \text{iff} \quad (f_i^+(x_1),\dots,f_i^+(x_n))\in R_{B_i^+},$$

thus making each

$$f_i^+ : A_i^+ \to B_i^+$$

a full homomorphism. Furthermore, it is obvious that the maps

$$\widetilde{f}_i : A \to B_i^+$$

defined by the same formula are homomorphisms, albeit not full, and hence we have a homomorphism

$$f : A \to \prod_{i=1}^{m} B_i^+$$

defined by requiring $p_i \cdot f = \widetilde{f}_i$ for the natural projections.

Now this f is full. Indeed, let $(f(x_1),\dots,f(x_n))$ be in the relation of the product. Then for every i,

$$(f_i^+(x_1),\dots,f_i^+(x_n)) = (p_i f(x_1),\dots,p_i f(x_n)) \in R_{B_i^+}.$$

Since $m > n$ there exists an i such that none of the x_j's is a_i, hence

$$(f_i^+(x_1),\dots,f_i^+(x_n)) = (f_i(x_1),\dots,f_i(x_n)) \in R_B$$

Since f_i is full, the statement follows. $\square$

3.3. Thus, $\mathcal{N}_0(\mathcal{B})$ is finite and we obtain as an immediate consequence

Theorem. *In* $\mathbf{Rel}_{\mathsf{full}}(\Delta)$ *there exists for every finite set of objects* $\mathcal{B}$ *a finite system of objects* $\mathcal{A}$ *and a finite duality*

$$\mathcal{A} \not\to X \quad \textit{iff} \quad X \to \mathcal{B}.$$

3.4. Let us briefly discuss the *inverse problem*: given a finite $\mathcal{A}$, does there exists a finite $\mathcal{B}$ such that $\mathbf{Forb}(\mathcal{A}) = \mathbf{CSP}(\mathcal{B})$? The answer is in general negative. For instance, in connected graphs there are only four such $\mathcal{A}$ containing less then three objects – see 5.3.1 and 5.4.

Nevertheless, we can isolate a necessary condition. The key to this is a definition of an "unavoidable" set of "complete systems".

Let $(X,<)$ be a linearly ordered set. Let $(a_1,\dots,a_k),(b_1,\dots,b_k)$ be two $k-$tuples of elements of X. We say that these tuples are *equivalent* if there exists a monotone (with respect to $<$) mapping ι :

$\{a_1, \dots, a_k\} \longrightarrow \{b_1, \dots, b_k\}$ such that $\iota(a_i) = b_i$ for every $i = 1, \dots, k$. This equivalence will be denoted by $\sim$. The equivalence classes of $\sim$ are called *types* (of the arity k). A type σ' is the mirror image of σ if σ' corresponds to the tuple $(a_k, \dots, a_1)$.

Let Σ be a set of order types (a *type-set*). By K_n^Σ we denote the following relational object (X, R):

$X = \{1, \dots, n\}$ and the relation structure consists from all tuples of X with a type $\sigma \in \Sigma$ (with respect to natural ordering of X). K_n^Σ is called a *complete* object (with type set Σ).

The type-set Σ and the complete object K_n^Σ are said to be *trivial* if (for every n) the object K_n^Σ is full homomorphism equivalent to the singleton complete object K_1^Σ. Note that there are many trivial type-sets ($2^{|T|}$ in $\mathbf{Rel}_{\mathsf{full}}(\Delta)$ with $|\Delta| = |T|$).

Lemma. *Let Σ, Σ' be sets of types. Then $K_m^\Sigma \longrightarrow K_n^{\Sigma'}$ iff one of the following possibilities occur:*

(i) *$\Sigma = \Sigma'$ is a trivial type-set;*
(ii) *$m \leq n$ and either $\Sigma = \Sigma'$ or Σ' is the mirror image of Σ.*

Proof follows by observing that from any non-trivial type-set Σ we can reconstruct the ordering of X (for any complete object K_X^Σ on X). $\square$

Finally, we say that a set Ξ of type sets is a *majorizing set* (in $\mathbf{Rel}_{\mathsf{full}}(\Delta)$) if for every non-trivial type-set Σ (of relations in $\mathbf{Rel}_{\mathsf{full}}(\Delta)$) there exists a set $\Sigma' \in \Xi$ such that either $\Sigma = \Sigma'$ or Σ' is the mirror image of Σ.

We have the following

Proposition. *For a finite set $\mathcal{A}$ of objects of $\mathbf{Rel}_{\mathsf{full}}(\Delta)$ the following holds:*

(i) *If there exists $B \in \mathbf{Rel}_{\mathsf{full}}(\Delta)$ such that $\mathbf{Forb}(\mathcal{A}) = \mathbf{CSP}(B)$; then $\mathcal{A}$ contains a set of non-trivial complete objects with majorizing set of set-types.*
(ii) *For every set $\mathcal{A}$ with majorizing order types there exists a finite set $\mathcal{A}'$ of non-trivial objects such that $\mathbf{Forb}(\mathcal{A} \cup \mathcal{A}') = \mathbf{CSP}(B)$ for some B.*

Proof. (i): Suppose the contrary. This equivalently means that there exists a type set Σ distinct from all the non-trivial set-types of all complete (arbitrarily ordered) objects in $\mathcal{A}$. As any subobject of any complete object K_n^Σ is again complete object with the same set-type we obtain that, using preceding lemma, that there is no finite duality with $\mathcal{B}$.

(ii): Let Ξ be a majorizing set of set-types. Let n be the maximal order (universum size) of an object in $\mathcal{A}$. Assume $\mathbf{Forb}(\mathcal{A})$ is non empty and let $B \in \mathbf{Forb}(\mathcal{A})$. Put $\mathcal{A}' = \mathcal{N}_0(\mathcal{B})$. $\mathcal{A}'$ is a finite set by 3.3 and clearly $\mathbf{Forb}(\mathcal{A} \cup \mathcal{A}') = \mathbf{CSP}(B)$. $\square$

Remark: We may choose B as the disjoint union of nontrivial complete objects K^{Σ}_{n-1} for $K^{\Sigma}_{n} \in \mathcal{A}$ together with the trivial forbidden objects in $\mathcal{A}$. Then the complete objects in $\mathbf{Forb}(\mathcal{A})$ and $\mathbf{Forb}(\mathcal{A} \cup \mathcal{A}')$ coincide. The structure of the non-complete Ramsey lists is more complex and it will be investigated in the next sections.

On the other side, by iterating the Ramsey's theorem we see easily that every large object of $\mathbf{Rel}_{\mathsf{full}}(\Delta)$ contains a large complete subsystem. The condition (i) of the Lemma is responsible for the difficulty in characterizing Ramsey lists. Let us finally remark that the properites of classes $\mathcal{N}(\mathcal{A})$ are closely related to the intensively studied Ramsey-type problems, particularly to Erdős - Hajnal problem; see [1].

4. One binary relation

The proof of Proposition 3.2 presents a finite system of objects containing the desirable $\mathcal{N}_0(\mathcal{B})$. It is, however, very large; listing the actual $\mathcal{N}_0(\mathcal{B})$ would be very hard.

In this section we will consider the simple (but important) case of one binary relation. Here, the listing is more feasible. In the next paragraph we will then discuss Ramsey lists in classical graphs and provide several concrete examples.

4.1. We will write $\mathbf{Rel}_{\mathsf{full}}$ for $\mathbf{Rel}_{\mathsf{full}}((2))$. The objects of $\mathbf{Rel}_{\mathsf{full}}$ can be interpreted as oriented graphs with possible loops.

4.2. The object $B+$. Let B be an object of $\mathbf{Rel}_{\mathsf{full}}$. Choose two distinct elements $\omega, \omega' \notin X_B \times \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$ and set

$$\begin{aligned}
X_{B+} &= (X_B \times \{\emptyset, \{0\}, \{1\}, \{0,1\}\}) \cup \{\omega, \omega'\},\\
R_{B+} &= \{(xu, yv) \mid xR_By,\ u, v \subseteq \{0,1\}\} \cup \{(\omega', \omega')\} \cup \\
&\quad \cup \{(x\{0\}, \omega), (x\{0\}, \omega') \mid x \in X_B\} \cup \\
&\quad \cup \{(\omega, x\{1\}), (\omega', x\{1\}) \mid x \in X_B\} \cup \\
&\quad \cup \{(x\{0,1\}, \omega), (\omega, x\{0,1\}), (x\{0,1\}, \omega'), (\omega', x\{0,1\}) \mid x \in X_B\}.
\end{aligned}$$

4.3. Proposition. *Let $A \in \mathcal{N}_0(\mathcal{B})$ in $\mathbf{Rel}_{\mathsf{full}}$. Then there is a $B \in \mathcal{B}$ such that $A \to B+$ (and A is isomorphic to a subobjectof $B+$).*

Proof. Choose an $a \in X_A$ and consider the object C carried by $X_A \setminus \{a\}$, with the relation inherited from A. Then, as A is core, C

is in $\to$ strictly smaller than A and hence there is a $B \in \mathcal{B}$ and a morphism

$$f : C \to B.$$

Define a mapping

$$g : A \to B+$$

by setting

$$g(a) = \begin{cases} \omega' & \text{if } \ aR_B a, \\ \omega & \text{otherwise,} \end{cases}$$

and for $x \in C$,

$$g(x) = \begin{cases} f(x)\emptyset & \text{if } \ x \notin aR_A \cup R_A a, \\ f(x)\{0\} & \text{if } \ x \in R_A a \setminus aR_A, \\ f(x)\{1\} & \text{if } \ x \in aR_A \setminus R_A a, \\ f(x)\{01\} & \text{if } \ x \in R_A a \cap aR_A. \end{cases}$$

Let xR_Ay. If $x, y \neq a$ then $f(x)R_Bf(y)$ and hence $g(x)R_{B+}g(y)$. If xR_Aa then $g(x)$ is $f(x)\{0\}$ or $f(x)\{0,1\}$, in both cases $\cdots R_{B+}\omega = g(a)$ or $\cdots R_{B+}\omega' = g(a)$. Similarly for aR_Ay.

Now let $g(x)R_{B+}g(y)$. If $g(x), g(y) \neq \omega, \omega'$ then $x, y \neq a$ and $f(x)uR_{B+}f(y)v$, hence $f(x)R_Bf(y)$ and finally xR_Ay. Let $g(x) = \omega$ or $g(x) = \omega'$ (so that $x = a$) and $g(y) \neq \omega, \omega'$. Then $g(y) = zu$ with $0 \in u$, and xR_Aa. Similarly if $g(x) = \omega$ or $g(y) = \omega'$ and $g(x) \neq \omega, \omega'$. when $g(x) = zu$ with $1 \in u$, and aR_Ax. The only remaining case is $g(x) = g(y) = \omega'$; Then $x = y = a$ and aR_Aa. $\square$

4.4. Thus constructed $B+$ can be applied to determining the Ramsey lists of finite $\mathcal{B}$ in categories such as

- $\mathbf{Graph}_{\mathsf{full}}$ of classical graphs, that is, symmetric antireflexive (X, R),
- $\mathbf{ConnGraph}_{\mathsf{full}}$ of connected classical graphs,
- $\mathbf{OrGraph}_{\mathsf{full}}$ of oriented graphs, that is, antisymmetric antireflexive (X, R),
- ,$\mathbf{Tour}_{\mathsf{full}}$ of tournaments, that is, antisymmetric antireflexive (X, R) in which for any two distinct x, y either xRy or yRx,
- $\mathbf{Poset}_{\mathsf{full}}$ of posets, that is, transitive antisymmetric (X, R),

and their variants with xRx allowed.

In fact, we typically do not even need to search the whole of the $B+$ since (unlike $B+$ itself) the images $g[A]$ stay in the category in question. Thus,

- in the antireflexive cases we can drop the ω',

- in the symmetric case we can do with $X_B \times \{\emptyset, 2\}$ instead of the whole of $X_B \times \mathfrak{P}(2)$,
- in the antisymmetric cases the $X_B \times \{\emptyset, \{0\}, \{1\}\}$ will do.

The object $B+$ from 4.2 typically does not stay in the category $\mathcal{C}$ in question but this does not impede the validity of the reasoning in 4.4 – with one exception. This concerns $\mathbf{ConnGraph}_{\mathsf{full}}$: while the properties of the whole of $B+$ are not relevant, it is essential that the object $C = A \setminus \{a\}$ does stay in $\mathcal{C}$. Now unlike all the other categories above, $\mathbf{ConnGraph}_{\mathsf{full}}$ does not have the property that every subset of an object carries an object. But luckily enough, in every connected A with more than one vertex there is an a such that $A \setminus \{a\}$ is connected. Thus, we can use the proof of 4.3 again, only the $a \in A$ cannot be chosen arbitrarily.

Consequently we have

4.4.1. Proposition. *Let $\mathcal{C}$ be any of the categories from 4.1. Let $A \in \mathcal{N}_0(\mathcal{B})$ in $\mathcal{C}$. Then there is a $B \in \mathcal{B}$ such that A is isomorphic to a subobject of $B+$.*

4.5. Note. Already in 3.3 (resp. 3.2) we had a finite collection of objects containing all the elements of $\mathcal{N}_0(\mathcal{B})$ as subobjects. Thus, one can say that we could list $\mathcal{N}_0(\mathcal{B})$ by means of a finite search; but of course the number of cases and individual checkings is prohibitive and one can seldom expect satisfactory results obtained by brutal force. In the binary case just discussed, and particularly in the case of classical graphs to be dealt with in the next section, the starting $B+$'s are simpler and we will be able to produce the lists in several basic cases. The existence of an efficient search procedure is an open problem, though.

5. Ramsey lists in symmetric graphs

5.1. First, observe that in the cases of $\mathbf{Graph}_{\mathsf{full}}$ and $\mathbf{ConnGraph}_{\mathsf{full}}$ the $B+$ from 4.2 and 4.3 can be reduced to the $B+'$ defined as follows.

Choose an element $\omega \notin B \times \{0, 1\}$ and set

$$X_{B+'} = (B \times \{0, 1\}) \cup \{\omega\},$$
$$R_{B+'} = \{(xi, yj) \mid xR_B y,\ i, j = 0, 1\} \cup \{(x1, \omega), (\omega, x1) \mid x \in X_B\}.$$

5.1.1 Now we can find all the elements of $\mathcal{N}_0(\mathcal{B})$ in among the subgraphs of the $B+$ with $B \in \mathcal{B}$. Such a search is not very effective, and requires a lot of checking. For simple $\mathcal{B}$'s, however, it does yield the lists fairly smoothly.

A more effective procedure remains an open problem.

5.1.2. Note that in our case an object is core iff

$$Rx = Ry \quad \Rightarrow \quad x = y.$$

5.2. Some particular graphs. We will use the following symbols for particular graphs (here, "ij" indicates that "both (i,j) and (j,i) are in the relation")

- $K_n = (\{0,1,\dots,n-1\}, \{ij \mid i \neq j\})$ is the complete graph with n vertices,
- P_n is the n-path $(\{0,1,\dots,n\}, \{01,12,\dots,(n-1)n\})$,
- C_n is the n-cycle $(\{0,1,\dots,n-1\}, \{01,12,\dots,(n-1)0\})$,
- $Y = (\{0,1,2,3\}, \{01,12,23,13\})$,
- $T = (\{0,1,2,3,4,5\}, \{01,12,23,34,25\})$,
- $A = (\{0,1,2,3,4,5\}, \{01,12,23,34,45,14\})$,
- and $B = (\{0,1,2,3,4,5\}, \{01,12,23,34,45,14,05\})$.

5.3. Lemma. *Every Ramsey list in* $\mathbf{ConnGraph}_{\mathsf{full}}$ *contains a complete graph K_n and a path P_m.*

Proof. All complete graphs are core. Hence some of the A_i has to exclude a complete graph K_k. Thus, $A_i \to K_k$ and hence $A_i = K_n$ since all subgraphs of a complete graph are complete.

Similarly with the paths, where all *connected* subgraphs of paths are paths, and the only one that is not core is P_2. $\square$

5.3.1. Corollary. *In* $\mathbf{ConnGraph}_{\mathsf{full}}$*, the only one-element Ramsey lists are $\{K_1\}$ (=$\{P_0\}$) and $\{K_2\}$ (=$\{P_1\}$).*

5.4. Proposition. *There are only two two-element Ramsey lists in* $\mathbf{ConnGraph}_{\mathsf{full}}$*, namely $\{K_3, P_3\}$ and $\{K_3, P_4\}$.*

Proof. By 5.3, a two element list is a $\{K_n, P_m\}$ with $n, m \geq 3$. Consider the graphs

$$S_k = (\{a, b_i, c_i \mid i = 1,\dots,k\}, \{ab_i, ac_i, b_ic_i \mid i = 1,\dots,k\})$$

where $a, b_1, c_1, b_2, c_2, \dots$ are distinct elements. S_n are core and infinitely many, and if $n \geq 3$ and $m \geq 4$ we have $K_n, P_m \nrightarrow S_k$. Thus, $\{K_3, P_3\}$ and $\{K_3, P_4\}$ are the only alternatives left. The first is dual to $\{P_1\}$ and the second to $\{P_3, A\}$ which is easy to check. $\square$

5.5. While by 4.3 for every finite $\mathcal{B}$ there is a finite $\mathcal{A}$ such that $\mathcal{A} \nrightarrow X$ iff $X \to \mathcal{B}$, the reverse does not hold, and indeed the finite $\mathcal{A}$ for which we can have a finite $\mathcal{B}$ to form a duality are rare.

Still, we have infinitely many three-element Ramsey lists.

Proposition. *We have the duality in* $\mathbf{ConnGraph}_{\mathsf{full}}$

$$\{K_{n+1}, P_3, Y\} \nrightarrow X \quad \textit{iff} \quad X \rightarrow K_n.$$

Proof. Let $M \rightarrow K_n+$ be minimal (core) such that $M \nrightarrow K_n$. Define M_i, $i = 0, 1$ by setting

$$M_i \times \{i\} = M \cap (K_n \times \{i\})$$

(thus, the set of vertices of M is $(M_0 \times \{0\}) \cup (M_1 \times \{1\}) \cup \{\omega\})$.

I. Let $M_0 = \emptyset$. Then $M_1 = K_n$ and $M \cong K_{n+1}$ (else $M \cong K_k$ with $k \leq n$ and $M \rightarrow K_n$).

If $M_0 \neq \emptyset$ then $M_1 \neq \emptyset$ as well, by connectedness.

II. Let $M_0 = \{x\}$. Then we cannot have $M_0 \cap M_1 = \emptyset$ since otherwise $x \sim \omega$ and M is not core. Thus, $x \in M_1$ and by connectedness there has to be another $y \in M_1 \setminus \{x\}$ and there is $x0, y1, x1, \omega$ isomorphic to Y.

III. Let $|M_0| \geq 2$. If there exist distinct x, y, z with $x, y \in M_0$ and $z \in M_1$ we have $x0, y0, z1, \omega$ isomorphic to Y.

Thus, we are left with $M_0 = \{x, y\} \supseteq M_1 \neq \emptyset$, $x \neq y$, say, $x \in M_1$. Then we have the path $x0, y0, x1, \omega$. $\square$

5.6. Lemma. *Every connected graph that contains C_4, that does not contain C_3, and that is core contains A or B (recall 5.2).*

Proof. Represent the 4-cycle as $(\{1, 2, 3, 4\}, \{12, 23, 34, 41\})$. One of the vertices $1, 3$, say 1, has to be connected with an x not connected with the other, and to avoid a triangle, it cannot be connected with 2 and 4 either. Similarly we can assume (by symmetry) a y connected just with with 2. We cannot have $x = y$ in which case there would be a triangle. Now if x and y are not connected we have A, if they are we have B. $\square$

5.7. Lemma. *Every tree that is not core is either a path or T.*

Proof. If it is not a path then there is a vertex x with degree at least three. If two of its neighbours were leaves, the would be equivalent, and our tree would not be core. $\square$

5.8. Proposition. *For paths we have the dualities*

$$\{P_4, C_3, A, C_5\} \nrightarrow X \quad \textit{iff} \quad X \rightarrow P_3,$$

and for $n \geq 4$,

$$\{P_{n+1}, T, C_3, A, B, C_5, \dots, C_{n+2}\} \nrightarrow X \quad \textit{iff} \quad X \rightarrow P_n.$$

Proof. $G \nrightarrow P_n$ if and only if it either contains a cycle or is a tree that cannot be mapped into P_n. Since $P_{n+1} \nrightarrow P_n$ and $P_{n+1} \rightarrow C_k$

for $k \geq n+3$, the C_k with $k \geq n+3$ are not minimal. This, together with Lemma 5.6 (for the case of a 4-cycle) accounts for the $C_3, A, C_5, \dots, C_{n+2}$ part of the left hand side (all the $C_3, A, C_5, \dots, C_{n+2}$ indeed are induced subgraphs of P_n+; B contains P_4 which will be excluded next).

It remains to determine the acyclic minimal $G \nrightarrow P_n$. There is, of course, P_{n+1}, and the only remaining candidate is T, by lemma . Now T does not fit into P_3+ (and even if fitted, one has $P_4 \to T$ and hence it would not be minimal anyway), but fits into any P_n+ with $n > 3$. □

5.9. By exactly the same reasoning we obtain

Proposition. *For cycles we have the dualities*

$$\{P_4, C_3, A\} \nrightarrow X \quad \textit{iff} \quad X \to C_5,$$

and for $n \geq 6$,

$$\{P_{n-1}, T, C_3, A, B, C_5, \dots, C_{n-1}\} \nrightarrow X \quad \textit{iff} \quad X \to C_n.$$

5.10. Remarks. 1. Note the similarities of the "left duals" of the paths and the cycles. Compare for instance the dualities

$$\{P_5, T, C_3, A, B, C_5, C_6\} \nrightarrow X \quad \text{iff} \quad X \to P_4$$

and

$$\{P_6, T, C_3, A, B, C_5, C_6\} \nrightarrow X \quad \text{iff} \quad X \to C_7.$$

2. In the cycles we have started with the C_5 (anomalous by the absence of T) and proceeded with the more regular C_n, $n \geq 6$, in analogy with the equally anomalous P_3 proceeded by the equally regular P_n, $n \geq 4$.

We have the extra cases of $n = 3, 4$. Now C_3 has been dealt with in 5.5, since $C_3 = K_3$, and we could say that C_4 is of no interest since it is not core. The latter is, however, just trying to escape the tedious analysis of $X \to A$ and $X \to B$: indeed, in all the formulas above, A is really the way to treat (and prohibit) the four-cycles (see 5.6), and should be viewed as such.

3. The duality of $X \to C_5$ appeared as one of the characteristics of monochromes in exact Gallai cliques in [3].

5.11. Another example. Tedious checking of the subgraphs of $A+$ (A from 5.2) yields the duality

$$\{P_4, C_3, C_5, E\} \nrightarrow X \quad \text{iff} \quad X \to A.$$

E stands for "exotic". It is

$$(\{0,1,2,3,4,5,6,7\},\{01,12,23,34,45,14,17,26,46,67\}),$$

a relatively complex graph (in this context).

5.11.1. Remark. This example indicates that even in simple cases the listings are not always quite easy. But there is also another important phenomenon. In all the previous cases, the objects of $\mathcal{N}_0(B)$ had at most $|B|+1$ vertices. Here we have eight vertices to the B's six, showing that the size of the critical graphs can increase by more. The estimate of the sizes of the $A \in \mathcal{N}_0(B)$ in terms of $|B|$ seems to be an interesting problem.

5.12. In the larger category $\mathbf{Graph}_{\mathsf{full}}$ the system $\mathcal{N}_0(K_n)$ is simpler than that of 5.5. It contains an element smaller than both Y and P_3, namely

$$P_0 + P_1,$$

where $G+H$ indicates (and will indicate in the sequel) the categorical sum (here, the disjoint union) of the two graphs.

Consequently we obtain

Proposition. *In* $\mathbf{Graph}_{\mathsf{full}}$ *we have the dualities*

$$\{K_{n+1}, P_0+P_1\} \not\to X \quad \textit{iff} \quad X \to K_n.$$

Thus, in contrast with Proposition 5.4, if we consider disconnected graphs, there are infinitely many two-element proper Ramsey lists.

5.13. Duals of paths in $\mathbf{Graph}_{\mathsf{full}}$. While admitting disconnected graphs simplified the dual Ramsey lists of the complete graphs, in the case of the paths the situation gets rather more complex. Let us see what happens.

The $\dots, T, C_3, A, B, C_5, \dots, C_{n+2}$ part of the Ramsey list from 5.8 remains intact: each proper subgraph of any of the graphs, connected or not, can be mapped into P_n (for the case with $n \geq 6$; for the shorter paths, the $P_0+P_1+P_1$ contained in T has to be discussed extra). Thus, we have to analyze the (possibly disconnected) $M \subseteq P_{n+1}$ minimal with respect to the property $M \not\to P_n$.

We have the following obvious observations:

5.13.1. • *both of the endpoints of P_{n+1} are in A, and no two of the vertices in $P_{n+1} \setminus A$ are neighbours (else we obtain a subgraph of P_n),*

• *none of the resulting connected intervals is isomorphic to P_2 (else the resulting A could be mapped into P_n),*

- *at most one of the resulting connected intervals consist of a single point,*
- *and the connected intervals constituting A can be arbitrarily permuted.*

Denote by

$$\mathcal{S}(n)$$

the collection of the (isomorphism types of) the $M \subseteq P_n$ minimal with respect to the property $M \nrightarrow P_{n-1}$ (such M's will be represented by means of sums of paths), and by

$$\mathcal{S}_0(n) \quad \text{resp.} \quad \mathcal{S}_1(n)$$

the sets of the elements of $\mathcal{S}(n)$ containing resp. not containing the summand P_0.

Further denote by

$$\mathcal{S}^{\square}(n)$$

the collection of the $M \subseteq P_n$ minimal with respect to the combined property

$$M \nrightarrow P_{n-1} \quad \text{and} \quad M \text{ has not } P_0 \text{ for a summand.}$$

Note that $\mathcal{S}^{\square}(n)$ is typically bigger than $\mathcal{S}_1(n)$: for instance we have

$$P_3 \in \mathcal{S}^{\square}(3),\ P_5 \in \mathcal{S}^{\square}(5)$$

but not in $\mathcal{S}_1(3)$ resp. $\mathcal{S}_1(5)$.

¿From 5.13.1 we easily infer that (if n is sufficiently large)

$$\mathcal{S}(n) = (P_0 + \mathcal{S}^{\square}(n-2)) \cup (P_1 + \mathcal{S}_1(n-3)),$$
$$\mathcal{S}^{\square}(n) = (P_1 + \mathcal{S}^{\square}(n-3)) \cup (P_3 + \mathcal{S}^{\square}(n-5)) \cup (P_5 + \mathcal{S}^{\square}(n-7))$$

(where $P + \mathcal{S}$ stands for $\{P + S \mid S \in \mathcal{S}\}$).

> **Note.** In the second formula one stops with the third summand since all the P_k with $k \geq 6$ already contain non-trivial sums without P_0. In fact, it seems that for n sufficiently large one obtains all the cases already in the first summand (the other two containing just repetitions).

As examples we can now compute the $\mathcal{S}(n)$ for small n (kG indicates $\overbrace{G+\cdots+G}^{n-\text{times}}$). An easy checking yields:

$$
\begin{aligned}
&\mathcal{S}(1)=\{P_1\}=\mathcal{S}_1(1)=\mathcal{S}^{\square}(1),\ \mathcal{S}_0(n)=\emptyset,\\
&\mathcal{S}(2)=\emptyset=\mathcal{S}_0(2)=\mathcal{S}_1(2)=\mathcal{S}^{\square}(2),\\
&\mathcal{S}(3)=\{P_0+P_1\}=\mathcal{S}_0(3),\ \mathcal{S}_1(3)=\emptyset,\ \mathcal{S}^{\square}(3)=\{P_3\},\\
&\mathcal{S}(4)=\{2P_1\}=\mathcal{S}_1(4)=\mathcal{S}^{\square}(4),\ \mathcal{S}_0(4)=\emptyset,\\
&\mathcal{S}(5)=\{P_0+P_3\}=\mathcal{S}_0(5),\ \mathcal{S}_1(5)=\emptyset\ \mathcal{S}^{\square}(5)=\{P_5\},\\
&\mathcal{S}(6)=\{P_0+2P_1\}=\mathcal{S}_0(6),\ \mathcal{S}_1(6)=\emptyset,\ \mathcal{S}^{\square}(6)=\{P_1+P_3\},\\
&\mathcal{S}(7)=\{P_0+P_5,\ 3P_1\},\ \mathcal{S}_0(7)=\{P_0+P_5\},\ \mathcal{S}_1(7)=\mathcal{S}^{\square}(7)=\{3P_1\}.
\end{aligned}
$$

Further we can proceed by the formulas above

$$
\begin{aligned}
\mathcal{S}(8)\quad &=\{P_0+P_1+P_3\},\\
&\mathcal{S}^{\square}(8)=\{P_1+P_5,2P_3\},\\
\mathcal{S}(9)\quad &=\{P_0+3P_1\},\\
&\mathcal{S}^{\square}(9)=\{2P_1+P_3\},\\
\mathcal{S}(10)\quad &=\{P_0+P_1+P_5,\ P_0+2P_3,\ 4P_1\},\\
&\mathcal{S}^{\square}(10)=\{4P_1,\ P_3+P_5\},\\
\mathcal{S}(11)\quad &=\{P_0+2P_1+P_3\},\\
&\mathcal{S}^{\square}(11)=\{2P_1+P_5,\ P_1+2P_3\},\\
\mathcal{S}(12)\quad &=\{P_0+4P_1,\ P_0+P_3+P_5\},\\
&\mathcal{S}^{\square}(12)=\{3P_1+P_3,\ 2P_5\},\\
\mathcal{S}(13)\quad &=\{P_0+2P_1+P_5,\ P_0+P_1+2P_3,\ 5P_1\},\\
&\mathcal{S}^{\square}(13)=\{5P_1,\ P_1+P_5,\ 3P_3\},\\
\mathcal{S}(14)\quad &=\{P_0+3P_1+P_3,P_0+2P_5\},\\
&\mathcal{S}^{\square}(14)=\{3P_1+P_5,\ 2P_1+2P_3\},\\
\mathcal{S}(15)\quad &=\{P_0+5P_1,\ P_0+P_1+P_3+P_5,\ P_0+3P_3\}
\end{aligned}
$$

etc. Thus, the resulting Ramsey lists corresponding to the paths do not seem to be more transparent than those in the connected case.

Note. After this paper was written we learned that some related results for graphs were independently obtained by P. Hell and its collaborators. See [6] for a survey of these results.

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