Optimal real number graph labelings of a subfamily of Kneser graphs*

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Abstract

A notion of real number graph labelings captures the dependence of the span of an optimal channel assignment on the separations that are required between frequencies assigned to close transmitters. We determine the spans of such optimal labelings for a subfamily of Kneser graphs formed by the complements of the line graphs of complete graphs. This subfamily contains (among others) the Petersen graph.

1 Introduction

Distance constrained labelings form a well-established graph theory model for the channel assignment problem. Since their introduction by Griggs and Yeh [23] in 1992 they attracted a considerable amount of interest of researchers. One can find papers on their structural aspects as well as on algorithms computing optimal or near-optimal labelings. Let us recall this notion: an $L(p_1, \dots, p_k)$*-labeling* of a graph G for non-negative integers $p_1, \dots, p_k$ is

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a labeling c of its vertices with non-negative integers such that the labels of any two vertices at distance j, $1 \le j \le k$, differ by at least p_j (p_j represents the required separation of the frequencies needed to avoid their interference). A labeling satisfying these properties is also called *proper*, and the maximal label used by a labelling is called the *span*. The smallest span of a proper $L(p_1, \ldots, p_k)$-labeling of G is denoted by $\lambda_{p_1,\ldots,p_k}(G)$.

Because of practical applications [24], the most studied distance constrained labelings are those with $k = 2$ and among these those with $p_1 = 2$ and $p_2 = 1$. A famous conjecture of Griggs and Yeh [23] asserts that every graph with maximum degree $\Delta \ge 2$ has an $L(2,1)$-labeling with span at most Δ^2. In their paper [23], Griggs and Yeh proved that every such graph has an $L(2,1)$-labeling with span at most $\Delta^2 + 2\Delta$. In a series of papers [10, 31, 17], this bound has been decreased successively to the current best bound $\Delta^2 + \Delta - 2$ of Gonçalves [17]. Let us also remark that the conjecture is known to be true for several special classes of graphs, such as graphs of maximum degree two, chordal graphs ([36], see also [11, 29]), Hamiltonian cubic graphs [25] and planar graphs with maximum degree $\Delta \ne 3$ [3]. There is a substantial body of results in this area which we are not able to survey here, but let us mention as examples the papers [4, 5] on real-world applications of this kind of labelings, the papers [33, 35] on distance constrained labelings of planar graphs and the papers [1, 6, 13, 14, 28, 34] on their algorithmic aspects.

Since the separations $p_1, \ldots, p_k$ might sometimes need to be determined ad hoc, it is interesting to study the optimal span $\lambda_{p_1,\ldots,p_k}(G)$ as the function of $p_1, \ldots, p_k$. This approach has been formalized by Griggs and Jin [18] and subsequently generalized to a more general notion of lambda graphs [2, 30]. Because of the kind of a problem that we address, it is enough for us to stay in the framework defined in [18] and thus we will not introduce the more general notion of lambda graphs.

In the setting proposed by Griggs and Jin [18], both the separations $p_1, \ldots, p_k$ and the labels of the vertices of G are non-negative reals. The smallest span of an optimal $L(p_1, \ldots, p_k)$ is then denoted by $\lambda(G; p_1, \ldots, p_k)$ to emphasize its dependence on the parameters $p_1, \ldots, p_k$. It is easy to see that $\lambda(G; p_1, \ldots, p_k)$ is a continuous function of $p_1, \ldots, p_k$. Griggs and Jin [19] also established other basic properties of $\lambda(G; p_1, \ldots, p_k)$ such as the scaling property, i.e., $\lambda(G; \alpha p_1, \ldots, \alpha p_k) = \alpha\lambda(G; p_1, \ldots, p_k)$ for every $\alpha \ge 0$, and that the function $\lambda(G; p_1, \ldots, p_k)$ is piecewise linear. In [30], it has been shown that the function $\lambda(G; p_1, \ldots, p_k)$ has only finitely many linear parts even if the graph G is infinite. The reader is referred to the survey [22] for a more comprehensive introduction to the subject.

In the fundamental case $k = 2$, the function $\lambda(G; p_1, p_2)$ is determined by its values for $p_2 = 1$ and thus it can also be viewed as a one-parameter function $\lambda(G; x, 1)$. Note that $\lambda(G; 1, 0) = \lim_{x\to\infty} \lambda(G; x, 1)/x$. We often refer to $\lambda(G; x, 1)$ as to the *lambda function* of the graph G. Note that $\lambda(G; 1, 0) = \chi(G) - 1$ and $\lambda(G; 1, 1) = \chi(G^2) - 1$. Since the problem of determining the lambda function of a graph includes determining its chromatic number as well as the chromatic number of its square, it is not surprising that the lambda functions are determined only for graph classes formed by well-structured graphs. Such graph classes include paths and cycles [15, 19], wheels [15, 19], or complete bipartite graphs [21]. Because of practical applications, the lambda functions of some particular infinite graphs are also known, for instance, the lambda functions of the infinite square and hexagonal planar grids have been determined [20, 8] and that of the triangular grid has been found [20, 8] for most of the values of x. Another class of graphs for which the lambda functions are known are infinite regular trees [16, 9]. In this paper, we extend this list of results by determining the lambda functions of some Kneser graphs. Notice that determining the lambda functions for all Kneser graphs would include determining the chromatic number of squares of Kneser graphs, a problem posed by Füredi, with only few partial results [12, 26].

Recall that a *Kneser graph* $K(n, k)$, $n > 2k$, is the graph with $\binom{n}{k}$ vertices defined as follows: each vertex is associated with a k-element subset of a base n-element set and two vertices are adjacent if the sets corresponding to them are disjoint. Note that the Kneser graph $K(5, 2)$ is the Petersen graph. The problem of determining the chromatic number of Kneser graphs is a well-known problem which was solved by Lovász using topological tools [32] (see also [27]) who showed the chromatic number of $K(n, k)$ is equal to $n-2k+2$. On the other, the chromatic number of squares of Kneser graphs is not known and there is even no conjecture on its value.

The problem of determining the lambda functions of all Kneser graphs includes the problems on the chromatic number of them as well as their squares as special cases; in this paper, we determine the lambda functions of all Kneser graphs $K(n, 2)$ with $n \geq 5$ (Theorem 8). As examples, the lambda functions of Kneser graphs $K(n, 2)$, $n = 5, 6, 7, 8$, can be found in Figure 1.

Before we start the exposition of our results, let us observe several properties of $K(n, 2)$. The graph $K(n, 2)$ is isomorphic to the complement of the

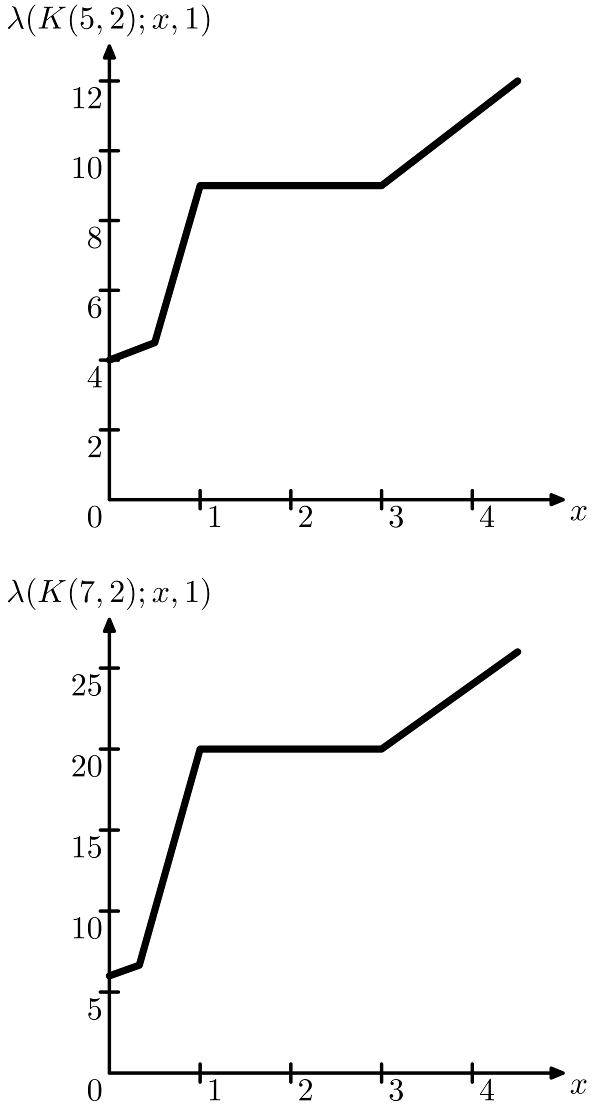

Figure 1: The lambda functions of the Kneser graphs $K(5,2)$, $K(6,2)$, $K(7,2)$ and $K(8,2)$.

line graph of the complete graph K_n: indeed, each vertex of $K(n,2)$ can be associated with an edge of K_n. Such an edge naturally corresponds to a two-element subset of an n-element set (which is formed by the vertices of K_n). Two such vertices of $K(n,2)$ are adjacent if and only if the corresponding sets are disjoint. The latter is equivalent to the fact that they are not adjacent in the line graph of K_n. It turned out to be very handful in our considerations to associate $n-1$ vertices of K_n with the numbers $0,\dots,n-2$ and the last one with the star, $*$. This distinction will be useful in most of our considerations.

A corollary of the just observed correspondence of $K(n,2)$ with the line graph of K_n is that the maximum clique size of $K(n,2)$ is $\lfloor n/2 \rfloor$ (the size of a maximum matching of K_n). Another fact that will be useful in our work is that the square of $K(n,2)$ is complete for all $n \geq 5$, i.e., the distance between any two vertices of $K(n,2)$ is at most two. In particular, $\lambda(K(n,2);1,1) = \chi(K(n,2)^2) - 1 = \binom{n}{2} - 1$.

2 Small values of x

In this section, we establish auxiliary lemmas that will be useful in determining the lambda function $\lambda(K(n,2);x,1)$ of Kneser graphs $K(n,2)$ for small values of x. We start with two lemmas asserting that vertices of Kneser graphs $K(n,2)$ can be ordered in such a way that large segments of consecutive vertices form cliques in $K(n,2)$. We first consider the case when n is odd.

Lemma 1. *For every $\ell \geq 2$, there exists an order $v_1,\dots,v_m$, $m = \binom{2\ell+1}{2}$, of the vertices of $K(2\ell+1,2)$ such that for every $i = 1,\dots,m-(\ell-1)$ the vertices $v_i,\dots,v_{i+\ell-1}$ form a clique of order ℓ in $K(2\ell+1,2)$.*

Proof. We construct an order of the vertices of $K(2\ell+1,2)$ that can be partitioned into ℓ blocks, $B_0,\dots,B_{\ell-1}$, each formed by $2\ell+1$ consecutive vertices. The first vertex $v_{k(2\ell+1)+1}$ of B_k is the vertex corresponding to the set $\{*,k\}$ (see Figure 2), the i-th vertex $v_{k(2\ell+1)+i}$, $i = 2,\dots,\ell$, is the vertex corresponding to the set $\{(-(i-1)+k) \bmod 2\ell, ((i-1)+k) \bmod 2\ell\}$, the $(\ell+1)$-th vertex $v_{k(2\ell+1)+\ell+1}$ is the vertex corresponding to the set $\{*,k+\ell\}$, and the i-th vertex $v_{k(2\ell+1)+i}$, $i = \ell+2,\dots,2\ell+1$, is the vertex corresponding to the set $\{((i-\ell-1)+k) \bmod 2\ell, (-(i-\ell-2)+k) \bmod 2\ell\}$.

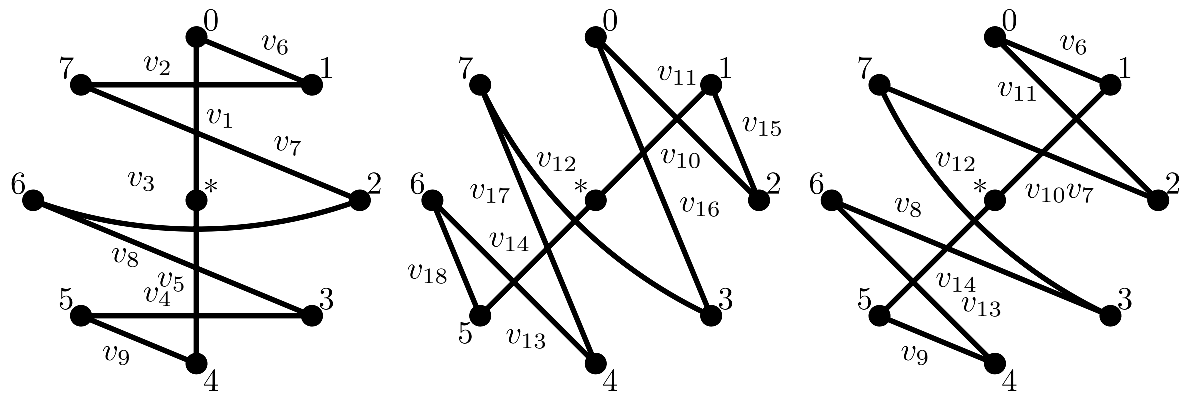

Figure 2: The edges corresponding to the vertices $v_1, \dots, v_9$ contained in the block B_1 and the vertices $v_{10}, \dots, v_{18}$ contained in the block B_2 of the order of the vertices of $K(9,2)$ constructed in the proof of Lemma 1 (the left and the middle part of the figure). In the right part of the figure, there is depicted the subgraph formed by the last four edges of the first block and the first five edges of the second block.

The constructed order of the vertices has a nice interpretation in the edge representation of Kneser graphs. In the drawing of a complete graph with the star vertex in the middle and the numbered vertices on a cycle around the star vertex, each block B_i corresponds to a Hamilton cycle and each ℓ consecutive edges of B_i form a matching as depicted in Figure 2. Consecutive blocks are then obtained by rotating the Hamilton cycle by $360/(2\ell)$ degrees clockwise.

We now verify that any ℓ consecutive vertices form a clique of $K(2\ell+1, 2)$, i.e., the sets corresponding to them are mutually disjoint. The easiest way to see this is to use the correspondence with Hamilton cycles of $K_{2\ell+1}$ explained in the previous paragraph. Observe that the last ℓ edges of each block B_i,$i = 0, \dots, \ell-2$, with the first $\ell+1$ edges of the block B_{i+1} also form a Hamilton cycle and, moreover, any ℓ consecutive edges out of these form a matching (see Figure 2). Since any ℓ consecutive edges lie in a single block or in two blocks B_i and B_{i+1} for some $i = 0, \dots, \ell-2$, we conclude that any ℓ edges that are consecutive in the constructed order form a matching in the complete graph and thus the corresponding vertices form a clique of $K(2\ell+1, 2)$. The statement of the lemma now follows. $\square$

Next, we focus on the Kneser graphs $K(n, 2)$ with even n.

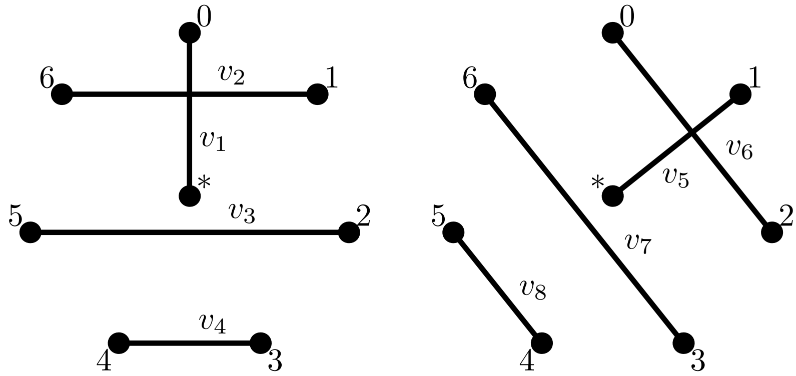

Figure 3: The edges corresponding to vertices of the first and the second block of the order constructed for $K(8,2)$ in the proof of Lemma 2.

Lemma 2. *For every $\ell \geq 2$, there exists an order $v_1, \ldots, v_m$, $m = \binom{2\ell+2}{2}$, of the vertices of $K(2\ell+2,2)$ such that for every $i = 1, \ldots, m-(\ell-1)$ the vertices $v_i, \ldots, v_{i+\ell-1}$ form a clique of order ℓ in $K(2\ell+2,2)$ and for every $i = 1, \ell+2, 2\ell+3, \ldots, m-\ell$, the vertices $v_i, \ldots, v_{i+\ell}$ form a clique of order $\ell+1$.*

Proof. We construct an order of the vertices of $K(2\ell+2,2)$ that consists of $2\ell+1$ blocks of $\ell+1$ vertices each. The blocks are numbered from 0 to 2ℓ. The first vertex $v_{k(\ell+1)+1}$ of the k-th block is the vertex corresponding to the set $\{*, k\}$ and the i-th vertex $v_{k(\ell+1)+i}$, $i = 2, \ldots, \ell+1$, is the vertex corresponding to the set $\{(-(i-1)+k) \bmod (2\ell+1), ((i-1)+k) \bmod (2\ell+1)\}$. Clearly, each block forms a clique of order $\ell+1$ as required.

The constructed order of the vertices has also a nice interpretation in the edge representation of Kneser graphs. In the drawing of a complete graph with the star vertex in the middle and the numbered vertices on a cycle around the star vertex, each block corresponds to a perfect matching consisting of an edge joining the star vertex to one of the numbered vertices and all the edges perpendicular to this edge (see Figure 3). Consecutive blocks are then obtained by rotating the perfect matching by $360/(2\ell+1)$ degrees clockwise.

We now verify that any ℓ consecutive vertices form a clique of $K(2\ell+2,2)$, i.e., the sets corresponding to them are mutually disjoint. Let us consider a vertex $v_{k(\ell+1)+i}$, $k = 0, \ldots, 2\ell$ and $i = 1, \ldots, \ell+1$. The set corresponding to it is disjoint from all the sets corresponding to the vertices within the same block, i.e., the sets corresponding to $v_{k(\ell+1)+i'}$ for $i' = 1, \ldots, \ell+1$ and $i' \neq i$.

If $k > 0$, the last vertex of the previous block whose set is *not* disjoint from the set corresponding to $v_{k(\ell+1)+i}$ is the vertex $v_{(k-1)(\ell+1)+i+1}$ if $i < \ell+1$ (we do not have to analyze the case $i = \ell+1$). Similarly, if $k < 2\ell+1$, the first vertex of the next block whose set is not disjoint from the set corresponding to $v_{k(\ell+1)+i}$ is the vertex $v_{(k-1)(\ell+1)+i-1}$ if $i > 0$. Hence, the set corresponding to $v_{k(\ell+1)+i}$ is disjoint from all the sets corresponding to the vertices $v_{k(\ell+1)+i+j}$ for $j = \pm 1, \ldots, \pm(\ell-1)$. The claim of the lemma now follows. $\square$

In the rest of this section, we focus on establishing lower bounds on the lambda function of $K(n,2)$ for $x \in \langle 0, 1/\ell \rangle$. Again, we have to distinguish two cases with respect to the parity of n. Let us start with the odd case.

Lemma 3. *The following estimate holds for every $\ell \geq 2$ and $x \in \langle 0, 1/\ell \rangle$:*

$$\lambda(K(2\ell+1,2); x, 1) \geq 2\ell + (\ell-1)x \ .$$

Proof. Let c be an $L(x,1)$-labeling of $K(2\ell+1,2)$ and let $v_1, \ldots, v_m$, $m = \binom{2\ell+1}{2}$, be an order of the vertices such that $c(v_1) \leq c(v_2) \leq \cdots \leq c(v_m)$.

We now show that $c(v_{i\ell+1}) \geq i$ for every $i = 0, \ldots, 2\ell$. The inequality clearly holds for $i = 0$. If $i > 0$, then there are two non-adjacent vertices among $v_{(i-1)\ell+1}, \ldots, v_{i\ell+1}$ since $K(2\ell+1,2)$ does not contain a clique of order $\ell+1$. Hence, the labels of these two vertices differ by at least one. We infer from $c(v_{(i-1)\ell+1}) \leq \cdots \leq c(v_{i\ell+1})$ that $c(v_{i\ell+1}) \geq c(v_{(i-1)\ell+1})+1 \geq (i-1)+1 = i$ and eventually conclude that $c(v_{2\ell^2+1}) \geq 2\ell$.

Since the difference of any two labels is at least x, the next estimate readily follows:

$$c(v_m) \geq c(v_{2\ell^2+1}) + \big(m - (2\ell^2+1)\big)\, x \geq 2\ell + (\ell-1)x \ .$$

This finishes the proof of the lemma. $\square$

We conclude this section with the lower bound on the lambda function of $K(n,2)$ with n even. This case turned out to be more complex than the case of odd n.

Lemma 4. *The following estimate holds for every $\ell \geq 2$ and $x \in \langle 0, 1/\ell \rangle$:*

$$\lambda(K(2\ell+2,2); x, 1) \geq 2\ell + 3\ell x \ .$$

Proof. Let c be an $L(x,1)$-labeling of $K(2\ell+2,2)$ and let $v_1,\ldots,v_m$, $m=\binom{2\ell+2}{2}$ be an order of the vertices such that $c(v_1)\le c(v_2)\le\cdots\le c(v_m)$.

We define recursively indices $i_0,\ldots,i_{2\ell}$. Set $i_0=1$. For $j>0$, let i_j be the largest index such that the vertices from the i_{j-1}-th to (i_j-1)-th one, i.e., $v_{i_{j-1}},\ldots,v_{i_j-1}$, form a clique. Since the maximal order of a clique of $K(2\ell+2,2)$ is $\ell+1$, $K(2\ell+2,2)$ cannot be vertex covered by less than $2\ell+1$ cliques, and thus all the indices $i_0,\ldots,i_{2\ell}$ are well-defined.

Let further C_j, $j=0,\ldots,2\ell-1$, be the clique of $K(2\ell+2,2)$ formed by the vertices $v_{i_j},\ldots,v_{i_{j+1}-1}$ and γ_j, $j=0,\ldots,2\ell$, be the number of cliques of order $\ell+1$ among $C_0,\ldots,C_{j-1}$. In particular, $\gamma_0=0$ and $\gamma_j\le j$.

We now prove the following claim:

Claim: *It holds that $c(v_{i_j})\ge j+\gamma_j x$ for every $j=0,\ldots,2\ell$.*

We proceed by the induction on j. The statement trivially holds for $j=0$. Hence, let us assume $j>0$. Let $k\in\{0,\ldots,|C_{j-1}|-1\}$ be the largest k such that the vertex $v_{i_{j-1}+k}$ of C_{j-1} is not adjacent to the vertex v_{i_j}. In addition, if the order of the clique C_{j-1} is $\ell+1$, then the edges of the complete graph corresponding to the vertices of C_{j-1} form a matching and the vertex v_{i_j} is not adjacent to at least two of the vertices of C_{j-1}. In particular, $k\ge 1$. Since the difference between the labels of any two vertices is at least x (recall that $x\le 1$) and v_{i_j} is not adjacent to $v_{i_{j-1}+k}$, we obtain:

$$c(v_{i_j})\ge c(v_{i_{j-1}+k})+1\ge c(v_{i_{j-1}})+1+kx\ge j+\gamma_{j-1}x+kx\ .$$

If $\gamma_j=\gamma_{j-1}$, we have obtained the desired inequality. Otherwise, the order of C_{j-1} is $\ell+1$, thus $k\ge 1$, and the inequality also follows. This finishes the proof of the claim.

The cliques $C_0,\ldots,C_{2\ell-1}$ contain at most $2\ell\cdot\ell+\gamma_{2\ell}$ vertices of $K(2\ell+2,2)$. Hence, $i_{2\ell}\le 2\ell^2+1+\gamma_{2\ell}$. The next estimate on $c(v_m)$ easily follows:

$$\begin{aligned}
c(v_m) &\ge c(v_{i_{2\ell}})+(m-i_{2\ell})\,x\\
&\ge 2\ell+\gamma_{2\ell}x+\big((2\ell^2+3\ell+1)-(2\ell^2+1+\gamma_{2\ell})\big)\,x\\
&\ge 2\ell+3\ell x\ .
\end{aligned}$$

The lemma is now established. $\square$

3 Large values of x

In this section, we prove lower and upper bounds on the lambda function of $K(n, 2)$ for $x \geq 1$. We start with the upper bound for $x \in \langle 1, 3 \rangle$.

Lemma 5. *For every $n \geq 5$ and $x \in \langle 1, 3 \rangle$, there exists an $L(x, 1)$-labeling of $K(n, 2)$ of span $\binom{n}{2} - 1$ such that all the assigned labels are (distinct) integers and the two largest labels are $\binom{n}{2} - 2$ and $\binom{n}{2} - 1$.*

Proof. The proof proceeds by induction on n. For $n = 5$, label the vertices of $K(5, 2)$ with labels $0, 1, \ldots, 9$ in the order in which they correspond to the following sets:

$$\{*, 0\}, \{*, 1\}, \{*, 2\}, \{*, 3\}, \{3, 2\}, \{3, 1\}, \{3, 0\}, \{0, 1\}, \{0, 2\} \text{ and } \{1, 2\} .$$

Since the vertices corresponding to the sets $\{0, 2\}$ and $\{1, 2\}$ are non-adjacent, the labeling also satisfies the additional assertion of the lemma.

Now assume that $n > 5$. By induction, there exists an $L(x, 1)$-labeling of the subgraph G of $K(n, 2)$ induced by the vertices corresponding to the sets not containing the star that has span $\binom{n-1}{2} - 1$ and two non-adjacent vertices of G are assigned the labels $\binom{n-1}{2} - 2$ and $\binom{n-1}{2} - 1$. By symmetry, we can assume that the vertices labeled with $\binom{n-1}{2} - 2$ and $\binom{n-1}{2} - 1$ correspond to the sets $\{0, 2\}$ and $\{0, 1\}$, respectively. Assign now the label $\binom{n-1}{2} + i$ to the vertex corresponding to the set $\{*, i\}$, $i = 0, \ldots, n - 2$. It is easy to verify that the labeling obtained in this way is an $L(x, 1)$-labeling of $K(n, 2)$ and there are two non-adjacent vertices, $\{*, n - 3\}$ and $\{*, n - 2\}$, labeled with the labels $\binom{n}{2} - 2$ and $\binom{n}{2} - 1$. □

Next, we prove the upper bound for $x \geq 3$.

Lemma 6. *For every $n \geq 5$ and $x \geq 3$, there exists an $L(x, 1)$-labeling of $K(n, 2)$ with span $(n-3)(x-3) + \binom{n}{2} - 1$ such that the three largest labels are $(n-3)(x-3) + \binom{n}{2} - 3$, $(n-3)(x-3) + \binom{n}{2} - 2$ and $(n-3)(x-3) + \binom{n}{2} - 1$.*

Proof. We proceed similarly as in the proof of Lemma 5. First, we label the vertices of $K(5, 2)$ with labels

$$0, 1, 2, 3, x + 1, x + 2, x + 3, 2x + 1, 2x + 2 \text{ and } 2x + 3$$

in the order in which they correspond to the sets

$$\{*, 0\}, \{*, 1\}, \{*, 2\}, \{*, 3\}, \{3, 2\}, \{3, 1\}, \{3, 0\}, \{0, 1\}, \{0, 2\} \text{ and } \{1, 2\} .$$

The labeling also satisfies the additional assertion of the lemma.

We assume $n > 5$ in the rest. Let us consider an $L(x,1)$-labeling of the subgraph of $K(n,2)$ induced by vertices corresponding to sets that do not contain the star such that three largest labels are $(n-4)(x-3)+\binom{n-1}{2}-3$, $(n-4)(x-3)+\binom{n-1}{2}-2$ and $(n-4)(x-3)+\binom{n-1}{2}-1$. By symmetry, we can assume that the label $(n-4)(x-3)+\binom{n-1}{2}-2$ is assigned to the vertex corresponding to the set $\{0,2\}$ and the label $(n-4)(x-3)+\binom{n-1}{2}-1$ to the one corresponding to $\{0,1\}$. Label now the vertex corresponding to the set $\{*,i\}$, $i=0,\ldots,n-2$, with $(n-3)(x-3)+\binom{n-1}{2}+i$. It is easy to verify that the labeling obtained in this way satisfies the assertion of the lemma. $\square$

We finish the section with establishing the lower bound for $x \geq 3$ that matches the upper bound shown in the previous lemma.

Lemma 7. *The following estimate holds for every $n \geq 5$ and $x \geq 3$:*

$$\lambda(K(n,2);x,1) \geq (n-3)(x-3)+\binom{n}{2}-1 \; .$$

Proof. Consider an $L(x,1)$-labeling c of $K(n,2)$ and let $v_1,\ldots,v_m$, $m=\binom{n}{2}$, be an order of the vertices such that $c(v_1) \leq c(v_2) \leq \cdots \leq c(v_m)$. Let i_1 be the largest index such that the vertices $v_1,\ldots,v_{i_1}$ form an independent set in $K(n,2)$, i_2 the largest index such that the vertices $v_{i_1+1},\ldots,v_{i_2}$ form an independent set, i_3 the largest index such that the vertices $v_{i_2+1},\ldots,v_{i_3}$ form an independent set, etc. Finally, let $A_j=\{v_{i_{j-1}+1},\ldots,v_{i_j}\}$ for $j=1,2,\ldots$, (setting $i_0=0$), and let k be the number of such sets A_j.

There are two types of independent sets A_j: those corresponding to stars in the edge representation of $K(n,2)$ and those corresponding to triangles. Let k_s be the number of sets of the former type. Since k_s vertices of a complete graph of order n are incident with $\frac{k_s(2n-1-k_s)}{2}$ edges and each independent set of the other type contains exactly three edges, we have the following bound on k:

$$k \geq k_s + \frac{\binom{n}{2}-\frac{k_s(2n-1-k_s)}{2}}{3} = \frac{k_s^2-(2n-7)k_s+n^2-n}{6} \; .$$

It is straightforward to verify using elementary tools from the mathematical analysis that the expression is minimized for $k_s \in \{n-4, n-3\}$. Hence, we can conclude that:

$$k \geq \frac{(n-3)^2-(2n-7)(n-3)+n^2-n}{6} = \frac{6n-12}{6} = n-2 \; .$$

We now prove the following claim:

Claim: *If the vertex v_i, $i \in \{1, \ldots, m\}$, is contained in A_j, then $c(v_i) \geq (j-1)(x-3) + i - 1$.*

We proceed by induction on i. Since $v_1 \in A_1$, the claim trivially holds for $i = 1$. Let us now consider a vertex v_i with $i > 1$. If v_{i-1} and v_i are contained in the same set A_j, then we have

$$c(v_i) \geq c(v_{i-1}) + 1 \geq (j-1)(x-3) + i - 1$$

since any two labels differ by at least one. Hence, we can assume in the rest that $v_{i-1} \in A_{j-1}$ and $v_i \in A_j$.

We claim that there exists $1 \leq i' \leq \min\{3, |A_{j-1}|\}$ such that the vertices $v_{i-i'}$ and v_i are adjacent (clearly, such $v_{i-i'} \in A_{j-1}$). If $|A_{j-1}| \leq 3$, the claim follows directly from the choice of the set A_{j-1}. On the other hand, if $|A_{j-1}| > 3$, then the edges corresponding to the vertices of A_{j-1} must form a star in the complete graph and the edge corresponding to v_i is incident to at most two such edges. Hence, v_i must be in $K(n, 2)$ adjacent to one of the vertices v_{i-3}, v_{i-2} and v_{i-1}.

Consider i' as defined in the previous paragraph. Since the vertices $v_{i-i'}$ and v_i are adjacent, their labels differ by at least $x \geq 3$ and we obtain the following bound on $c(v_i)$:

$$\begin{aligned} c(v_i) &\geq c(v_{i-i'}) + x = x + (j-2)(x-3) + (i-i') - 1 \\ &\geq (j-1)(x-3) + i - 1 + 3 - i' \geq (j-1)(x-3) + i - 1 \,. \end{aligned}$$

This finishes the proof of the claim.

Finally, since $c(v_m)$ is at least

$$(k-1)(x-3) + m - 1 \geq (n-3)(x-3) + \binom{n}{2} - 1 \,,$$

the statement of the lemma readily follows. □

4 Main result

We are now ready to prove our main theorem.

Theorem 8. *The lambda functions of the graphs $K(2\ell+1,2)$ and $K(2\ell+2,2)$ for every $\ell \geq 2$ are the following:*

$$\lambda(K(2\ell+1,2);x,1) = \begin{cases} 2\ell + (\ell-1)x & \text{for } x \in \langle 0, 1/\ell), \\ (2\ell^2+\ell-1)x & \text{for } x \in \langle 1/\ell, 1), \\ 2\ell^2+\ell-1 & \text{for } x \in \langle 1, 3), \\ (2\ell-2)x + 2\ell^2 - 5\ell + 5 & \text{for } x \geq 3, \end{cases}$$

and

$$\lambda(K(2\ell+2,2);x,1) = \begin{cases} 2\ell + 3\ell x & \text{for } x \in \langle 0, 1/\ell), \\ (2\ell^2+3\ell)x & \text{for } x \in \langle 1/\ell, 1), \\ 2\ell^2+3\ell & \text{for } x \in \langle 1, 3), \text{ and} \\ (2\ell-1)x + 2\ell^2 - 3\ell + 3 & \text{for } x \geq 3. \end{cases}$$

Proof. We first determine the lambda functions for $x \geq 1$. Let m be the number of vertices of $K(n,2)$ where n is equal to $2\ell+1$ or $2\ell+2$. Since $n \geq 5$, the distance between any two vertices of $K(n,2)$ is at most two. Hence, $\lambda(K(n,2);x,1) \geq m-1$. This simple lower bound and Lemma 5 determine the lambda function for $x \in \langle 1, 3\rangle$. For $x \geq 3$, the matching upper and lower bounds are given in Lemmas 6 and 7.

Next, we proceed separately for the cases of $K(2\ell+1,2)$ and $K(2\ell+2,2)$. Let us start with determining the rest of the lambda function of $K(2\ell+1,2)$. Note that $m = \binom{2\ell+1}{2}$ on this case. If $x \in \langle 0, 1/\ell)$, the value of $\lambda(K(2\ell+1,2);x,1)$ is at least $2\ell + (\ell-1)x$ by Lemma 3. For the upper bound, we consider the order of the vertices $v_1, \ldots, v_m$ as described in Lemma 1 and assign the vertex v_i the label $(i-1)x + \lfloor (i-1)/\ell \rfloor (1-\ell x)$. The labeling can be interpreted as follows: the vertices in the order form ℓ cliques of order $2\ell+1$ and the label of the i-th vertex of the k-th clique is $k + (i-1)x$.

We now argue that the obtained labeling is proper. Consider two vertices v_i and v_j, $1 \leq i < j \leq m$. If $j - i < \ell$, then the vertices v_i and v_j are adjacent by the choice of our order. Since their labels differ by at least x, the edge $v_i v_j$ is properly labeled. Otherwise, $j - i \geq \ell$ and the labels of v_i and v_j differ by at least one, thus they are properly labelled. Since

$$c(v_m) = (m-1)x + \left\lfloor \frac{m-1}{\ell} \right\rfloor (1-\ell x) = (2\ell^2+\ell-1)x + 2\ell(1-\ell x) = 2\ell + (\ell-1)x,$$

the upper bound follows.

Suppose that $x \in \langle 1/\ell, 1)$. Since the labels of any two vertices must differ by at least x,

$$\lambda(K(2\ell+1,2); x, 1) \geq (m-1)x = (2\ell^2+\ell-1)x \ .$$

For the upper bound, we consider the order $v_1, \ldots, v_m$ as in Lemma 1 and assign the vertex v_i the label $(i-1)x$. The arguments that this labeling is proper follow those presented in the previous paragraph.

It remains to determine the rest of the lambda function of $K(2\ell+2,2)$. Recall that $m = \binom{2\ell+2}{2}$ in this case. If $x \in \langle 0, 1/\ell)$, the value of $\lambda(K(2\ell+2,2); x, 1)$ is at least $2\ell+3\ell x$ by Lemma 4. For the upper bound, we consider the order of the vertices $v_1, \ldots, v_m$ as described in Lemma 2 and assign the vertex v_i the label $(i-1)x + \lfloor (i-1)/(\ell+1) \rfloor (1-\ell x)$. We can interpret the labeling with the aim of the proof of Lemma 2 as follows: the label of the i-th vertex of the block B_k (as defined in the proof) is $k(1+x)+(i-1)x$.

We now analyze the obtained labeling. The labels of two vertices v_i and v_j, $1 \leq i < j \leq m$, differ by at least one if $j-i > \ell$. If $j-i < \ell$, the vertices v_i and v_j are adjacent by the choice of the order and since their labels differ by at least x, the vertex corresponding to v_iv_j is properly labelled. If $j-i=\ell$ and $i = 1 \bmod (\ell+1)$, the vertices v_i and v_j are also adjacent and thus the edge v_iv_j is also properly labeled. Finally, if $i \neq 1 \bmod (\ell+1)$, the labels of v_i and v_j differ by at least one. We conclude that the labeling is a proper $L(x,1)$-labeling of $K(2\ell+1,2)$. Since

$$c(v_m) = (m-1)x + \left\lfloor \frac{m-1}{\ell+1} \right\rfloor (1-\ell x) = (2\ell^2+3\ell)x + 2\ell(1-\ell x) = 2\ell + 3\ell x,$$

the upper bound follows.

Let us now consider the case that $x \in \langle 1/\ell, 1)$. Since the labels of any two vertices must differ by at least x, we infer

$$\lambda(K(2\ell+2,2); x, 1) \geq (m-1)x = (2\ell^2+3\ell)x \ .$$

For the upper bound, we consider the order $v_1, \ldots, v_m$ as in Lemma 2 and assign the vertex v_i the label $(i-1)x$. The arguments that this labeling is proper are analogous to those presented in the previous paragraph. $\square$

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