On Forbidden Subdivision Characterization of Graph Classes

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Abstract

We provide a characterization of several graph parameters (the acyclic chromatic number, the arrangeability, and a sequence of parameters related to the expansion of a graph) in terms of forbidden subdivisions.

Let us start with several definitions. Throughout the paper, we consider only simple undirected graphs. A graph $G' = \mathrm{sd}_t(G)$ is the *t-subdivision* of a graph G, if G' is obtained from G by replacing each edge by a path with exactly t inner vertices. Similarly, G' is a $\leq t$*-subdivision* of G if the graph G' can be obtained from G by subdividing each edge by at most t vertices (the number of vertices may be different for each edge).

A coloring of vertices of a graph G is *proper* if no two adjacent vertices have the same color. The minimum k such that the graph G has a proper coloring by k colors is called the *chromatic number* of G and denoted by $\chi(G)$. A proper coloring of a graph G is *acyclic* if the union of each two color classes induces a forest, i.e., there is no cycle colored by two colors. The minimum k such that the graph G has an acyclic coloring by k colors is called the *acyclic chromatic number* of G and denoted by $\chi_a(G)$.

In this paper, we present an exact characterization of graph classes whose acyclic chromatic number is bounded by a constant. As a motivation, let us consider several older results. Borodin [2] have proved that the acyclic chromatic number of every planar graph is at most 5. Nešetřil and Ossona de Mendez [5] have proved that every graph G has a minor H such that

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$\chi_a(G) \leq O\left((\chi(H))^2\right)$. This implies that the acyclic chromatic number is bounded by a constant for every nontrivial minor-closed class of graphs. However, this result does not describe all graph classes with bounded acyclic chromatic number, e.g., $\mathrm{sd}_1(K_{n,n})$ has acyclic chromatic number 3, but it contains K_n as a minor.

On the other hand, Wood [15] has proved that the acyclic chromatic number of $\mathrm{sd}_1(G)$ is bounded by a function of the chromatic number of the graph G and vice versa:

Theorem 1 (Wood [15], Corollary 3) *For each graph G,*

$$\sqrt{\frac{\chi(G)}{2}} < \chi_a(sd_1(G)) \leq \max(3, \chi(G)).$$

A simple corollary of this theorem is the following characterization:

Corollary 2 *Let G be a graph with $\chi_a(G) = c$. If H is a graph such that $\chi(H) \geq 2c^2$, then G does not contain $sd_1(H)$ as a subgraph.*

In Section 1, we prove that this statement essentially describes all graphs with bounded acyclic chromatic number, i.e., that if G has high acyclic chromatic number, it contains as a subgraph a ≤ 1-subdivision of a graph with high chromatic number:

Theorem 3 *Let $c \geq 4$ be an arbitrary integer and $d = 56(c-1)^2\frac{\log(c-1)}{\log c - \log(c-1)}$. Let G be a graph with acyclic chromatic number greater than $c(c-1)^{c\binom{d}{2}}$, i.e., $\chi_a(G) \approx \exp(c^7 \log^3 c)$. If $\chi(G) \leq c$, then G contains a subgraph $G' = sd_1(G'')$ such that the chromatic number of the graph G'' is c.*

This result generalizes the result of Nešetřil and Ossona de Mendez [5], although our bound is much weaker than the quadratic one. The consequence of Theorem 3 and Corollary 2 is the following characterization of graph classes with the bounded acyclic chromatic number:

Corollary 4 *Let $\mathcal{G}$ be any class of graphs such that the chromatic number of every graph in $\mathcal{G}$ is bounded by a constant. The acyclic chromatic number of graphs in $\mathcal{G}$ is bounded by a constant if and only if there exists a constant c such that every graph G such that $sd_1(G)$ is a subgraph of a graph in $\mathcal{G}$ satisfies $\chi(G) \leq c$.*

The acyclic chromatic number of a graph is also related to several other graph parameters – the arrangeability, the greatest reduced average density, and the game chromatic number. We provide similar characterizations for the first two of these parameters, and outline the relationship with the game chromatic number in the Final Remarks section.

For a vertex v of a graph G, let $N(v)$ denote the open neighborhood of v. Given a linear ordering L of the vertices of a graph G, let $L^+(v)$ be the set of vertices of G that are after v in this ordering, and $L^-(v)$ the set of vertices that are before it. A graph G is p-arrangeable if there exists a linear ordering L of vertices of G such that every vertex v of G satisfies

$$\left| L^-(v) \cap \bigcup_{u \in N(v) \cap L^+(v)} N(u) \right| \leq p.$$

The arrangeability of the graph is an important parameter that bounds its acyclic chromatic number, the game chromatic number ([11]), and the Ramsey number ([3]) in a natural way. In Section 2, we show a precise characterization of graphs G with bounded arrangeability in terms of average degrees of graphs whose ≤ 1-subdivisions are subgraphs of G, analogical to Theorem 3 and Corollary 4. Rödl and Thomas [13] have shown that every graph with arrangeability p^8 contains a subdivision of the clique K_p as a subgraph; a result similar to ours is implicit in their proof. However, the result we obtained is slightly stronger. Komlós and Szemerédi [14] have proved that every simple graph with average degree at least d^2 contains a subdivision of K_d as a subgraph, hence Theorem 9 implies that every graph with arrangeability $\Omega(p^6)$ contains a subdivision of K_p as a subgraph.

Another parameter that admits a similar characterization is the *expansion* of a graph. A graph H is a *rank r contraction* of a graph G if there exists a set S of vertex disjoint induced connected subgraphs of G such that each member of S has radius at most r, and H is the simple graph obtained from G by contracting all edges of the subgraphs in S (the arising parallel edges are suppressed). For example, the only rank 0 contraction of G is the graph G itself, and a rank 1 contraction is obtained from G by contracting edges of a star forest. The *maximum average degree* $\mathrm{mad}(G)$ of graph G is the maximum of average degrees over all subgraphs of G. The *rank r greatest reduced average density* [2] of G (denoted by $\nabla_r(G)$) is the maximum of $\frac{\mathrm{mad}(G')}{2}$ over all rank r contractions G' of the graph G. A class of graphs $\mathcal{G}$ has *bounded*

[2]The greatest reduced average density is usually defined in terms of densities of the subgraphs. In this paper, we want to stress the relationship with the average and minimum degree of specific subgraphs, thus we define it using maximum average degree.

expansion with bounding function f, if for each $G \in \mathcal{G}$, $\nabla_r(G) \le f(r)$. For example, nontrivial minor closed classes have the expansion bounded by a constant function. The class of graphs with maximum degree $\le c$ has expansion bounded by the function c^r.

This graph parameter was recently introduced and studied by Nešetřil and Ossona de Mendez, see [7], [8] and [9]. In particular, in [6] they proved that the arrangeability of a graph G is bounded by a function of $\nabla_1(G)$ (this fact is also an easy consequence of Theorem 9). Therefore, the expansion of a graph bounds also the acyclic chromatic number and the other discussed parameters. If G is a $\le 2r$-subdivision of a graph with minimum degree d, then $\nabla_r(G) \ge d$. In Section 3, we prove that on the other hand, graphs with large $\nabla_r(G)$ contain $\le 2r$-subdivisions of graphs with high minimum degree.

1 Acyclic Chromatic Number

The goal of this section is to prove Theorem 3. We use probabilistic arguments. Let $\mathrm{Prob}[K]$ denote the probability of the event K and $\mathrm{E}[X]$ the expected value of the random variable X. We use the following variants of the well-known estimates, see e.g. [1] for reference.

Lemma 5 (Markov Inequality) *If X is a nonnegative random variable and $a > 0$, then*

$$Prob[X \ge a] \le \frac{E[X]}{a}.$$

Lemma 6 (Chernoff Inequality) *Let $X_1, \ldots, X_n$ be independent random variables, each attaining values 1 with probability p and 0 with probability $1-p$. Let $X = \sum_{i=1}^n X_i$. For any $t \ge 0$,*

$$Prob[X \ge np + t] < e^{-\frac{t^2}{2(np+t/3)}},$$

and

$$Prob[X \le np - t] < e^{-\frac{t^2}{2(np+t/3)}}.$$

We first prove a lemma regarding the graphs with high density. Note that a graph G with acyclic chromatic number less than c cannot have high density, as G is a union of $\binom{c}{2}$ forests. A graph G is *d-degenerated* if each subgraph of G contains a vertex of degree at most d. Note that the average degree of a d-degenerated graph is at most $2d$, hence every graph with average degree at least t contains a subgraph whose minimum degree is at least $\frac{t}{2}$.

Lemma 7 *Let $c \geq 4$ be an integer and let G be a graph with the minimum degree $d > 56(c-1)^2 \frac{\log(c-1)}{\log c - \log(c-1)}$, (i.e., $d = \Omega(c^3 \log c)$). Then the graph G contains a subgraph G' that is the 1-subdivision of a graph with chromatic number c.*

Proof. Every graph contains a bipartite subgraph with at least half of the edges of the original graph, i.e., G contains a bipartite subgraph G_1 with average degree more than $\frac{d}{2}$. The graph G_1 cannot be $\frac{d}{4}$-degenerated, since otherwise the average degree of G_1 would be at most $\frac{d}{2}$. Let G_2 be a subgraph of G_1 with minimum degree at least $d_2 = \frac{d}{4}$. The graph G_2 is bipartite, let $V(G_2) = A \cup B$ be a partition of its vertices to two independent sets such that $|A| \leq |B|$. Let $a = |A|$ and $b = |B|$.

Let $q = 7\frac{\log(c-1)}{\log c - \log(c-1)}$. We construct a graph G_3 in the following way: if $b \geq qa$, then let $G_3 = G_2$, $A' = A$ and $B' = B$. Otherwise, we choose sets A' and B' as described in the next paragraph, and let G_3 be the subgraph of G_2 induced by A' and B':

Let A' be a subset of A obtained by taking each element of A randomly independently with probability $p = \frac{b}{qa}$. Note that $\frac{d_2}{q} \geq 10$. Also, since the minimum degree of G_2 is at least d_2, it follows that $b \geq d_2$. The expected size of A' is $ap = \frac{b}{q}$, and by Chernoff Inequality, the size of A' is more than $\frac{2b}{q}$ with probability less than $e^{-\frac{3b}{8q}} \leq e^{-\frac{3d_2}{8q}} \leq e^{-\frac{15}{4}} < 0.5$. Consider a vertex v of B with degree $s \geq d_2$ in G_2, and let s' be the number of neighbors of v in B' and $r(v) = \frac{s'}{s}$. The expected number of neighbors of v in A' is ps. By Chernoff Inequality, the probability that $s' < \frac{p}{2}s$ is less than $e^{-\frac{3ps}{28}} \leq e^{\frac{3}{28}\cdot\frac{b}{a}\cdot\frac{d_2}{q}} \leq e^{-\frac{15}{14}} < 0.35$. Let B' be the set of vertices v of B such that $r(v) \geq \frac{p}{2}$. The expected value of $|B \setminus B'|$ is less than $0.35b$, and by Markov Inequality, $\text{Prob}\left[|B \setminus B'| \geq 0.7b\right] \leq 0.5$. Therefore, the probability that the set A' has size at most $\frac{2b}{q}$ while the set B' has size at least $0.3b$ is greater than zero. We let A' and B' be a pair of sets that satisfies these properties.

Let $a' = |A'|$ and $b' = |B'|$. Observe that the degree of every vertex of B' in G_3 is at least $\frac{b}{2qa}d_2 \geq \frac{1}{2q}d_2 = (c-1)^2 = d_3$, and that $b' \geq 0.3b \geq 0.15qa'$. Let $D_1, \ldots, D_{b'} \geq d_3$ be the degrees of vertices of B'.

We show that the graph G_3 contains as a subgraph the 1-subdivision of a graph with chromatic number c. Suppose for contradiction that each graph whose 1-subdivision is a subgraph of G_3 has chromatic number at most $c-1$. Let us consider only the subgraphs whose vertices of degree 2 created by subdividing edges belong to B'. There are exactly $N_G = \prod_{i=1}^{b'} \binom{D_i}{2}$ such subgraphs and $N_C = (c-1)^{a'}$ colorings of A' by $c-1$ colors.

Let us for each coloring φ of A' determine the number of subgraphs H such that all vertices of B' have degree 2 in H, and φ is a proper coloring of the graph obtained from H by suppressing the vertices in B'. Let us consider a vertex v in B' of degree D. Since φ is proper, the two edges incident with v in H lead to vertices with different colors. Let M be the neighborhood of v, $|M| = D$. Let m_i be the number of vertices of M colored by φ with the color i. The number s of the pairs of neighbors of v that have different colors satisfies

$$s = \sum_{1\le i<j\le c-1} m_i m_j = \frac{1}{2} \sum_{1\le i,j\le c-1, i\neq j} m_i m_j = \frac{1}{2} \sum_{i=1}^{c-1} m_i(D - m_i)$$

$$s = \frac{1}{2}\left(D^2 - \sum_{i=1}^{c-1} m_i^2\right) \le \frac{1}{2}\left(D^2 - \frac{D^2}{c-1}\right).$$

Therefore, the number of the subgraphs of G_3 for that φ is proper is at most $N_P = \left(\frac{1}{2}\left(1 - \frac{1}{c-1}\right)\right)^{b'} \prod_{i=1}^{b'} D_i^2$. For each subgraph of G_3 there exists at least one proper coloring, hence $N_G \le N_C N_P$, and we obtain

$$(c-1)^{a'} \left(\frac{1}{2}\left(1 - \frac{1}{c-1}\right)\right)^{b'} \prod_{i=1}^{b'} D_i^2 \ge \prod_{i=1}^{b'} \binom{D_i}{2}$$

$$(c-1)^{a'} \left(1 - \frac{1}{c-1}\right)^{b'} \ge \prod_{i=1}^{b'} \left(1 - \frac{1}{D_i}\right) \ge \left(1 - \frac{1}{d_3}\right)^{b'}$$

Since $b' \ge 0.15qa'$ and $\left(1 - \frac{1}{d_3}\right)\left(1 - \frac{1}{c-1}\right)^{-1} = \frac{c}{c-1} > 1$, it follows that

$$(c-1)^{a'} \ge \left(\left(1 - \frac{1}{d_3}\right)\left(1 - \frac{1}{c-1}\right)^{-1}\right)^{0.15qa'}$$

$$(c-1) \ge \left(\frac{c}{c-1}\right)^{0.15q}$$

This is a contradiction, since

$$\left(\frac{c}{c-1}\right)^{0.15q} > \left(\frac{c}{c-1}\right)^{\frac{\log(c-1)}{\log c - \log(c-1)}} = c - 1.$$

□

Let us now prove the main theorem of this section.

Proof of Theorem 3 We prove the contravariant implication: "Let G be a graph with $\chi(G) \le c$. If all graphs whose 1-subdivision is a subgraph of G have chromatic number at most $c-1$, then G has acyclic chromatic number at most $c_1 = c(c-1)^{c\binom{d}{2}}$."

Let us assume that G is a graph with chromatic number at most c, such that all graphs whose 1-subdivision is a subgraph of G have chromatic number at most $c-1$. By Lemma 7, the graph G is d-degenerated. Let $L = v_1, \ldots, v_n$ be an ordering of the vertices of G in such a way that each vertex has at most d neighbors after it; let $N_t = L^+(v_t) \cap N(v)$ be the set of the neighbors of v_t that are after v_t in the ordering L, and let $v_{t,j}$ be the j-th of these neighbors.

Suppose that there exists a proper coloring φ such that each set N_t is rainbow (i.e., no two vertices in N_t have the same color). Let us consider an arbitrary cycle C in G. Let v be the vertex of C that appears first in the ordering L, and let u and w be the neighbors of v in C. The colors $\varphi(u)$, $\varphi(v)$ and $\varphi(w)$ are mutually distinct, hence C is not colored by two colors. Therefore, the coloring φ is acyclic.

Let us now construct a coloring φ that satisfies this property. Let φ_0 be a fixed proper coloring of G by c colors. For $i = 1, \ldots, c$ and $1 \le j_1 < j_2 \le d$, we define the graph G_{i,j_1,j_2} in the following way: the vertices of G_{i,j_1,j_2} are the vertices of G, and for each t such that $\varphi_0(v_t) = i$, we join by an edge the pair of vertices v_{t,j_1} and v_{t,j_2} (if both of the vertices exist). Note that the 1-subdivision of G_{i,j_1,j_2} is a subgraph of G, hence G_{i,j_1,j_2} can be colored by $c-1$ colors. Let φ_{i,j_1,j_2} be such a coloring.

We color each vertex v of G with the $c\binom{d}{2} + 1$-tuple $\varphi(v)$ consisting of $\varphi_0(v)$ and $\varphi_{i,j_1,j_2}(v)$ for $i = 1, \ldots, c$ and $1 \le j_1 < j_2 \le d$. Each N_t is rainbow in this coloring, as the vertices $v_{t,a}$ and $v_{t,b}$ get distinct colors in the coloring $\varphi_{\varphi_0(v_t),a,b}$. Also, the coloring φ is proper since the coloring φ_0 is proper. Therefore, we found the acyclic coloring φ of G by c_1 colors, hence the claim of the theorem holds. □

2 Arrangeability

Let us start with a simple observation regarding the arrangeability. Let G be the 1-subdivision of a graph with minimum degree d, and consider the ordering L that shows that its arrangeability is at most p. Let v be the last vertex of degree greater than two in this ordering. Note that $|N(v) \cap L^-(v)| \le p+1$ and $|N(v) \cap L^+(v)| \le p$. Since the degree of v is at least d, the graph G is not p-arrangeable for $p < \frac{d-1}{2}$. The goal of this section is to prove that on the other hand, every graph with large arrangeability contains a ≤ 1-subdivision of a graph with large minimum degree.

Let us define several concepts and notation we use in the proofs. A *double-star* is the 1-subdivision of a star. The *ray vertices* of a double-star are its vertices of degree one, and the *middle vertices* are the vertices created by subdividing the edges; the remaining vertex is the *center* of the double-star. The *middle edges* of a double star are the edges that are incident with the center, while the remaining edges are the *ray edges*. Given a double-star S and a set X of vertices, let $d_2^X(S)$ be the number of ray vertices of S in X. Given a fixed ordering L of the vertices of a graph G, let the *back-degree* $d^-(v)$ of a vertex v be the number of neighbors of v before it in L, $d^-(v) = |N(v) \cap L^-(v)|$. Let the *double back-degree* $d_2^-(v)$ be the maximum of $d_2^{L^-(v)}(S)$ over all double-stars S that are subgraphs of G and have center v. First, we show the following characterization of graphs with small arrangeability:

Lemma 8 *If G is a p-arrangeable graph, then there exists ordering L of vertices of G such that each vertex has the back-degree at most $p+1$ and the double back-degree at most $2p+1$. On the other hand, if there exists an ordering L of vertices of a graph G such that the back-degree of each vertex is at most d_1 and the double back-degree is at most d_2, then the graph G has arrangeability at most d_1d_2.*

Proof. Suppose first that the graph G is p-arrangeable, and let L be an ordering of the vertices of G that witnesses its arrangeability. Let S be a double-star in G with center v. The back-degree of v is at most $p+1$, hence S has at most $p+1$ middle vertices in $L^-(v)$. By the p-arrangeability of G, the double-star S also has at most p middle vertices in $L^+(v)$ whose ray vertex belongs to $L^-(v)$. Therefore, $d_2^-(v) \leq 2p+1$.

Let us now assume that L is an ordering of vertices of the graph G such that for each v, $d^-(v) \leq d_1$ and $d_2^-(v) \leq d_2$. Consider an arbitrary vertex v, and let $X = L^-(v) \cap \bigcup_{u \in N(v) \cap L^+(v)} N(u)$. For each vertex $x \in X$, let us choose one of its neighbors $u_x \in N(v) \cap L^+(v)$ arbitrarily, and let $U = \{u_x : x \in X\}$. By the definition of the double back-degree, $|U| \leq d_2$. On the other hand, each vertex in U has at most d_1 neighbors before it, hence $|X| \leq d_1d_2$. Therefore, the ordering L witnesses that G is d_1d_2-arrangeable. □

Let us formulate the main theorem of this section:

Theorem 9 *Let $d \geq 1$ be an arbitrary integer and $p = 4d^2(4d+5)$. Let G be a d-degenerated graph. If G is not p-arrangeable, then G contains a subgraph $G' = sd_1(G'')$ such that the minimum degree of G'' is at least d.*

Proof. Let G be a d-degenerated graph that is not p-arrangeable and let n be the number of vertices of G. Let $d_1 = 4d$ and $d_2 = d(4d+5)$ and consider the following algorithm that attempts to construct an ordering of vertices of G such that for each vertex v, $d^-(v) \leq d_1$ and $d_2^-(v) \leq d_2$: We set $G_0 = G$. In the step $i > 0$, if there exists a vertex v of G_{i-1} such that the degree of v in G_{i-1} is at most d_1 and $d_2^{V(G_{i-1})}(S) \leq d_2$ for each double-star S in G with center v (note that we consider also the double-stars that are not subgraphs of G_{i-1}), then let $v_{n-i+1} = v$ and $G_i = G_{i-1} - v$. Obviously, if this algorithm succeeds in each step, the ordering $L = v_1, v_2, \ldots, v_n$ satisfies the required properties.

By Lemma 8, since $p = d_1 d_2$ and G is not p-arrangeable, this algorithm fails on G. This means that there exists i such that each vertex v of G_{i-1} has more than d_1 neighbors in G_{i-1}, or is a center of a star in G with more than d_2 ray vertices in $V(G_{i-1})$. Let V_1 be the set of vertices of degree greater than d_1 in G_{i-1}, and let $V_2 = V(G_{i-1}) \setminus V_1$. Let $n_1 = |V_1|$ and $n_2 = |V_2|$. Each vertex v in V_2 has degree at most d_1 in G_{i-1}, hence v is a center vertex of a double-star with more than $d_2 - d_1$ ray vertices in $V(G_{i-1})$ and all middle vertices in $V(G) \setminus V(G_{i-1})$; let us choose such a star S_v for each vertex $v \in V_2$ arbitrarily. Let X be the set of the chosen double-stars, $X = \{S_v | v \in V_2\}$.

Let M be the set of middle vertices of double-stars in X, let $m = |M|$, and let G_X be the bipartite graph on $V_2 \cup M$ (note that V_2 and M are disjoint) whose set of edges consists of all the middle edges of the double-stars in X. Since the graph G is d-degenerated, the average degree of G_X is at most $2d$. On the other hand, each vertex in V_2 has degree at least $d_2 - d_1$ in G_X, hence $d(m + n_2) \geq |E(G_X)| \geq n_2(d_2 - d_1)$. It follows that $m \geq \frac{n_2(d_2-d_1-d)}{d} = 4dn_2 = d_1 n_2$. For each vertex $u \in M$, let us choose an arbitrary double-star $T_u \in X$ such that u is a middle vertex of T_u. Let us remove u together with the corresponding ray vertex from each double-star in X except for T_u. Let X' be the set of double-stars obtained this way. The double-stars in X' have disjoint middle vertices, and in total m rays (the ray vertices do not have to be disjoint).

Let us consider the graph H with the vertex set $V_1 \cup V_2$ in that the vertices u and v are connected by an edge if:

1. $u \in V_1$ and uv is an edge of G_{i-1}, or

2. u is a center of a star $T \in X'$, and v is a ray vertex of T.

A ≤ 1-subdivision of H is a subgraph of G. Observe that the graph H has $n_1 + n_2$ vertices, and at least $\frac{1}{2}(n_1 d_1 + m)$ edges. Using the lower bound $m \geq d_1 n_2$, we conclude that $|E(H)| \geq \frac{d_1}{2}(n_1 + n_2)$.

Let H' be the subgraph of H, whose edges are only the edges connecting the ray vertices of the double-stars in X' with their centers. The graph G is d-degenerated, hence H' has at least $(\frac{d_1}{2} - d)(n_1 + n_2) = d(n_1 + n_2)$ edges. Therefore, the average degree of H' is at least $2d$. It follows that H' has a subgraph G'' with the minimum degree at least d. The graph $G' = \mathrm{sd}_1(G'')$ is a subgraph of G that satisfies the claim of the theorem. □

3 Expansion

We now focus on the characterization of graphs with bounded expansion. A non-empty graph H is *an average (resp. minimum) degree* (r, d)*-witness* if there exists a (r, d)*-witness decomposition* $D = \{(S_1, s_1), (S_2, s_2), \ldots, (S_k, s_k)\}$ of H, i.e., a set of disjoint nonempty induced subgraphs S_1, S_2, ..., S_k that cover $V(H)$, and vertices $s_i \in V(S_i)$ such that

- the subgraphs S_i are trees, and
- for each vertex $v \in S_i$, the distance of v from s_i in S_i is at most r, and
- for each $i \neq j$, there is at most one edge between S_i and S_j in H, and
- the average (resp. minimum) degree of the graph obtained from H by identifying all the vertices of each tree S_i with s_i is at least d.

Observe that $\nabla_r(G) \geq d$ if and only if G contains an average degree $(r, 2d)$-witness as a subgraph. Therefore, if $\nabla_r(G) \geq d$ then G contains a minimum degree (r, d)-witness as a subgraph.

The *size* of the decomposition is the number of its trees. The vertices s_i of a witness decomposition are called *centers*. The edges that belong to the trees of the decomposition are called *internal* and the remaining edges are *external*. For a non-center vertex $v \in V(S_i)$, the unique internal edge from v on the shortest path to s_i is called the *parent edge* and its vertex different from v is called the *parent vertex*.

We always assume that each leaf of a tree S_i is incident to at least one external edge (if this is not the case, the leaf vertex may be removed from the witness). When an external edge is removed from the decomposition, we also repeatedly remove leaves that are incident to no external edges. Similarly, the operation of *removal* of a tree from the decomposition D of a graph H is defined in the following way: The decomposition $D' = \{(S'_1, s'_1), \ldots\}$ of a graph $H' = \bigcup_i V(S'_i)$ is obtained from the decomposition $\{(S_1, s_1) \ldots, (S_{t-1}, s_{t-1}), (S_{t+1}, s_{t+1}), \ldots\}$ by repeatedly removing leaves of trees that are incident with no external edges. Given an internal edge $e \in S_t$, the decomposition D' of the graph H is obtained from D by *splitting on* e if D' consists of trees $S_1, \ldots, S_{t-1}, S_{t+1}, \ldots$, and the two trees S'_t and S''_t obtained from S_t by removing e. If $s_t \in S'_t$ then the center of S'_t is s_t and the center of S''_t is the common vertex of e and S''_t. The edge e becomes external by this operation. *Expunging* of a center vertex v is performed by first splitting on all the internal edges incident to v, and then removing the tree consisting of v.

Lemma 10 *Let H be a minimum degree (r,d)-witness with a decomposition $D = \{(S_1, s_1), (S_2, s_2), \ldots\}$ and let $d_1 = \sqrt[r+1]{\frac{d}{4}}$. There exists a minimum degree (r, d_1)-witness $H' \subseteq H$ with a decomposition $D' = \{(S'_1, s'_1), (S'_2, s'_2), \ldots\}$ such that the degree of each center is at least d_1.*

Proof. We construct the new decomposition D' by repeatedly expunging the vertices v such that v is a center and its degree is less than d_1, as long as any such vertices exist. Let us show that the decomposition D' obtained by this construction is non-empty.

Let k be the size of D and let e be the number of external edges of D. Note that $e \geq \frac{d}{2}k$. Let us count the number of external edges that get removed by expunging the vertices. If an edge e is removed by expunging a vertex v, let us assign e to the tree in D that contains the vertex v. When a vertex is expunged, its degree is less than d_1. The depth of each tree in the decomposition D is at most r, thus there are at most d_1^{r+1} edges assigned to each tree. Therefore, at most $d_1^{r+1}k$ external edges are removed. Since $e \geq \frac{d}{2}k > d_1^{r+1}k$, the decomposition D' is non-empty, and it obviously satisfies the claim of the lemma. □

Consider a minimum degree (r,d)-witness H with decomposition D such that the degree of each center is at least d. Given a non-center vertex $v \in S_i$, let $B(v)$ be the component of $S_i - s_i$ that contains v. The non-center vertex $v \in S_i$ is called *lonely* if there is only one external edge incident to the vertices in $B(v)$ and this edge is incident to v. Note that in this case, $B(v)$ is a path with the end vertex v. An external edge $e = \{u, v\}$ is called *critical* if u or v is lonely, and *bicritical* if both u and v is lonely. Observe that there exists a (r,d)-witness $H' \subseteq H$ with a decomposition D' such that the degree of each center is at least d and each external edge is critical.

Theorem 11 *Let $r, d \geq 1$ be arbitrary integers and let $p = 4\,(4d)^{(r+1)^2}$. If $\nabla_r(G) \geq p$, then G contains a subgraph G' that is a $\leq 2r$-subdivision of a graph with minimum degree d.*

Proof. Let G be a graph with $\nabla_r(G) \geq p$. As we noted before, there exists a minimum degree (r,p)-witness $H \subseteq G$. Let $d_1 = \sqrt[r+1]{\frac{p}{4}} = (4d)^{r+1}$. By Lemma 10, there exists a minimum degree (r,d_1)-witness $H' \subseteq H$ with a decomposition $D' = \{(S'_1, s'_1), (S'_2, s'_2), \ldots\}$ such that the degree of each center is at least d_1. Furthermore, we may assume that each external edge in the decomposition D' is critical.

Let $b = \sqrt[r+1]{d_1} = 4d$. We create a new decomposition D'' of a graph $H'' \subseteq H'$ by splitting on parent edges of all non-center vertices whose degree is greater than b. After splitting on edge uv, where u is the parent vertex of v, if u is not lonely, then we also remove the edge e. Let us call the center vertices of D' the *old centers*, and the center vertices of D'' that are non-center in D' the *new centers*. The decomposition D'' satisfies the following properties:

1) All old centers have degree at least d_1 and all new centers have degree at least b, and

2) all non-center vertices have degree at most b, and

3) all external edges are critical, and

4) all external edges between trees with the new centers are bicritical (all such edges were internal in D''), and

5) the lonely vertex of each external edge that is not bicritical belongs to a tree whose center is old.

We construct a graph G'' in the following way: For each tree S with the center s in the decomposition D'' and for each component C of $S - s$ that is incident with more than one external edge, we select one external edge incident to a vertex in C arbitrarily. Let W be the set consisting of all vertices incident with the selected edges and all vertices of the bicritical edges of D''. The graph G'' is the induced subgraph of H'' with the vertex set that consists of the centers of D'', the vertices in W and the vertices on the paths that join the vertices of W with the centers of their trees. Observe that all the non-center vertices of G'' have degree exactly 2, i.e., the graph G'' is a $\leq 2r$-subdivision of some graph F''.

Let us compute average degree of F''. Let n_{old} be the number of old centers, n_{new} the number of new centers, $n = n_{\text{old}} + n_{\text{new}}$ the number of vertices of F'' and m the number of edges of F''. The properties 1), 3), 4) and 5) of D'' imply that the degree of each new vertex in F'' is at least b, hence $m \geq \frac{b}{2} n_{\text{new}}$. On the other hand, the total number of external edges in

D'' is at least $\frac{d_1}{2}n_{\text{old}}$, and by the properties 2) and 3) of D'', the number of external edges is decreased at most b^r times during the construction of G'', i.e., $m \geq \frac{d_1}{2b^r}n_{\text{old}} = \frac{b}{2}n_{\text{old}}$. Hence $m \geq \frac{b}{4}n$, and the average degree of F'' is at least $\frac{b}{2}$.

Therefore, there exists a subgraph $F' \subseteq F''$ such that the minimum degree of F' is at least $\frac{b}{4} = d$. The corresponding subgraph $G' \subseteq G''$ is a $\leq 2r$-subdivision of F', hence the claim of the theorem holds. □

4 Final Remarks

The *graph coloring game* with k colors and a graph G has the following rules: There are two players, Alice and Bob, who take turns. Each move of Alice or Bob consists of coloring a so far uncolored vertex of G by one of the k colors in such a way that the obtained partial coloring of G is proper. Alice wins if the whole graph G is colored, while Bob wins if he prevents this, i.e., manages to ensure that there is an uncolored vertex such that all k colors are used in its neighborhood. The *game chromatic number* of a graph G is defined as the minimum k such that Alice has a winning strategy. Zhu and Dinsky [4] have proved that the game chromatic number of a graph is bounded by a function of the acyclic chromatic number, and conjectured that each graph with high acyclic chromatic number contains a subgraph with high game chromatic number. The consequence of Corollary 4 is that this conjecture is implied by the following conjecture:

Conjecture 1 *There exists a function f such that for each graph G, if $\chi(G) \geq f(c)$ then the game chromatic number of $sd_1(G)$ is at least c.*

It is easy to prove that the game chromatic number of $\text{sd}_1(K_n)$ is at least $\log_4 n$. Rödl [12] has proved that a graph with large chromatic number contains a large clique or a triangle-free subgraph with large chromatic number. Hence, it would suffice to prove the following equivalent claim:

Conjecture 2 *There exists a function f such that for each triangle-free graph G, if $\chi(G) \geq f(c)$ then the game chromatic number of $sd_1(G)$ is at least c.*

A simpler parameter related to the game chromatic number is a *game coloring number*. The game coloring number of a graph G is the minimum number k for that Alice wins a *marking game*: Alice and Bob are marking vertices of a graph in such a way that in the moment when a vertex is marked, it has at most $k-1$ marked neighbors. Alice wins if all the vertices of the graph are marked, while Bob wins if this becomes impossible. It is easy to see that the game coloring number is the upper bound on the game chromatic number. There are known examples of graphs with acyclic chromatic number 3 and arbitrarily large game coloring number – Kierstead and Trotter [10] have proved that the game coloring number of $\mathrm{sd}_1(K_{n,n})$ is $\theta(\log n)$. On the other hand, the game coloring number is bounded by the arrangeability of a graph. It is natural to conjecture the following:

Conjecture 3 *There exists a function f such that for each graph G, if G is not $f(c)$-arrangeable then the game coloring number of G is at least c.*

By Theorem 9, the equivalent statement is that there exists a function f' such that $\delta(G) \geq f'(c)$ implies that the game chromatic number of $\mathrm{sd}_1(G)$ is at least c.

The bounds of all our theorems can be improved. While the bounds of Theorem 9 and Theorem 11 have basically the correct magnitude (there are graphs G with maximum degree d, arrangeability $\Omega(d^2)$ and $\nabla_r(G) = \Omega(d^{r+1})$), the gap between the bounds for Theorem 3 is quite large. The graph $\mathrm{sd}_1(K_n)$ has acyclic chromatic number $\theta\left(\sqrt{n}\right)$ and contains the 1-subdivision of a graph with chromatic number n, but the bound of Theorem 3 is exponential. It would be interesting to decrease the upper bound or to find an example showing that an exponential bound is necessary.

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References

[1] N. Alon, J.H. Spencer, *The probabilistic method (2ed)*, Wiley-Interscience, New York, 2000.

[2] O.V. Borodin, *On acyclic colorings of planar graphs*, Discrete Math. **25** (1979), 211-236.

[3] G. Chen and R.H. Schelp, *Graphs with linearly bounded Ramsey numbers*, J. Combinatorial Theory, Ser. B **57** (1993), 138-149.

[4] T. Dinski and X. Zhu, *A bound for the game chromatic number of graphs*, Discrete Math. **196** (1999), 109-115.

[5] J. Nešetřil and P. Ossona de Mendez, *Colorings and homomorphisms of minor closed classes*, in: J. Pach, et al. (Eds.), J. Goodman and R. Pollack Festschrift, Springer, Berlin, 2003, pp. 651-664.

[6] J. Nešetřil and P. Ossona de Mendez, *Fraternal augmentations, arrangeability and linearly bounded Ramsey numbers*, submitted to Combinatorica.

[7] J. Nešetřil and P. Ossona de Mendez, *Grad and classes with bounded expansion I – decompositions*, submitted to European Journal of Combinatorics.

[8] J. Nešetřil and P. Ossona de Mendez, *Grad and classes with bounded expansion II – algorithmic aspects*, submitted to European Journal of Combinatorics.

[9] J. Nešetřil and P. Ossona de Mendez, *Grad and classes with bounded expansion II – restricted dualities*, submitted to European Journal of Combinatorics.

[10] H. Kierstead, private communication.

[11] H. Kierstead, W.T. Trotter, *Planar graph coloring with an uncooperative partner*, J. Graph Theory **18** (1994), 569-584.

[12] V. Rödl, *On the chromatic number of subgraphs of a given graph*, Proc. Amer. Math. Soc. **64** (1977), no. 2, 370-371.

[13] V. Rödl, R. Thomas, *Arrangeability and clique subdivisions*, The mathematics of Paul Erdos, II, Algorithms Combin. **14**, Springer, Berlin (1997), 236-239.

[14] E. Szemerèdi, Colloquium at Emory University, Atlanta, GA, April 22, 1994.

[15] David R. Wood, *Acyclic, Star and Oriented Colourings of Graph Subdivisions*, Discrete Mathematics and Theoretical Computer Science **7** (2005), 37-50.