Non-rainbow colorings of 3-, 4- and 5-connected plane graphs*

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Abstract

We study vertex-colorings of plane graphs that do not contain a rainbow face, i.e., a face with vertices of mutually distinct colors. If G is 3-connected plane graph with n vertices, then the number of colors in such a coloring does not exceed $\lfloor \frac{7n-8}{9} \rfloor$. If G is 4-connected, then the number of colors is at most $\lfloor \frac{5n-6}{8} \rfloor$, and for $n \equiv 3 \pmod 8$, it is at most $\lfloor \frac{5n-6}{8} \rfloor - 1$. Finally, if G is 5-connected, then the number of colors is at most $\lfloor \frac{43}{100}n - \frac{19}{25} \rfloor$. The bounds for 3-connected and 4-connected plane graphs are the best possible as we exhibit constructions of graphs with colorings matching the bounds.

1 Introduction

Colorings of graphs embedded on surfaces with face-constraints have recently drawn a substantial amount of attention. There are two natural questions derived from hypergraph colorings that one may ask in this setting:

1. *What is the minimal number of colors needed to color an embedded graph in such a way that each of its faces is incident with vertices of at least two different colors, i.e., there is no monochromatic face?*

2. *What is the maximal number of colors that can be used in a coloring of an embedded graph that contains no face with vertices of mutually distinct colors, i.e., that contains no rainbow face?*

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The first question can be traced back to work of Zykov [24] and was further explored by Kündgen and Ramamurthi [17]. It can be shown [8] that every graph embedded on a surface of genus ε has a coloring with $O(\sqrt[3]{\varepsilon})$ colors that avoids a monochromatic face. Let us remark that this type of coloring can be also formulated in the terms of colorings of so-called face hypergraphs of embedded graphs. Also let us mention that colorings that avoid both monochromatic and rainbow faces have been also studied, see, e.g., [7, 14, 16]. For instance, the results of Penaud [20] and Diwan [5] imply that each plane graph with at least five vertices has a coloring with two colors as well as a coloring with three colors that avoid both monochromatic and rainbow faces.

In this paper, we focus on the second question. A *non-rainbow coloring* of a plane graph G is a vertex-coloring such that each face of G is incident with at least two vertices with the same color. Unlike in the case of ordinary colorings, the goal is to maximize the number of used colors and the maximum number of colors that can be used in a non-rainbow coloring of a plane graph G is denoted by $\chi_f(G)$. Let us remark at this point that the graphs considered in this paper can contain parallel edges unless they form a bigon in the embedding of G. Though it may take some time to digest this definition, the setting is so natural that it has recently appeared independently in the work of Ramamurthi and West [22] and Negami [19] (see also [1, 2, 18] for some even earlier results of this favor). Negami addressed the following anti-Ramsey extremal question (equivalent to our second one):

Problem 1. *What is the smallest number $k(G)$ of colors such that every vertex-coloring of a plane graph G with $k(G)$ colors contains a rainbow face?*

It is easy to see that $\chi_f(G) = k(G) - 1$ and the results obtained in either of the scenarios translate smoothly to the other one.

Let us briefly survey some results from this area. Ramamurthi and West [21] noticed that every plane graph G has a non-rainbow coloring with at least $\alpha(G)+1$ colors, in particular, every plane graph G of order n has a coloring with at least $\lceil \frac{n}{4} \rceil + 1$ colors by the Four Color Theorem. Also, Grötzsch's theorem [9, 23] implies that every triangle-free plane graph has a non-rainbow coloring with $\lceil \frac{n}{3} \rceil + 1$ colors. Ramamurthi et al. [21] conjectured that this bound can be improved to $\lceil \frac{n}{2} \rceil + 1$. Partial results on this conjecture were obtained in [15] and the conjecture has been eventually proven in [13]. More generally, Jungić et al. [13] proved that every planar graph of order n with girth $g \geq 5$ has a non-rainbow coloring with at least $\left\lceil \frac{g-3}{g-2}n - \frac{g-7}{2(g-2)} \right\rceil$ colors if g is odd, and $\left\lceil \frac{g-3}{g-2}n - \frac{g-6}{2(g-2)} \right\rceil$ colors if g is even. All these bounds are the best possible.

Negami [19] investigated non-rainbow colorings of plane triangulations G and showed that $\alpha(G) + 1 \leq \chi_f(G) \leq 2\alpha(G)$. Jendrol' and Schrötter [12] determined the number $\chi_f(G)$ for all semiregular polyhedra. In addition, Jendrol' [10] established that $\frac{n}{2} + \alpha_1^* - 2 \leq \chi_f(G) \leq n - \alpha_0^*$ for 3-connected cubic plane graphs G

where α_0^* is the independence number of the dual graph G^* of G and α_1^* is the size of the largest matching of G^*. He also conjectured [10, 11] the following (let us remark that the former conjecture was proven in [6]):

Conjecture 1. *Every cubic 3-connected plane graph G of order n has $\chi_f(G) = \frac{n}{2} + \alpha_1^* - 2$.*

Conjecture 2. *A non-rainbow coloring of a plane 3-connected graph G of order n uses at most $\left\lfloor \frac{3n-1}{4} \right\rfloor$ colors.*

Motivated by Conjecture 2, we establish tight upper bounds on the maximal numbers of colors used in non-rainbow colorings of plane 3-connected and 4-connected graphs and close lower and upper bounds on the maximal number of such colors for 5-connected plane graphs. We show that a non-rainbow coloring of a plane 3-connected graph of order n always use at most $\left\lfloor \frac{7n-8}{9} \right\rfloor$ colors and a non-rainbow coloring of a plane 4-connected graph always use at most $\left\lfloor \frac{5n-6}{8} \right\rfloor$ colors, and for $n \equiv 3 \pmod 8$, it is at most $\left\lfloor \frac{5n-6}{8} \right\rfloor - 1$ colors. All these bounds are the best possible. Let us also remark that the optimal bound for 2-connected plane graphs is $n-1$ and is achived for a cycle.

For 5-connected plane graphs G, we show that the number of colors of a non-rainbow coloring of G does not exceed $\left\lfloor \frac{43}{100}n - \frac{19}{25} \right\rfloor \approx .430\, n$ where n is the order of G. On the other hand, we construct plane 5-connected graphs G of order n with non-rainbow colorings with almost $\frac{171}{400}n \approx .428\, n$ colors. We were not able to close the gap between our lower and upper bounds in this case and conjecture (see Conjecture 3 at the end of the paper) that the correct bound is $\frac{3}{7}n + \text{const.}$.

Let us now briefly introduce several definitions that will be useful in our further considerations. Most of them are standard graph theory definitions, but we like to include them for the sake of completeness. A *color class* of a vertex-coloring is the set of vertices with the same color. A *monochromatic* path or cycle is a path or a cycle such that all its vertices have the same color. We often refer to a cycle of length k as to a k*-cycle*.

If G is a graph embedded in the plane and C a cycle of G, then $\mathrm{Int}(C)$ is the subgraph of G lying in the closed region bounded by C, in particular, the subgraph $\mathrm{Int}(C)$ includes the cycle C. Similarly, $\mathrm{Ext}(C)$ is the subgraph of G lying outside the open region bounded by C. If both $\mathrm{Int}(C)$ and $\mathrm{Ext}(C)$ contain more vertices than C, then the cycle C is said to be *separating*. Observe that if G is a 3-connected plane graph and C is a separating triangle, then both $\mathrm{Int}(C)$ and $\mathrm{Ext}(C)$ are 3-connected. Conversely, the following is also true: if G and H are two graphs with 3-cycles $u_1u_2u_3$ and $v_1v_2v_3$, then the graph obtained from G and H by identifying the 3-cycles $u_1u_2u_3$ and $v_1v_2v_3$ is called the 3-*sum* of G and H. If both G and H are 3-connected, then their 3-sum is also 3-connected.

2 Counting argument

Our upper bounds are proved using the counting argument based on the lemmas established in this section. Let G be a colored plane graph of minimum degree at least d. We define a *d-weight* $w_d(H)$ of a connected monochromatic subgraph H of G as $k - \frac{1}{2}\sum_{v\in V(H)}(\deg_G(v) - d)$, where k is the number of faces of G that share at least one edge with H, and $\deg_G(v)$ is the degree of v in G. The next lemma provides a simple (and in most cases good enough) upper bound on the d-weight of a monochromatic subgraph of G. We say that a subgraph H is a *maximal connected monochromatic subgraph of* G if all the vertice of H have the same color, H is connected and there is no supergraph of H with these two properties.

Lemma 1. *Let G be a colored plane graph of minimum degree at least d. If H is a maximal connected monochromatic subgraph of G, then*

$$w_d(H) \leq \frac{1}{2}\sum_{v\in V(H)} \min\{2\deg_H(v), d\} \leq \frac{d}{2}|V(H)| \;.$$

Proof. The second inequality of the statement of the lemma obviously holds and thus we focus on proving the first one. For a vertex v of H, let k_v be the number of faces of G that contain an edge e of H incident with v. By the definition, the d-weight of H is at most

$$\frac{1}{2}\sum_{v\in V(H)} (k_v + d - \deg_G(v))$$

since each face incident with a monochromatic edge of H is counted in at least two variables k_v (note that a single face can be incident with more edges of H).

Observe that $k_v \leq \min\{2\deg_H(v), \deg_G(v)\}$. We infer from $k_v \leq 2\deg_H(v)$ that $k_v + d - \deg_G(v) \leq k_v \leq 2\deg_H(v)$ and from $k_v \leq \deg_G(v)$ that $k_v + d - \deg_G(v) \leq d$. Hence, $k_v + d - \deg_G(v) \leq \min\{2\deg_H(v), d\}$. The assertion of the lemma now follows. $\square$

The following lemma provides an upper bound on the number of colors used in a non-rainbow coloring. Note that we assume that each face of a given graph contains a monochromatic edge in the statement of Lemma 2 and not only that c is a rainbow coloring. As we see in the rest of the paper, this does not decrease the generality of our considerations.

Lemma 2. *Let G be a plane connected graph of order n and with minimum degree at least d, let c be a vertex-coloring of G such that each face of G contains a monochromatic edge, and let $H_1, \ldots, H_t$ be all maximal connected monochromatic* *subgraphs of G. If there exist $\alpha > 0$ and $\beta_1, \ldots, \beta_t \geq 0$ such that $w_d(H_i) \leq \alpha(|V(H_i)| - 1) - \beta_i$ for every $i = 1, \ldots, t$, then the coloring c uses at most*

$$\left(1 - \frac{d-2}{2\alpha}\right) n - \frac{2 + \sum_{i=1}^{t} \beta_i}{\alpha}$$

colors.

Proof. Let n_i be the number of vertices of H_i and k_i the number of faces of G incident with an edge of H_i.

Since each face of G is incident with a monochromatic edge, the number f of faces of G is at most $\sum_{i=1}^{t} k_i$. By Euler's formula, we have the following:

$$\sum_{i=1}^{t} k_i \geq f = \frac{1}{2} \sum_{v \in V(G)} \deg(v) - n + 2 = \frac{1}{2} \sum_{v \in V(G)} (\deg(v) - d) + \frac{d-2}{2} n + 2 \,.$$

We now plug our assumption that the d-weight of H_i is at most $\alpha(|V(H_i) - 1) - \beta_i$ to the above estimate:

$$\begin{aligned} \frac{d-2}{2} n + 2 &\leq \sum_{i=1}^{t} k_i - \frac{1}{2} \sum_{v \in V(G)} (\deg_G(v) - d) \\ &\leq \sum_{i=1}^{t} \left(k_i - \frac{1}{2} \sum_{v \in V(H_i)} (\deg_G(v) - d) \right) \\ &\leq \sum_{i=1}^{t} (\alpha(n_i - 1) - \beta_i) \,. \end{aligned}$$

We infer from this inequality that

$$\alpha\, t \leq \alpha\, n - \frac{d-2}{2}\, n - 2 - \sum_{i=1}^{t} \beta_i$$

which yields that

$$t \leq \left(1 - \frac{d-2}{2\alpha}\right) n - \frac{2 + \sum_{i=1}^{t} \beta_i}{\alpha} \,.$$

This finishes the proof of the lemma since the number of colors used by c is at most t. □

In Sections 3–5, we apply Lemma 2 with different values of α and β_i (setting $\beta_i = 0$ in most cases).

3 3-connected plane graphs

In this section, we prove our lower and upper bounds on the number of colors of non-rainbow colorings of 3-connected plane graphs. The upper bound is rather easy once we have established Lemma 2.

Theorem 3. *If G is a plane 3-connected graph with $n \geq 4$ vertices, then the number of colors used by any non-rainbow coloring c of G does not exceed $\left\lfloor \frac{7n-8}{9} \right\rfloor$.*

Proof. By adding edges to G, we can assume without loss of generality that each face of G is incident with a monochromatic edge. In addition, we can assume that each color class induces a connected subgraph of G; otherwise, we can recolor one of the components to increase the number of used colors. Let H be a subgraph induced by one of the color classes (note that H is a maximal connected monochromatic subgraph of G) and let n' be the number of its vertices.

Since G is 3-connected, it has minimum degree three and thus we can apply Lemma 2 with $d = 3$. We now estimate the 3-weight of H. If $n' = 1$, then the 3-weight of H is non-positive. If $n' = 2$, then H is a single edge and thus $w_3(H) \leq 2 \leq \frac{9}{4}(n'-1)$ by Lemma 1. If $n' \geq 3$, then $w_3(H) \leq \frac{3}{2}n' \leq \frac{9}{4}(n'-1)$ again by Lemma 1. Therefore, the assumption of Lemma 2 is satisfied for $\alpha = \frac{9}{4}$, $\beta_i = 0$ and $d = 3$. The upper bound $\frac{7n-8}{9}$ on the number of colors used by c easily follows. $\square$

In the rest of this section, we show that the bound established in Theorem 3 is the best possible. Let us start with the following lemma that allows us to construct larger examples that match the bound from smaller ones. Notice that in Lemma 4, the monochromatic triangle can be both facial or separating.

Lemma 4. *Let G be a plane 3-connected graph with n vertices that has a non-rainbow coloring c with k colors. If G contains a monochromatic triangle, then there exists a plane 3-connected graph G' with $n+9$ vertices that has a non-rainbow coloring c' with $k+7$ colors and which also contains a monochromatic triangle.*

Proof. Let $v_1v_2v_3$ be a monochromatic triangle contained in G. Split the triangle into two copies, $v_1'v_2'v_3'$ and $v_1''v_2''v_3''$, and keep the rest of the graph (see Figure 1 for illustration). Next, insert a cycle $w_1w_2w_3w_4w_5w_6$ of length six between the cycles $v_1'v_2'v_3'$ and $v_1''v_2''v_3''$ as in the figure, and insert the following edges: $v_1'w_1$, $v_1''w_2$, $v_2'w_3$, $v_2''w_4$, $v_3'w_5$, and $v_3''w_6$. Let G' be the resulting graph. The vertices of $\mathrm{Int}(v_1''v_2''v_3'')$ with same color as v_1'' are recolored by a new color not previously used by c. Six new colors are also assigned to the vertices $w_1, \ldots, w_6$. We conclude that the resulting coloring c' uses $k+7$ colors. It follows that if c is a non-rainbow coloring of G, then c' is a non-rainbow coloring of G'.

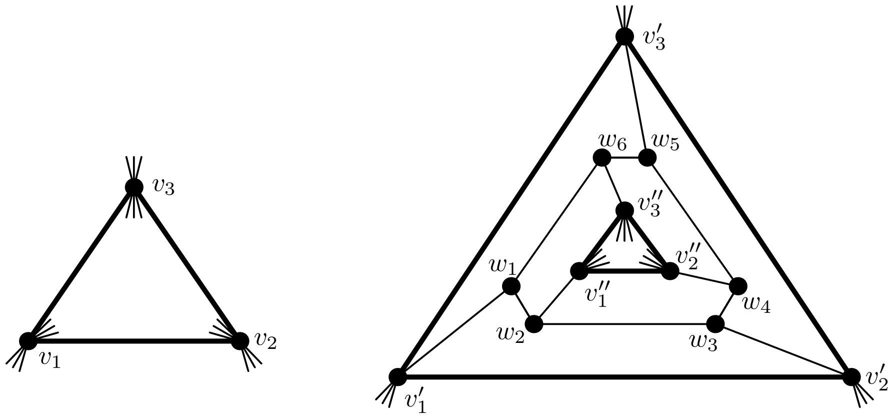

Figure 1: A construction presented in Lemma 4. The monochromatic edges in the configurations are drawn bold.

It remains to verify that the graph G' is 3-connected. The 3-connectivity of G implies that both $\mathrm{Int}(v_1v_2v_3)$ and $\mathrm{Ext}(v_1v_2v_3)$ are 3-connected. Since the graph G' can be viewed as the 3-sum of $\mathrm{Int}(v_1v_2v_3)$, $\mathrm{Ext}(v_1v_2v_3)$ and a 12-vertex 3-connected graph, G' is also 3-connected. $\square$

We can now provide constructions of 3-connected graphs that witness that the bound established in Theorem 3 is tight:

Theorem 5. *For every $n \geq 4$, there exists a plane 3-connected graph G with $n \geq 4$ vertices that has a non-rainbow coloring with $\left\lfloor \frac{7n-8}{9} \right\rfloor$ colors.*

Proof. The reader can find the graphs G for $n = 4, \ldots, 12$ in Figure 2. Since each of the graphs depicted in Figure 2 contains a monochromatic triangle, the existence of graphs G for all $n \geq 13$ follows from Lemma 4. $\square$

4 4-connected plane graphs

In this section, we prove our lower and upper bounds on the number of colors of non-rainbow colorings of 4-connected plane graphs.

Theorem 6. *Let G be a plane 4-connected graph with $n \geq 6$ vertices. The number of colors in a non-rainbow coloring of G does not exceed $\left\lfloor \frac{5n-6}{8} \right\rfloor$. Moreover, if $n \equiv 3 \pmod 8$, then the number of colors does not exceed $\left\lfloor \frac{5n}{8} \right\rfloor - 1$.*

Proof. Fix a non-rainbow coloring c of G. Without loss of generality, we can assume by adding edges that each face is incident with at least one monochromatic edge. In addition, we can also assume that the vertices of each color induce a connected subgraph of G; otherwise, recoloring one of the components with a new

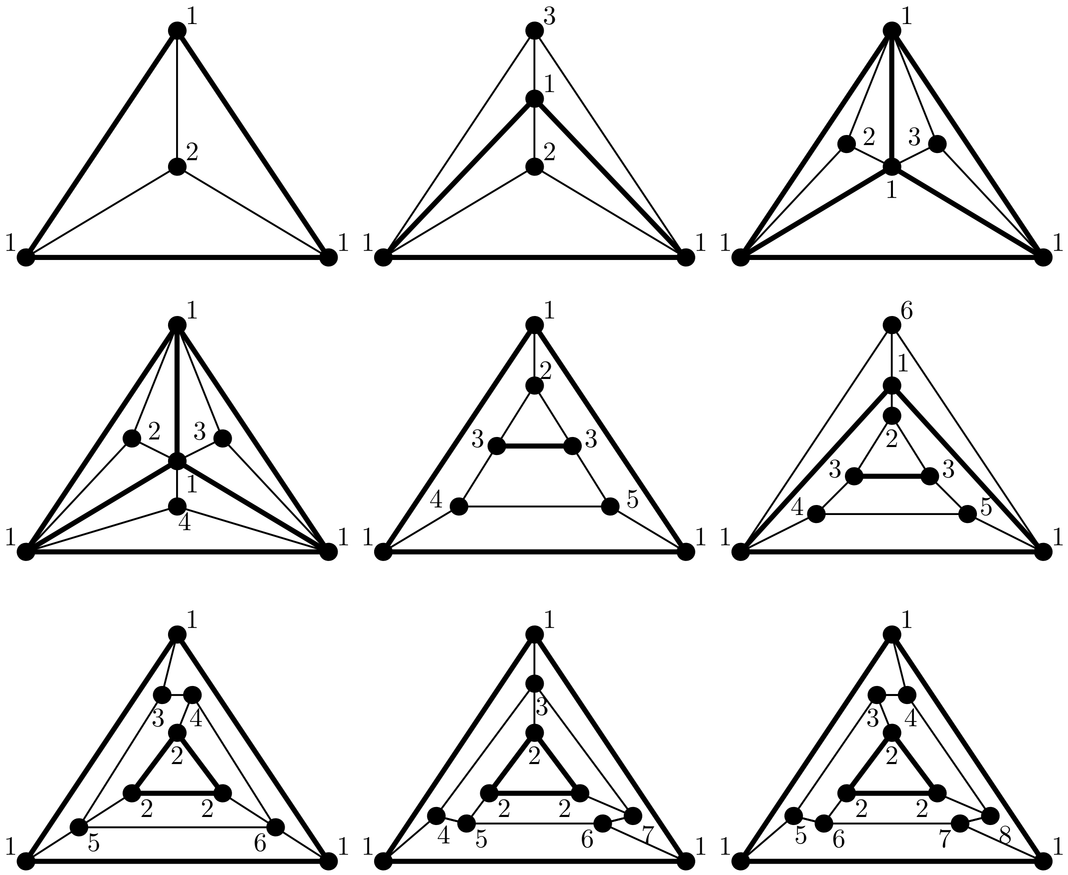

Figure 2: 3-connected plane graphs G with n vertices, $n = 4, \ldots, 12$, that have non-rainbow colorings with $\left\lfloor \frac{7n-8}{9} \right\rfloor$ colors. The edges of G that are monochromatic in such a coloring are drawn bold. The colors assigned to the vertices are represented by numbers.

color yields a non-rainbow coloring of G with more colors. Let H be a subgraph of G induced by the vertices of one of the colors and n' the number of its vertices. Our aim is to show that the 4-weight $w_4(H)$ of H is at most $\frac{8}{3}(n'-1)$.

The inequality $w_4(H) \le \frac{8}{3}(n'-1)$ clearly holds if $n'=1$. If $n' \ge 4$, $w_4(H) \le 2n' \le \frac{8}{3}(n'-1)$ by Lemma 1. If $n'=2$, H is a single edge and thus the degree of both the vertices of H is one. Hence, by Lemma 1, we have $w_4(H) \le 2 < \frac{8}{3}(n'-1)$. If $n'=3$, H is either a 3-vertex path or a triangle. In the former case, $w_4(H) \le 1+2+1 \le \frac{8}{3}(n'-1)$ by Lemma 1. If H is a triangle, then it bounds a 3-face of G since G is 4-connected. Therefore, the edges of H are incident with at most four distinct faces of G, and $w_4(H) \le 4 < \frac{8}{3}(n'-1)$ by the definition of the d-weight. We infer from Lemma 2 applied with $\alpha = \frac{8}{3}$, $\beta_i = 0$, and $d=4$ that the number of colors used by c is at most $\frac{5n-6}{8}$. This establishes the theorem for $n \not\equiv 3 \pmod 8$.

Let us consider the case $n \equiv 3 \pmod 8$. It is straightforward to verify that unless H is a single vertex or $n'=4$, the estimates established in the previous paragraph yield $w_4(H) \le \frac{8}{3}(n'-1) - \frac{2}{3}$. If $n'=4$, then $w_4(H) \le 6 \le \frac{8}{3}(n'-1) - \frac{2}{3}$ unless H is a 4-cycle. Therefore, if there is a maximal connected monochromatic subgraph H_1 of G different from a vertex or a 4-cycle, we can apply Lemma 2 with $\alpha = \frac{8}{3}$, $\beta_1 = \frac{2}{3}$, $\beta_i = 0$ with $i \neq 1$, and $d=4$ to obtain the desired bound. We conclude that if the number of colors used by c is greater than $\frac{5n}{8} - 1$, then each color class is either a single vertex or a 4-cycle. We analyze this case in the next paragraph.

Let f the number of faces of G and s the number of monochromatic 4-cycles of G. Since each face of G is incident with a monochromatic edge, it follows that $f \le 8s$. On the other hand, since G is 4-connected, its minimum degree is at least four, and thus the number of its edges is at least $2n$. Hence, by Euler's formula, we get the following:

$$8s \ge n + f - n \ge (2n+2) - n = n+2 \; .$$

Since $n \equiv 3 \pmod 8$, we infer the following:

$$s \ge \left\lceil \frac{n+2}{8} \right\rceil = \frac{n+5}{8} \; .$$

We conclude that the number of colors used by c, which is equal to $n - 3s$, is at most

$$n - 3s \le n - 3 \cdot \frac{n+5}{8} = \frac{5n-15}{8} \le \frac{5n}{8} - 1 \; .$$

□

In the rest of this section, we show that the bound established in Theorem 6 is tight. We start with a lemma that allows us to construct larger examples of graphs for which the bound of the theorem is tight from smaller ones.

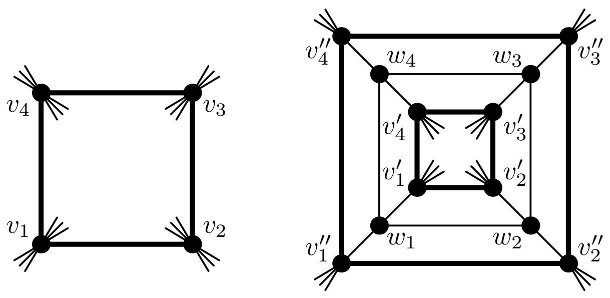

Figure 3: A construction presented in Lemma 7. The monochromatic edges in the configurations are drawn bold.

Lemma 7. *Let G be a plane 4-connected graph with n vertices that has a non-rainbow coloring c with k colors. If G contains a separating monochromatic 4-cycle, then there exists a 4-connected plane graph with $n+8$ vertices that has a non-rainbow coloring with $k+5$ colors and with a separating monochromatic 4-cycle.*

Proof. Let $v_1v_2v_3v_4$ be a monochromatic 4-cycle of G. Split the cycle to two cycles $v_1'v_2'v_3'v_4'$ and $v_1''v_2''v_3''v_4''$. Each vertex of G adjacent to v_i is adjacent to v_i' or to v_i'' in such a way that the resulting graph is still plane, see Figure 3. In addition, add a new cycle $w_1w_2w_3w_4$ between the two cycles $v_1'v_2'v_3'v_4'$ and $v_1''v_2''v_3''v_4''$, and add edges $v_i'w_i$ and $v_i''w_i$, $i=1,2,3,4$. Let G' be the obtained graph. The vertices v_i' keep the color of the vertices v_i, the vertices v_i'' and all the vertices with the same color in $\mathrm{Int}(v_1''v_2''v_3''v_4'')$ are recolored by a new color and each of the vertices w_i also receives a new color. Hence, G' is a graph of order $n+8$ and the constructed coloring is a non-rainbow coloring of G' with $k+5$ colors.

It remains to verify that G' is 4-connected. Let A' be a vertex cut of G' formed by at most three vertices. Note that each component of $G' \setminus A'$ contains at least one original vertex of G, i.e., a vertex different from v_i', v_i'' and w_i. Let A be the set obtained from A' by replacing v_i', v_i'' or w_i by the vertex v_i. Note that A contains at most three vertices. It is easy to verify that if $G' \setminus A'$ is disconnected, the graph $G \setminus A$ is also disconnected. This contradicts our assumption that G is 4-connected. □

We finish this section with a construction of graphs for which the bound proven in Theorem 6 is tight.

Theorem 8. *For every $n \geq 6$, $n \not\equiv 3 \pmod 8$, there exists a plane 4-connected graph G with n vertices that has a non-rainbow coloring with $\left\lfloor \frac{5n-6}{8} \right\rfloor$ colors. Moreover, for $n \geq 6$ and $n \equiv 3 \pmod 8$, there exists such a graph G that has a non-rainbow coloring with $\left\lfloor \frac{5n}{8} \right\rfloor - 1$ colors.*

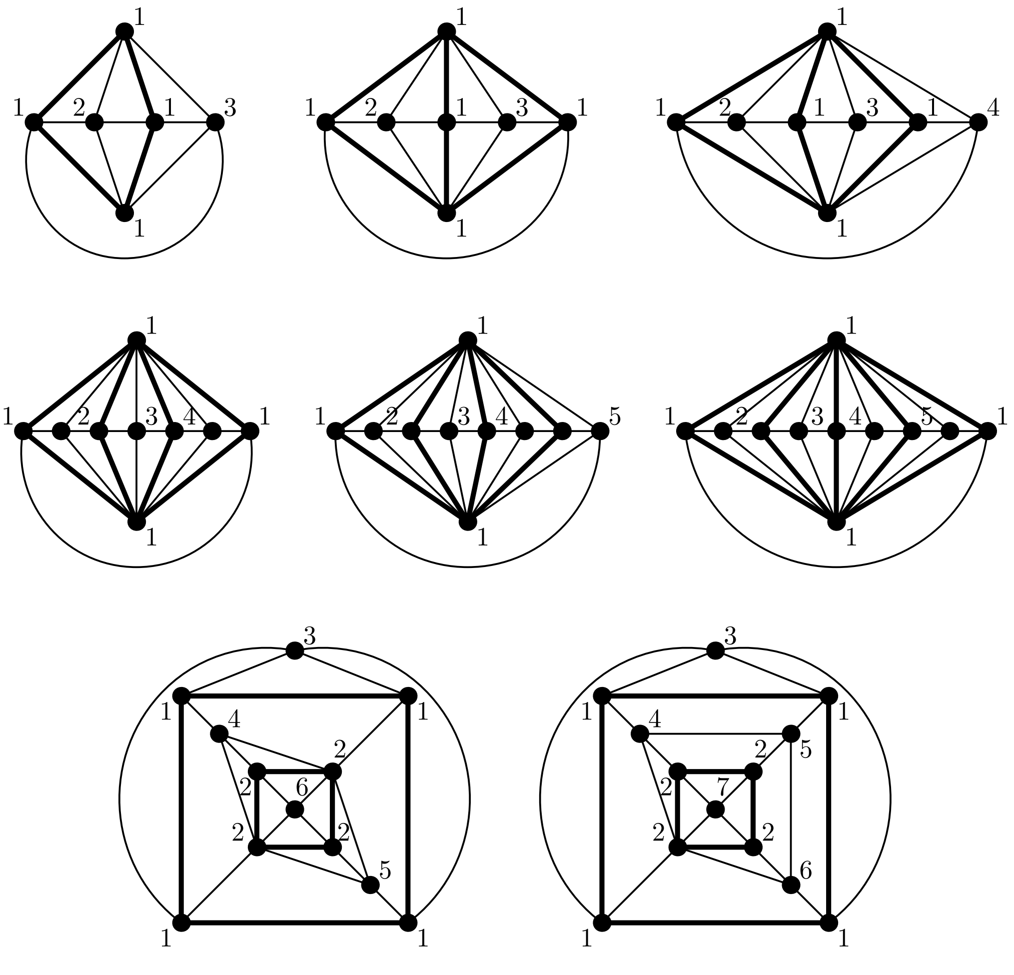

Figure 4: 4-connected plane graphs G with n vertices, $n = 6, 7, \ldots, 13$, that have non-rainbow colorings with $\left\lfloor \frac{5n-6}{8} \right\rfloor$ colors if $n \neq 11$, i.e., $n \not\equiv 3 \pmod 8$, and with $\left\lfloor \frac{5n}{8} \right\rfloor - 1 = 5$ colors, if $n = 11$. The edges of G that are monochromatic in such a coloring are drawn bold. The colors assigned to the vertices are represented by numbers (some vertices are not labeled with numbers; those vertices are colored with 1).

Proof. The construction of graphs G for $n = 6, 7, \ldots, 13$ can be found in Figure 4, and the existence of the graphs G for $n \geq 14$ follows from Lemma 7 by induction on n. $\square$

5 5-connected plane graphs

In the last section of the paper, we focus on bounds on 5-connected plane graphs. We start with showing the upper bound on the number of the colors.

Theorem 9. *Let G be a plane 5-connected graph with n vertices. The number of colors in a non-rainbow coloring of G does not exceed $\left\lfloor \frac{43}{100}n - \frac{19}{25} \right\rfloor$.*

Proof. By adding edges if necessary, we can assume that each face of G is incident with a monochromatic edge. Our aim is to apply Lemma 2 with $\alpha = \frac{50}{19}$ and $d = 5$. In particular, we have to show that $w_5(H) \leq \frac{50}{19}(n' - 1)$ for each maximal connected monochromatic subgraph H of G, where n' is the number of vertices of H.

If $n' = 1$, the estimate on $w_5(H)$ clearly holds. If $n' \geq 20$, then $w_5(H) \leq \frac{5}{2}n' \leq \frac{50}{19}(n' - 1)$ by Lemma 1. In the rest, we consider the case $1 < n' < 20$.

Let k_v be the number of faces of G that contain a monochromatic edge of H incident with a vertex v. As explained in the proof of Lemma 1, it holds that

$$w_5(H) \leq \frac{1}{2} \sum_{v \in V(H)} (k_v + 5 - \deg_G(v)) \quad \text{where} \tag{1}$$

$$k_v + 5 - \deg_G(v) \leq \min\{2\deg_H(v), 5\} \quad \text{for every } v \in V(H). \tag{2}$$

In particular, a vertex v contributes at most one to the sum in (1) if $\deg_H(v) = 1$, at most two if $\deg_H(v) = 2$, and it contributes at most $\frac{5}{2}$ otherwise.

Since the graph G is 5-connected, each face f of H such that the boundary of f is a single 3-cycle or 4-cycle is also a face of G. For each such face f of H, choose one of its edges to be its representative; let e_f be the chosen edge. Since each edge of H is incident with at most two faces and there are at least three edges incident with each face, we infer from Hall's theorem that all the edges e_f can be chosen to be pairwise distinct. Observe that for each edge $e_f = uv$, we can decrease both k_u and k_v in (1) by 1/2 without violating the inequality, since we count the face f at least three times (at each of the three or four vertices incident with it) instead of counting it only twice.

Let H' be the subgraph of H obtained by removing all the edges e_f. Note that H' need not to be connected. We claim that if a vertex v has degree two in H', then its contribution to the sum in (1) is at most two. In order to see this, we distinguish two cases based on the degree δ of v in H. If $\delta = 2$, then the claimed bound matches the estimate given in the formula (2). If $\delta \geq 3$, then for each of $\delta - 2$ removed edges we can decrease k_v by 1/2, and thus v contributes at most

$\frac{5}{2} - (\delta - 2)/2 \leq 2$ to the sum. Similarly, the contribution of a vertex of degree one in H' is at most $3/2$ and that of an isolated vertex of H' at most 1.

Since each face of H' is incident with at least five distinct vertices (we have removed from each face of size three or four at least one edge and we have removed different edges from different faces), the number of edges of H' is at most $\frac{5n'-10}{3}$ by Euler's formula (note that the bound holds even if H' is not connected) unless $n' = 2$ or $n' = 3$. If $n' = 2$, then the monochromatic edge of H is contained in at most two faces and thus $w_5(H) \leq 2$. In particular, $w_5(H) \leq \frac{50}{19}(n'-1)$. If $n' = 3$, then $w_5(H) \leq 4$ as there are at most four faces containing a monochromatic edge of H and $w_5(H) \leq \frac{50}{19}(n'-1)$ as desired. Hence, we can assume in the rest that $n' > 3$, in particular, that the number of edges of H' does not exceed $\frac{5n'-10}{3}$.

Let n'_0 be the number of isolated vertices of H', n'_1 the number of vertices of degree one, n'_2 the number of vertices of degree two and m' the number of its edges. Since the sum of the degrees of the vertices of H' is $2m'$, it holds that $3n' - 3n'_0 - 2n'_1 - n'_2 \leq 2m'$. An easy counting argument now yields that

$$3n'_0 + 2n'_1 + n'_2 \geq 3n' - 2m' \geq 3n' - 2 \cdot \frac{5n'-10}{3} = \frac{20-n'}{3} .$$

Hence, the 5-weight of H can be bounded as follows:

$$w_5(H) \leq n'_0 + \frac{3}{2}n'_1 + 2n'_2 + \frac{5}{2}(n' - n'_0 - n'_1 - n'_2) = \frac{5}{2}n' - \frac{3}{2}n'_0 - n'_1 - \frac{n'_2}{2} =$$

$$\frac{5}{2}n' - \frac{1}{2}(3n'_0 + 2n'_1 + n'_2) \leq \frac{5}{2}n' - \frac{20-n'}{6} = \frac{8n'-10}{3} .$$

We leave to the reader to verify that $\frac{8n'-10}{3} \leq \frac{50}{19}(n'-1)$ for $n' \leq 20$. Finally, the upper bound of $\frac{43}{100}n - \frac{19}{25}$ on the number of colors follows from Lemma 2 applied with $\alpha = \frac{50}{19}$, $\beta_i = 0$ for all i, and $d = 5$. $\square$

Unfortunately, we were not able to find matching lower and upper bounds on the number of colors in non-rainbow colorings of 5-connected plane graphs. The best lower bound construction that we have found is given in the next theorem.

Theorem 10. *For every real number $\varepsilon > 0$, there exists a plane 5-connected graph G with n vertices that has a non-rainbow coloring with $\left(\frac{171}{400} - \varepsilon\right) n$ colors.*

Proof. We construct plane 5-connected graphs by combining plane gadgets of two different types, A-gadgets and B-gadgets.

An A-gadget is obtained as follows (the gadget is depicted in Figure 5): consider three concentric cycles in the plane, the inner and outer ones of length 10 and the middle one of length 20, and join the vertices of the middle cycle in an alternating way to the vertices of the inner and outer cycles. The graph obtained in this way is the graph formed by bold edges in the left part of Figure 5. Into each of the 20 pentagonal faces of the obtained graph, add a single vertex and

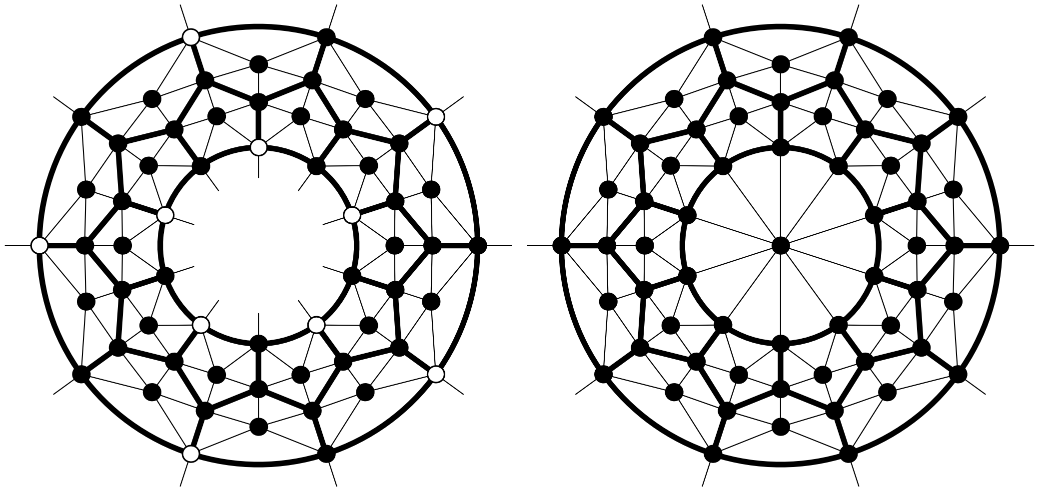

Figure 5: An A-gadget and its coloring (the monochromatic edges are bold). The interconnecting edges between an A-gadget and two B-gadgets are depicted in the left part of the figure and the interconnection between an A-gadget with an additional vertex of degree 10 and a B-gadget surrounding it in the right part of the figure. The vertices used to find five vertex-disjoint paths in the proof of 5-connectivity are drawn with empty circles.

join it with all the five vertices on its boundary. This completes the construction of the gadget. Note that an A-gadget has 60 vertices.

We now describe the coloring of an A-gadget. The 40 vertices of the three original cycles are colored with the same color and the 20 new vertices with mutually distinct colors. In this way, a coloring of the gadget with 21 colors avoiding a rainbow face is obtained (see Figure 5 for illustration).

The construction of a B-gadget is more complex. First, start with a copy of the dodecahedron and subdivide each edge of two antipodal faces. Next, place a copy of the dodecahedron into each of the ten pentagonal faces of the obtained graph and join it by five edges to the rest of the graph as depicted in the left part of Figure 6 (the gray pentagons represent the copies of the dodecahedron). Finally, add a vertex to each of the eleven faces of each copy of the dodecahedron and join this vertex to all the five vertices on the boundary of the face (see Figure 6 for illustration). The obtained graph has $30 + 10 \cdot 20 + 10 \cdot 11 = 340$ vertices.

We next describe the coloring of the vertices of the B-gadget. The vertices of each copy of the dodecahedron are colored with the same color but vertices in different copies receive different colors. The remaining vertices are colored with mutually distinct colors. In this way, a coloring with $30+10+10\cdot 11 = 150$ colors that avoids a rainbow face is obtained. The coloring is also depicted in Figure 6.

We are now ready to construct plane graphs G_k. The graph G_k is obtained

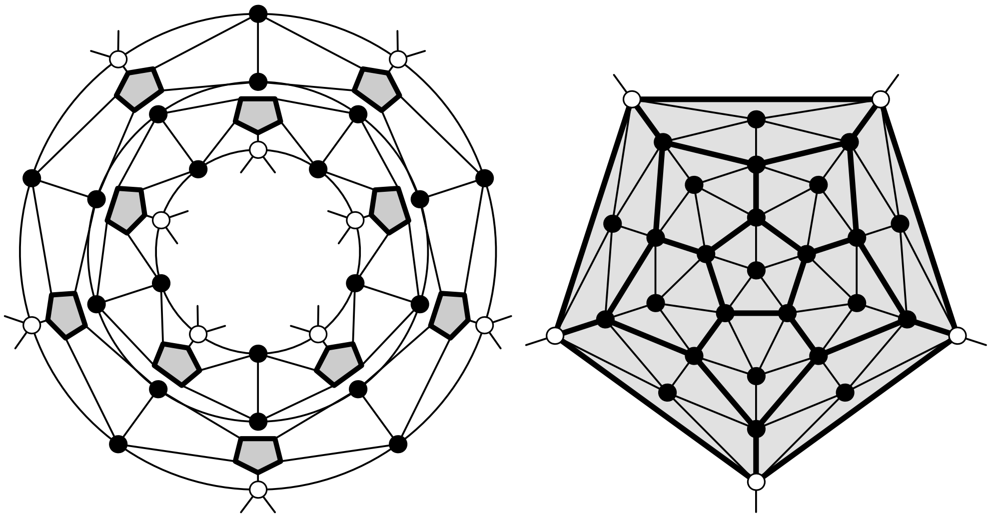

Figure 6: A B-gadget (depicted in the left part of the figure) and its coloring where the monochromatic edges are bold. Each of the gray parts of the gadget is a copy of the drawing depicted in the right. The vertices used to find five vertex-disjoint paths in the proof of 5-connectivity are drawn with empty circles.

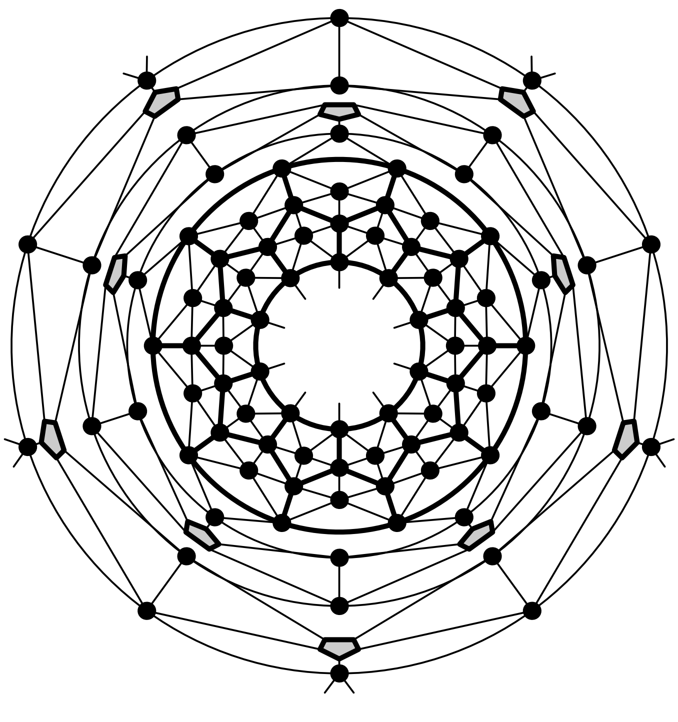

Figure 7: Edges joining an A-gadget and a B-gadget.

from $k+1$ A-gadgets and k B-gadgets by placing the gadgets concentrically in an alternating way, i.e., each B-gadget is surrounded by two A-gadgets. Next, ten edges between each pair of two neighboring gadgets are added in such a way that each vertex of the A-gadget is incident with one such edge and the vertices of the B-gadget have degree at least five, say this is done in the way drawn in Figure 7. A new vertex is placed in the most inner face and joined by ten edges to the vertices in its boundary as depicted in the right part of Figure 5. Similarly, a new vertex is added to the outer face. Hence, the graph G_k has $n = 60(k+1) + 340k + 2 = 400k + 62$ vertices in total.

A coloring of G_k with non-rainbow faces is obtained from the colorings of the gadgets and vertices of different gadgets are colored with distinct colors. The two vertices not contained in any of the gadgets are assigned colors different from the colors of all the other vertices. In this way, a coloring of G_k with $21(k+1)+150k+2 = 171k+23$ colors is obtained. Hence, for a sufficiently large integer k, the coloring uses more than $(\frac{171}{400} - \varepsilon)n$ colors.

It remains to verify that the graph G_k is 5-connected. Since a complete proof of this fact is very technical, we sketch only the main idea and the reader is invited to check the missing details. In order to verify that the graph G_k is 5-connected, it is enough to construct five vertex-disjoint paths between any pair of vertices u and v of G_k. Assume first that u and v are contained in different gadgets. If u is contained in an A-gadget, find five vertex-disjoint paths from u to the five vertices depicted with empty cycles in right part of Figure 5 that surrounds the part of G_k containing the vertex v. If u is contained in a copy of the dodecahedron in a B-gadget, find first five vertex disjoint paths to the five vertices on the outer face of the dodecahedron and then extend them to vertex-disjoint paths to the vertices on the boundary of the gadget. Since the five vertices on the inner boundary and the five vertices on the outer boundary that are drawn with empty cycles in Figures 5 and 6 can be joined by five vertex-disjoint paths, there exist five vertex-disjoint paths between u and v. We leave the remaining details to the reader. Note that it is also necessary to verify the existence of five vertex-disjoint paths between u and v if u and v are in the same gadget, or if one or both are the vertices of degree 10. $\square$

We were not able to close the gap between the multiplicative constants in the bounds that we provide in Theorems 9 and 10. We leave determining the optimal multiplicative constant in the bounds as an open problem.

Conjecture 3. *There exists a constant C such that a rainbow coloring of a 5-connected plane graph with n vertices uses at most $\frac{3}{7}n + C$ colors and there exist 5-connected plane graphs with n vertices (for arbitrarily large n) with non-rainbow colorings with at least $\frac{3}{7}n - C$ colors.*

Note that the conjectured multiplicative constant of 3/7 is sandwiched between the bounds given in Theorems 9 and 10.

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