A step towards the Bermond-Thomassen conjecture about disjoint cycles in digraphs

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Abstract

In 1981, Bermond and Thomassen conjectured that every digraph with minimum out-degree at least $2k-1$ contains k disjoint cycles. This conjecture is trivial for $k=1$, and was established for $k=2$ by Thomassen in 1983. We verify it for the next case, by proving that every digraph with minimum out-degree at least five contains three disjoint cycles. To show this, we improve Thomassen's result by proving that every digraph whose vertices have out-degree at least three, except at most two with out-degree two, indeed contains two disjoint cycles.

1 Introduction

Our notations will mainly follow that of [2]. By *cycle* we mean oriented cycle, that is an oriented path starting and ending at the same vertex. A cycle of length d is called a *d-cycle*. A 1-cycle is a *loop* and a 3-cycle is also called a *triangle*. All digraphs contained in this paper can have loops and 2-cycles but no parallel arcs. A digraph without cycles of length at most two is called an *oriented graph*.

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Fix a digraph $D = (V, A)$. Its *order* is the size of the vertex-set V. Given a subset X of V, the sub-digraph of D *induced by* X is the digraph $D[X] := (X, A')$ where A' is the set of all arcs in A that start and end in X. Two sub-digraphs D_1 and D_2 of D are *disjoint* if their vertex-sets are. We write $v \to u$ to denote an arc from the vertex v to the vertex u.

We denote by D^* the digraph obtained from D by reversing the direction of every arc. For every vertex $v \in V$ let $N_D^+(v) := \{x \in V : v \to x \in A\}$ be the *out-neighbourhood of* v *in* D, and let $d_D^+(v) := |N_D^+|$ be the *out-degree of* v *in* D. The vertices of $N_D^+(v)$ are the *out-neighbours of* v. The *in-neighbourhood of* v *in* D is $N_D^-(v) := N_{D^*}^+(v)$, and its *in-degree* is $d_D^-(v) := |N_D^-(v)|$. The vertices of $N_D^-(v)$ are the *in-neighbours of* v. If the context is clear, we may omit the subscript and just write $N^+(v)$ and $N^-(v)$.

Given two disjoint subsets $X, X' \subset V$ we say that X *dominates* X' if X' is contained in the out-neighbourhood of each vertex of X. If the set X is comprised of only one vertex v we simply say that v *dominates* X' . The set X' is *dominated* if there exists a vertex dominating it. The set X *dominates* a sub-digraph D' of D if it dominates its vertex-set $V(D')$.

An arc is said to be *d-dominated* if it is dominated by a vertex of out-degree d.

We are interested in the following conjecture stated by Bermond and Thomassen in 1981.

Conjecture 1 ([3]). *For every positive integer k, every digraph with minimum out-degree at least $2k - 1$ contains k disjoint cycles.*

It is an obvious observation if k is one, and Thomassen gave a nice and simple proof of it when k is two in 1983.

Theorem 1 ([6]). *Every digraph with minimum out-degree at least three contains two disjoint cycles.*

Thomassen [6] also established the existence of a finite integer $f(k)$ such that every digraph of minimum out-degree at least $f(k)$ contains k disjoint cycles. As noted in [3], such an integer cannot be less than $2k - 1$, so Conjecture 1 is optimal. Alon [1] proved that for every integer k, the value $64k$ is suitable for $f(k)$ in 1996. Recently, Conjecture 1 has been verified for (almost) regular tournaments [5, 4].

Our main result is the following theorem, which proves Conjecture 1 when k is three.

Theorem 2. *Every digraph with minimum out-degree at least five contains three disjoint cycles.*

We note that the method used in [1] allows to upper bound the order of a minimum counter-example to Conjecture 1. For instance, when k is three the order of a minimum counter example is at most 42. However, as pointed out in [1], this bound is out of reach for a brute-force attack. Thus we need to develop new tools to study this conjecture and prove Theorem 2. One of them is to strengthen Theorem 1.

Theorem 3. *Let D be a digraph whose vertices have out-degree at least three, except at most two which have out-degree two. The digraph D contains two disjoint cycles.*

The paper is organised as follows. In the next section we slightly improve Thomassen's result by proving Theorem 3 which is a crucial ingredient in our proof of Theorem 2. Section 3 is devoted to the proof of a property of a certain class of digraphs, which may be of independent interest. In Section 4 we establish Theorem 2. The proof proceeds by contradiction: we consider a minimum counter-example D—with respect to the number of vertices—to the statement of the theorem, and exhibit some of its structural properties. Then, the argument is split into two cases: in Sub-section 4.1 we suppose that D does not contain a triangle while in Sub-section 4.2 we establish the result if D contains a triangle.

2 Improving Theorem 1

As mentioned earlier, Thomassen proved that Conjecture 1 is true if k is two, namely every digraph with minimum out-degree three contains two disjoint cycles. The goal of this section is to strengthen this result, by proving Theorem 3.

Proof of Theorem 3. Contrary to the statement, let $D = (V, A)$ be a minimum counter-example with respect to the number of vertices. We also assume that each vertex has out-degree at most three. First, observe that D cannot contain a loop. If C is a loop, the digraph obtained from D by removing the vertex of C has minimum out-degree at least one, thus it contains a cycle C'. The cycles C and C' of D are disjoint, a contradiction. So the order of D is at least four. We now establish two properties of D. Recall that a sub-digraph is 2-dominated if there exists a vertex of out-degree two dominating it.

(A) *Every 2-cycle of D is 2-dominated. In particular D contains at most two 2-cycles.*

Suppose that $C := uv$ is a 2-cycle. Let D' be the digraph obtained from D by removing u and v. Then D' cannot have minimum out-degree at least one, otherwise it would contain a cycle which would be disjoint from C, a contradiction. Therefore there exists a vertex of D of out-degree two dominating C, as asserted. From this fact it directly follows that D does not contain more than two 2-cycles, since each vertex of out-degree two can dominate at most one 2-cycle and D contains at most two vertices of out-degree two.

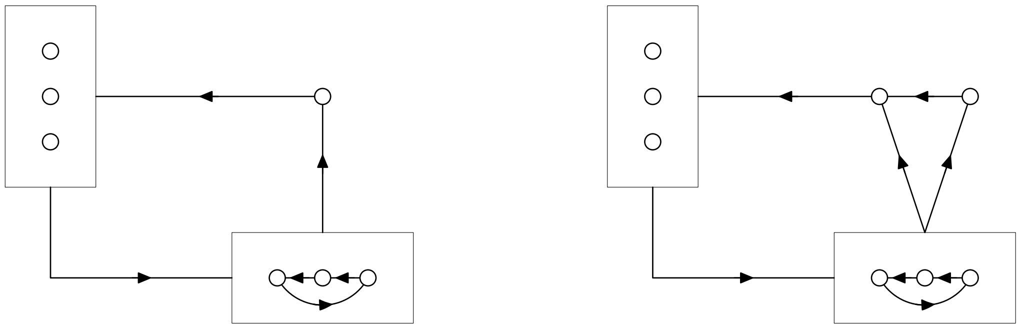
The next property, proved in [6], is still valid under our weaker assumptions.

(B) *Every arc of* D *is dominated.*
Suppose that $u \to v \in A$ is not dominated. By Property (A), we can assume that $v \to u$ is not an arc of D. Denote by D' the digraph obtained from D by first removing all arcs out-going from u except $u \to v$, and then contracting the arc $u \to v$ into a new vertex w. The out-degree of w in D' is equal to the out-degree of v in D. Moreover, the out-degree of each other vertex of D' is the same as its out-degree in D. Hence, by the minimality of D, the digraph D' contains two disjoint cycles, which yield two disjoint cycles in D, a contradiction.

Fix a vertex v and let x be an in-neighbour of v. Note that $d^-_D(v) \geq 1$ by the minimality of D. As the arc $x \to v$ is dominated, there exists a vertex $y \in V$ with $\{x, v\} \subseteq N^+(y)$. Consequently the digraph $D[N^-(v)]$ has out-degree at least one and thus contains a cycle. In particular the size of the in-neighbourhood of each vertex is at least two. Observe now that if $d^-_D(v) \geq 3$ for every $v \in V$, then D indeed contains two disjoint cycles: just apply Theorem 1 to D^*.

Therefore, there exists a vertex of in-degree two in D, and hence a 2-cycle $C_1 := uv$. By Property (A), let z be a vertex of out-degree two dominating u and v. The sub-digraph $D[N^-(z)]$ contains a cycle, which must intersect C_1. So we can assume that $u \to z \in A$, and we denote by C_2 the cycle zu. Again by Property (A), there exists a vertex z' of out-degree two that dominates C_2. Note that $z' \neq v$, otherwise D would contain three 2-cycles, thereby contradicting Property (A). Observe also that neither z nor u can dominate z', otherwise D would contain three 2-cycles. Therefore the cycle contained in $D[N^-(z')]$ is disjoint from the 2-cycle uz, a contradiction. This contradiction concludes the proof. $\square$

We note that this result is optimal, since a symmetrically oriented triangle—i.e. three vertices x_1, x_2, x_3 with an arc from x_i to x_j whenever $i \neq j$—does not have two disjoint cycles. It is also optimal if we restrict ourselves to oriented graphs, since there exist oriented graphs on seven vertices with three vertices of out-degree two, four vertices of out-degree three and no two disjoint cycles. See Figure 1(a) for an example. Moreover, the oriented graph



(*a*) (*b*)

Figure 1: (*a*) An oriented graph with three vertices of out-degree two, four vertices of out-degree three and no two disjoint cycles, and (*b*) an oriented graph whose vertices all have out-degree three, except one which has out-degree one, and yet without two disjoint cycles. An arc from/to a box goes from/to every vertex of the box.

of Figure 1(*b*) has no two disjoint cycles, yet every vertex has out-degree three except one which has out-degree one.

3 Arc-dominated oriented graphs

We say that a digraph $D = (V, A)$ is *arc-dominated* if every arc of A is dominated. As we will see, a minimum counter-example to Theorem 2—and more generally, to Conjecture 1—must be arc-dominated, and it must be an oriented graph—i.e. it contains neither a loop nor a 2-cycle. We put the following proposition in a dedicated section because we believe that it might be of independent interest.

Proposition 4. *Let $D = (V, A)$ be an arc-dominated oriented graph, and let $X \subset V$ such that $D[X]$ is either acyclic or an induced cycle of D. There exists a cycle C disjoint from $D[X]$ such that every vertex of C has at least one out-neighbour in X.*

Proof. We set $X' := V \setminus X$. Let S be the set of vertices of X' having at least one out-neighbour in X. Observe that it is enough to prove that $D[S]$ contains a cycle. To this end, it suffices to establish that every vertex of S has at least one in-neighbour in S. Suppose on the contrary that there exists a vertex $v \in S$ with no in-neighbour in S. We set $Y := \left(N^-_D(v) \cup N^+_D(v)\right) \cap X$. By the definition of S, v has an out-neighbour x in X, so in particular $Y \neq \emptyset$. Since for every $y \in Y$ there is an arc between v and y, and since D is arc-dominated, there exists a vertex z which dominates $\{v, y\}$. It follows that $z \in$

$X \cap N_D^-(v) \subset Y$. In particular this proves that $D[Y]$ and hence $D[X]$ contains a cycle. This is not possible if $D[X]$ is acyclic and concludes the proof in this case. If $D[X]$ is an induced cycle C' of D, then $D[X] = D[Y] = C'$. Consider the out-neighbour y of x in C'. By what precedes, it is dominated by a vertex of $N_D^-(v) \cap X$, which must be x since C' is induced. This is a contradiction since $\{v, x\}$ would induce a 2-cycle in D. $\square$

Corollary 5. *Let $D = (V, A)$ be an arc-dominated oriented graph. Suppose that C is a cycle of D, and C' an induced cycle disjoint from C. If there is no arc from a vertex of C to a vertex of C' then D contains three disjoint cycles.*

Proof. We apply Proposition 4 with X being $V(C')$. We deduce that there exists a cycle C_1 disjoint from C' such that every vertex of C_1 has an out-neighbour in C'. As there is no arc from C to C', the cycle C_1 is certainly disjoint from C. Thus, C, C' and C_1 are three disjoint cycles of D. $\square$

4 Proof of Theorem 2

Our goal in this section is to establish Theorem 2. We proceed by contradiction: we suppose that the statement of the theorem is false, and consider a counter-example with the minimum number of vertices. We first establish some fundamental properties of such a digraph, that will be extensively used in the sequel. Until the end, we let $D = (V, A)$ be a counter-example to the statement of Theorem 2 with the smallest number of vertices, and subject to this with the smallest number of arcs. In particular, every vertex has out-degree exactly five. We denote by n the order of D. Note that $n \geq 5$.

Lemma 6. *The following hold.*

- (i) *The digraph D is an oriented graph, i.e. it has no loop and no 2-cycle.*
- (ii) *Every arc of D is dominated. In particular, the in-neighbourhood of every vertex contains a cycle.*
- (iii) *Every triangle of D is dominated by three different vertices.*
- (iv) *If a vertex v dominates a cycle C, there exists a triangle vuw with $u \in V(C)$ and $w \notin V(C)$.*

Proof. (i) Suppose that C is a cycle of D of length at most two. Note that the induced sub-digraph D' of D obtained by removing the vertices of C has minimum degree at least three. Thus, by Theorem 1, D'

contains two disjoint cycles, which are certainly disjoint from the cycle C. Hence, D contains three disjoint cycles, a contradiction.

(ii) It is proved exactly as Property (B) in the proof of Theorem 3, so we do not repeat it here.

(iii) Let C be a triangle of D, and consider the digraph D' obtained from D by removing the vertices of C. The digraph D' has minimum out-degree at least two. Moreover every vertex of D' that does not dominate C in D has out-degree at least three in D'. As D' cannot contain two disjoint cycles—otherwise D would contain three disjoint cycles—the contrapositive of Theorem 3 implies that at least three vertices of D' have out-degree two, and hence these vertices dominate C in D.

(iv) Denote by C' a cycle contained in $N^-(v)$. As v dominates a cycle C, by (i) the cycles C and C' are disjoint. According to Corollary 5, there exists an arc from C to C', which yields the sought triangle.

□

According to Item (i) of the preceding lemma, D is actually an oriented graph. So, as every vertex has out-degree five, we deduce that the order n of D is at least 11. The proof is now split into two parts, regarding whether D contains a triangle.

4.1 The digraph D does not contain a triangle

In this sub-section, we assume that D does not contain a triangle. In particular, every 4-cycle of D is induced. We first establish some useful properties of D.

Lemma 7. *For every vertex v of D the sub-digraph induced by the out-neighbours of v is acyclic.*

Proof. Since D has no triangle this follows directly by Lemma 6(iv). □

We define a spanning sub-digraph D' of D as follows. Recall that, by Lemma 6(ii), the in-neighbourhood of every vertex u of D contains an induced cycle C_u. We let $D' = (V, A')$ be the spanning sub-digraph of D where A' is comprised of all arcs $v \to u$ of D with $v \in V(C_u)$. The obtained digraph D' has some useful properties, stated in the next lemma.

Lemma 8. *The following hold.*

(i) *If $v \to u$ belongs to A' then $N^+_D(v) \cap N^-_{D'}(u) \neq \emptyset$.*

(*ii*) *The digraph D' is 4-regular, i.e. $d^+_{D'}(v) = 4 = d^-_{D'}(v)$ for every vertex v. In particular, D contains a 4-cycle.*

(*iii*) *If the arc $v \to u$ belongs to $A \setminus A'$ then $N^+_D(v) \cap N^-_D(u) = \emptyset$.*

Proof. (*i*) Let $v \in V(C_u)$. By the definition of C_u, the out-neighbour of v in C_u dominates u in D' and belongs to $N^+_D(v)$.

(*ii*) By Lemma 6(*ii*), for every vertex v we have $d^-_{D'}(v) \geq 4$ since D contains no triangle. Therefore, to prove the statement we only need to show that $d^+_{D'}(v) \leq 4$ for every vertex v. Suppose on the contrary that v is a vertex of D with out-degree five in D'. Hence, $N^+_D(v) = N^+_{D'}(v)$. Let $u \in N^+_{D'}(v)$. By (*i*), $N^+_{D'}(v) \cap N^-_{D'}(u) \neq \emptyset$. So the sub-digraph of D' induced by the out-neighbours of v has minimum in-degree at least one, and hence it contains a cycle. This contradicts Lemma 7.

(*iii*) Suppose that $v \to u$ is an arc of D contradicting the statement. Again, we shall prove that the out-neighbourhood of v in D contains a cycle, thereby contradicting Lemma 7. Let $z \in N^+_D(v)$, it suffices to prove that z is dominated by a vertex of $N^+_D(v)$. If $z = u$ this is clear by the definition of v and u, so suppose that $z \neq u$. By (*ii*), the vertex v has out-degree four in D', thus $v \to z \in A'$ and hence (*i*) yields the conclusion.

□

We prove a last preliminary lemma before turning to the proof of Theorem 2.

Lemma 9. *Let C be a 4-cycle of D. The following hold.*

(*i*) *There exist at least three vertices with each exactly three out-neighbours in C;*

(*ii*) *at least one of the arcs of C is not in D'.*

Proof. (*i*) By Lemma 7 every vertex of D has at most three out-neighbours in C. Suppose that at most two vertices of D have three out-neighbours in C. Then, every vertex of the sub-digraph of D obtained by removing C has out-degree at least three, except at most two vertices that have out-degree two. By Theorem 3, it contains two disjoint cycles. These two cycles together with C yield three disjoint cycles in D, a contradiction.

(*ii*) Suppose on the contrary that $C := xyzt$ is a 4-cycle of D'. By the preceding item, there exist three vertices a, b and c with each three out-neighbours in C. Note that no vertex of C can dominate a vertex of $\{a, b, c\}$, otherwise D would contain a triangle or a 2-cycle. As there are 9 arcs from $\{a, b, c\}$ to C, at least one vertex of C, say y, is dominated by $\{a, b, c\}$. Furthermore, one of the arcs $a \to y, b \to y, c \to y$ is not in D'. Otherwise, as $x \to y \in A'$ and $d^-_{D'}(y) = 4$ by Lemma 8(*ii*), the cycle C_y would be comprised of the vertices a, b, c and x. This is not possible since there is no arc from x to $\{a, b, c\}$. Without loss of generality we can assume that $a \to y \notin A'$. By Lemma 8(*iii*), we deduce that the vertex x is not an out-neighbour of a in D. It follows that $a \to z$ and $a \to t$ are in A, and hence in A' by Lemma 8(*iii*) since a dominates $\{y, z\}$.

We assert that $\{b, c\}$ dominates $\{x, t\}$ in D. By symetry it is enough to prove that b dominates $\{x, t\}$. If it is not the case then b dominates z in D. As $y \in N^+_D(b) \cap N^-_D(z)$, Lemma 8(*iii*) implies that $b \to z \in A'$. Hence the induced cycle C_z contains the vertices a, y and z, which is a contradiction since $\{a, y\}$ dominates z. This proves the assertion.

Now, note that the arcs $b \to x$ and $c \to x$ must belong to A' by Lemma 8(*iii*). Consequently, the induced cycle C_x contains the vertices b, c and t, which is a contradiction since $\{b, c\}$ dominates t in D. This concludes the proof.

□

We now switch to the proof of Theorem 2. We shall obtain a contradiction by proving that D' contains a 4-cycle. To this end, we first prove Property (C) below, which states that D contains a 4-cycle with two consecutive arcs in D'. As we shall see, this implies that D' contains a 4-cycle.

(C) *There exists a* 4*-cycle of* D *with two consecutive arcs belonging to* A'.

By Lemmas 8(*ii*) and 9(*ii*), let $C := xyzt$ be a 4-cycle of D with $x \notin V(C_y)$. Consequently, C and C_y are disjoint. Let us write $C_y = abcd$ with $a \notin V(C_b)$. So, the cycles C_y and C_b are disjoint. As D does not have three disjoint cycles, we deduce that C_b must contain a vertex of C. This vertex cannot be x, since by Lemma 8(*iii*) x has no out-neighbour in $N^-_D(y)$. Moreover, it can be neither y nor z—otherwise D would contain a 2-cycle or a triangle. Hence $t \in V(C_b)$. The situation is depicted in Figure 2(a). Note that $tbyz$ is a 4-cycle with two consecutive arcs in D', namely $t \to b$ and $b \to y$. This establishes Property (C).

We are now in position to conclude the proof, by showing that there exists a 4-cycle of D included in D' and thereby contradicting Lemma 8(*ii*). By

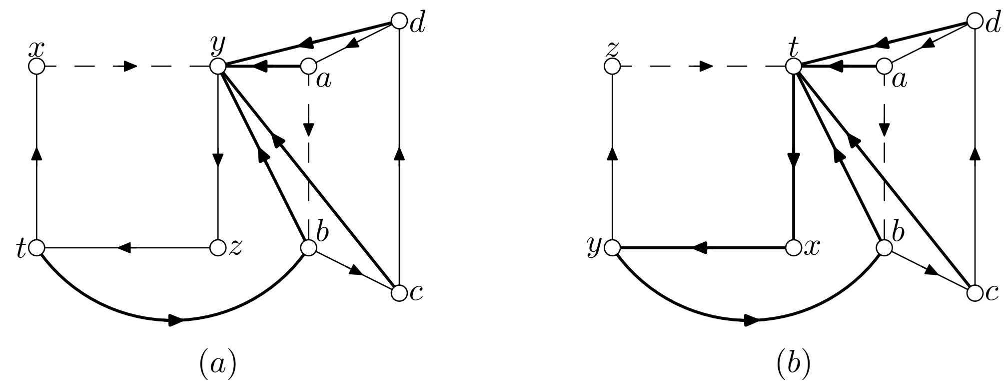


Figure 2: The arcs belonging to A' are drawn in bold, and the arcs not in A' are dashed. The remaining ones are only known to be in A.

Property (C) let $C := xyzt$ be a 4-cycle of D with two consecutive arcs in D'. By Lemma 9(ii) at least one of the arcs of C is not in D'. Therefore, up to renaming the vertices, we can assume that $t \to x \in A, x \to y \in A$ and $z \to t \notin A'$. Thus C and $C_t := abcd$ are disjoint. By Lemma 9(ii), assume that $a \notin C_b$. The cycles C_t and C_b being disjoint, C_b must intersect the cycle C. As none of x, z and t has an out-neighbour in C_b, we infer that $y \in V(C_b)$. Therefore $txyb$ is a 4-cycle of D which is included in D', see Figure 2(b). This contradiction concludes the proof when D does not contain a triangle.

4.2 The digraph D contains a triangle

For every vertex $u \in V$, we let $\varphi(u)$ be the greatest integer r for which there exist triangles $T_1, T_2, \ldots, T_r$ such that

- the intersection of every two triangles is the vertex u; and
- the in-neighbour of u in T_i dominates T_{i-1} for every $i \in \{2, 3, \ldots, r\}$.

Thus, $\varphi(u) = 0$ if and only if u is not contained in a triangle, and $1 \leq \varphi(u) \leq 5$ otherwise.

Lemma 10. *Either D contains two disjoint triangles, or all the triangles of D share a common vertex x. In the latter case $\varphi(x) \geq 3$.*

Proof. Let $\Phi := \max_{u \in V} \varphi(u)$. As D contains a triangle, we deduce from Lemma 6(iii) and (iv) that $\Phi \geq 2$.

We suppose first that $\Phi = 2$. We shall establish that D contains two disjoint triangles. Suppose on the contrary that it is not the case. Then, the following holds.

(D) *Every vertex $x \in V$ such that $\varphi(x) = 2$ is dominated by a triangle.*

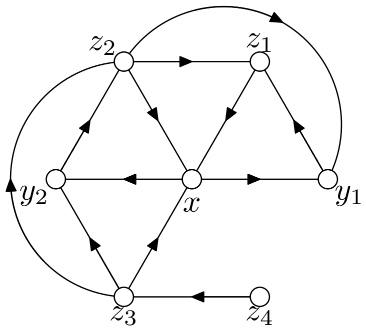


Figure 3: Configuration obtained when $\varphi(x) = 2$.

By the definition of φ, there exist four vertices y_1, y_2, z_1, z_2 such that $T_1 := xy_1z_1$ and $T_2 := xy_2z_2$ are two triangles and z_2 dominates T_1. According to Lemma 6(iii), there exists a vertex $z_3 \notin \{y_1, z_1\}$ dominating T_2. Thus, Lemma 6(iv) implies that there exists a triangle $T_3 := z_3a_1b_1$, with $a_1 \in V(T_2)$ and $b_1 \notin V(T_2)$. There are three distinct vertices that dominate T_3. Among the vertices so far defined, only y_1 and z_1 may dominate T_3. Thus, there exists $z_4 \notin \{y_1, z_1\}$ that dominates T_3. Moreover, there exists a triangle $T_4 := z_4a_2b_2$ with $a_2 \in V(T_3)$ and $b_2 \notin V(T_3)$. The situation is depicted in Figure 3. We set $X := \{x, y_1, z_1, y_2, z_2, z_3, z_4\}$.

If $z_1 \to z_3 \in A$, then $z_3z_2z_1$ is a triangle which dominates x, which would establish Property (D). We thus assume in the remaining that z_1 does not dominate z_3. The vertex b_1 dominates z_3, thus either $b_1 = y_1$ or $b_1 \notin X$. We consider these two cases separately.

$b_1 \notin X$. Then a_1 must be x, otherwise $z_3a_1b_1$ and one of T_1, T_2 are disjoint. Now, T_1, T_2 and z_3xb_1 show that $\varphi(x) \geq 3$, a contradiction.

$b_1 = y_1$. Consider $T_4 = z_4a_2b_2$. Note that z_4 dominates $b_1 = y_1$. Notice also that the vertex b_2 does not lie in $\{y_2, z_2\}$, otherwise $z_4z_3b_2$ and T_1 would be two disjoint triangles. If $b_2 = x$ then T_1, T_2 and z_4z_3x show that $\varphi(x) \geq 3$, a contradiction. If $b_2 = z_1$, then $z_4y_1z_1$ and T_2 are two disjoint triangles. Thus, as $b_2 \neq b_1 = y_1$ (since $b_2 \notin V(T_3)$), we deduce that $b_2 \notin X$. As T_4 must intersect T_1, T_2 and T_3, we infer that $a_2 = x$. Consequently, $z_3z_2y_1$ and T_4 are two disjoint triangles, a contradiction.

This establishes Property (D). Note that we also have showed that z_1 must indeed dominate z_3. Hence, $\varphi(z_2) \geq 2$, by considering the triangles T_2 and $z_3z_2z_1$.

Now consider a vertex x such that $\varphi(x) = 2$, and let T_1 and T_2 be two triangles as before. In particular, we can assume that the vertex z_2 satisfies $\varphi(z_2) = 2$, thus is dominated by a triangle T. Observe that T_1 and T are two disjoint triangles, a contradiction.

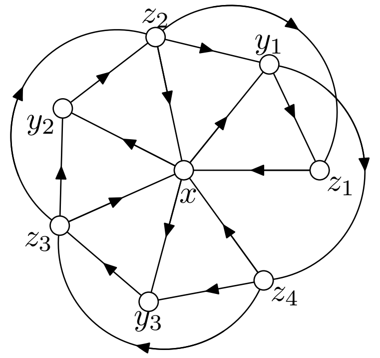


Figure 4: Configuration obtained when $\varphi(x) \geq 3$ and $b = y_1$.

In conclusion, we have proved that D contains two disjoint triangles if Φ is two.

We assume now that $\Phi \geq 3$, and we let x be a vertex such that $\varphi(x) = \Phi$. By contradiction, suppose that D does not contain two disjoint triangles, and yet contains a triangle T not containing x. There exist three triangles $T_i := xy_iz_i$, $i \in \{1,2,3\}$, such that $V(T_i) \cap V(T_j) = \{x\}$ if $i \neq j$, and z_i dominates T_{i-1} if $i > 1$. As D does not contain two disjoint triangles, we deduce that T contains a vertex from each set $\{y_i, z_i\}$, for $i \in \{1,2,3\}$.

According to Lemma 6(iii), there exists a vertex z_4, distinct from all the vertices defined so far, that dominates the triangle T_3. Thus, there exists a triangle $T_4 := z_4ab$, with $a \in V(T_3)$ and $b \notin V(T_3)$. Notice that $b \neq x$. Hence, if $a \neq x$, we obtain two disjoint triangles; indeed, the triangle T_4 intersects at most two triangles among T_1, T_2 and T_3, because $x \notin V(T_4)$ and $z_4 \notin V(T_1) \cup V(T_2) \cup V(T_3)$. Thus, among the triangles T_i, $i \in \{1,2,3,4\}$, at least two are disjoint, a contradiction.

Therefore, $a = x$. Let $X := \{x, y_1, z_1, y_2, z_2, y_3, z_3, z_4\}$. Note that b either belongs to $\{y_2, y_1\}$ or does not belong to X. The latter case is not possible, since T_4 and T would then be two disjoint triangles—because, as noted earlier, $V(T) \subset \{y_1, z_1, y_2, z_2, y_3, z_3\}$. If $b = y_2$, then T_1 and $z_4z_3y_2$ are two disjoint triangles. Therefore, we infer that b is y_1, so $T_4 = z_4xy_1$. The situation is depicted in Figure 4.

As D does not contain two disjoint triangles, $V(T)$ must intersect the set $\{y_1, z_4\}$. So, y_1 is a vertex of T. Now, observe that the triangles T_2, T_3 and T_4 fulfil the same conditions as do T_1, T_2 and T_3. Consequently, we deduce as previously that $y_2 \in V(T)$. So, the triangle T either is $z_3y_2y_1$ or is comprised of the vertices y_1, y_2 and y_3. If the former case, let $u \notin X$ be a vertex dominating T_4. This is possible since at least three vertices dominate T_4. There exists a triangle T_5 comprised of u, a vertex $u_1 \in V(T_4)$ and a vertex $u_2 \notin V(T_4)$. If $u_1 \in \{y_1, z_4\}$, then T_5 and either T_2 or T_3 are two disjoint triangles, since $x \notin V(T_5)$. So, $u_1 = x$ and u_2 is either y_2, y_3

or a new vertex. In all cases, T_5 and $z_3z_2y_1$ are two disjoint triangles, a contradiction. Consequently, $V(T) = \{y_1, y_2, y_3\}$. Thus, none of the vertices $z_i, i \in \{1, 2, 3, 4\}$, dominates T. As T is dominated by at least three vertices, we can choose a vertex u that dominates T and is different from x. Now, there exists a triangle $T' := uu_1u_2$ with $u_1 \in \{y_1, y_2, y_3\}$, and $u_2 \notin V(T)$. Note that $u_2 \neq x$. Consequently, T' and one triangle among T_1, T_2 and T_3 are disjoint, a contradiction. This concludes the proof. □

We define now two subsets of V. Let Y be the set of vertices contained in a triangle, and Z the set of vertices dominating a triangle. We set $D_Y := D[Y]$, and $D_Z := D[Z]$. From Lemma 6(iv) we deduce that D_Z is an induced subdigraph of D_Y. The following lemma will prove to be useful.

Lemma 11. *The following hold.*

(i) *Every vertex of Y has at least five in-neighbours in D, with at least four lying in D_Y;*

(ii) *the minimum in-degree of the digraph D_Z is at least three.*

Proof. (i) Let $T := xyz$ be a triangle containing x. By Lemma 6(iii), there exist three vertices u, v and w that dominate T. By the definition of Y, the vertices u, v, w and z, which are all in-neighbours of x, belong to Y. Thus, it only remains to show that there exists a fifth in-neighbour of x in D. To this end, suppose on the contrary that $d^-_D(x) = 4$. Consider the cycle C_x. Since z is dominated by $\{u, v, w\}$, it cannot belong to C_x. Thus, C_x is a triangle whose vertices are u, v and w. In particular T and C_x are two disjoint cycles, and there is no arc from the triangle T to the cycle C_x, which contradicts Corollary 5.

(ii) Let x be a vertex of D_Z. By Lemma 6(iv) there exists a triangle $T := xyz$, along with three vertices u, v, w dominating T. Thus, $\{u, v, w\} \subseteq N^-_{D_Z}(x)$, which proves the desired statement.

□

We finish the proof of Theorem 2 right after having established the following bound.

Lemma 12. *Suppose that T and T' are two disjoint triangles of D. If ℓ denotes the number of arcs between T and T' then $n \leq 22 - \ell$.*

Proof. Let $X := V(T) \cup V(T')$ and $X' := V \setminus X$. We shall obtain the desired inequality by counting the number L of arcs from a vertex of X to a vertex of X'. Since every vertex has out-degree five, L is $4 \times 6 - \ell = 24 - \ell$. We now prove that $L \geq n+2$, which will imply that $n+2 \leq 24-\ell$, and hence $n \leq 22-\ell$. Note that every vertex of X' has an in-neighbour in X, otherwise D would contain three disjoint cycles by Lemma 6(ii). As the digraph $D[X']$ is acyclic (and of order at least $n-6 \geq 5$), there exists a vertex $v \in X'$ having no in-neighbour in X', and another vertex w with at most one in-neighbour in X'. All together, these two vertices have at least $3+2=5$ in-neighbours in X. Now, note that T and T' are two disjoint triangles in D_Y. By Lemma 11(i), D_Y has minimum in-degree at least four—and so its order is at least nine. Consequently, there exists three vertices a, b and c of $Y \setminus X$ having at least four, three and two in-neighbours in X, respectively—otherwise D_Y, and hence D, would contain three disjoint cycles, a contradiction. According to Lemma 11(i), every vertex of Y has in-degree at least five in D. If $\{v, w\} \subset Y$, we infer from what precedes that $L \geq 5+4+2+n-6-3=n+2$. If only one of v, w lies in Y, we deduce that $L \geq 5+3+2+2+n-6-4=n+2$, while if none of them is in Y, we have $L \geq 3+2+4+3+2+n-6-5=n+3$. □

We now obtain a contradiction by proving that D indeed contains three disjoint cycles. Recall that the order of D is at least 11. According to Lemma 10, either all the triangles of D share a common vertex, or D contains two disjoint triangles. We consider the two cases separately.

Case 1: *D does not contain two disjoint triangles.* In this case, all triangles of D share a common vertex, say x, and we have $\varphi(x) \geq 3$. All the vertices of D_Z are in-neighbours of x, since x is contained in every triangle. By Lemma 11(ii), the digraph D_Z has minimum in-degree at least three. We assert that D_Z has also minimum out-degree at least three. To see this, suppose the contrary, and let z be a vertex with out-degree at most two in D_Z. Note that $x \notin Z$, so $z \neq x$. We set $D_1 := D_Z - z$. Observe that the digraph D_1^* fulfils the hypothesis of Theorem 3, since all its vertices have out-degree at least three (by Lemma 11(ii)) except at most two vertices which have out-degree two. Thus, the digraph D_1^* contains two disjoint cycles. They yield two disjoint cycles of D_1, say C_1 and C_2. As $z \in D_Z$, there exists a triangle $T := zuv$ in D. By the definition of x, we have $u = x$. As noticed earlier, $Z \subseteq N_D^-(x)$, hence the triangle T is disjoint from both C_1 and C_2, a contradiction. Therefore D_Z has minimum out-degree at least three. Let us set $m := |Z|$. We shall lower bound m as a function of n. As D_Z has minimum out-degree three, every vertex of Z has at least four out-neighbours in $Z \cup \{x\}$, and thus at most one in $Z' := V \setminus (Z \cup \{x\})$. So the following holds.

(E) *The number of arcs from a vertex of Z to a vertex of Z' is at most m.*

Furthermore, by Theorem 1, D contains two disjoint cycles C_1 and C_2 comprised of vertices of Z. Observe that every vertex of Z' has at least one in-neighbour in Z: otherwise, by Lemma 6(ii), D would contain a cycle comprised of vertices of $Z' \cup \{x\}$, which together with C_1 and C_2 would yield three disjoint cycles, a contradiction. As $\varphi(x) \geq 3$, there exist three out-neighbours y_1, y_2 and y_3 of x in Z', each having at least three in-neighbours in Z, by Lemma 6(iii). Consequently, the following is true.

(F) *The number of arcs from a vertex of Z to a vertex of Z' is at least $9 + (n - 1 - m - 3) = n - m + 5$.*

It follows from Properties (E) and (F) that

$$2m \geq n + 5. \tag{1}$$

We now aim at bounding $|A|$, the number of arcs of D, in terms of m. Recall that $|A| = 5n$, since every vertex of D has out-degree five. We partition V into the sets Z, Z' and $\{x\}$. Recall that every vertex has in-degree at least three, by Lemma 6(ii). As $Z \subseteq Y$, each vertex of Z has at least five in-neighbours in D by Lemma 11(ii). So

$$\sum_{v \in Z} d_D^-(v) \geq 5m. \tag{2}$$

Recall also that $Z \subseteq N_D^-(x)$, thus

$$|N_D^-(x)| \geq m. \tag{3}$$

Moreover, according to Lemma 11(i), every vertex of Y has in-degree at least five in D, and $|Y \cap Z'| \geq 3$ since $\varphi(x) \geq 3$. In particular, x has at most two out-neighbours not in Z. As x belongs to every triangle of D, every vertex not in $N_D^+(x)$ has in-degree at least four in D, by Lemma 6(ii). Therefore we obtain

$$\sum_{v \in Z'} d_D^-(v) \geq 3 \times 5 + 2 \times 3 + (n - 1 - m - 5) \times 4 = 4n - 4m - 3. \tag{4}$$

By Equations (2), (3) and (4), we infer that the number of arcs of D is at least $5m + m + 4n - 4m - 3 = 4n + 2m - 3$. As $|A| = 5n$, we obtain

$$2m \leq n + 3. \tag{5}$$

Equations (1) and (5) are contradictory, which concludes the first case of our proof.

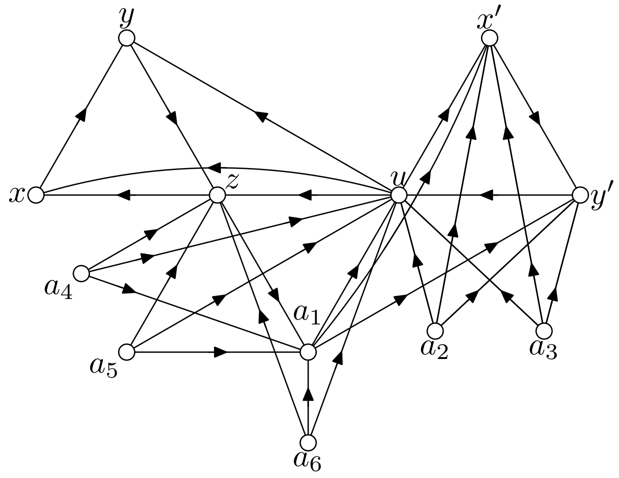


Figure 5: The sub-digraph F of D.

Case 2: *D has two disjoint triangles.* Let $T := xyz$ and $T' := x'y'z'$ be two disjoint triangles. Consider the sub-digraph D_1 of D obtained by removing T and T'. As D does not contain three disjoint cycles, D_1 is acyclic, thus has a vertex u of out-degree zero. Hence, the vertex u has five out-neighbours among x, y, z, x', y' and z'. Without loss of generality, let $N_D^+(u) = V(T) \cup \{x', z'\}$. Necessarily, $y' \in N_D^-(u)$, otherwise T, T' and C_u would be three disjoint cycles of D, a contradiction. So T and $T_1 := ux'y'$ are two disjoint triangles of D. By Lemma 6(iii), there exists an arc from a vertex of T to a vertex of T_1. Moreover, there are at least three arcs from a vertex of T_1 to a vertex of T, since the vertex u dominates T. So Lemma 12 implies that $n \leq 22 - 4 = 18$. By Lemma 6(iii), there exist three vertices a_1, a_2 and a_3 that dominate T_1. Clearly, none of these vertices belongs to $V(T) \cup V(T_1)$. Moreover at least one of them, say a_1, has no in-neighbour in $\{a_1, a_2, a_3\}$, since otherwise T, T_1 and $D[\{a_1, a_2, a_3\}]$ would be three disjoint cycles of D. By Lemma 6(ii), the vertex a_1 must have an in-neighbour in T, otherwise T, T_1 and C_{a_1} would be disjoint, a contradiction. Without loss of generality, we assume that $z \in N_D^-(a_1)$. The triangle $T_2 := uza_1$ is dominated by three vertices a_4, a_5 and a_6. Clearly, none of these vertices belongs to $V(T) \cup V(T_1) \cup \{a_1, a_2, a_3\}$. More precisely, among the vertices not in T_2, only y', a_2 and a_3 dominate u, and none of them dominates a_1. Thus, we obtain the sub-digraph F of D, depicted in Figure 5. For convenience, every vertex of D not in F is called *extern*.

Note that all the vertices of F belong to Y, and hence have in-degree at least five in D by Lemma 11(i). As D does not contain three disjoint cycles, there exists $i \in \{2, 3, \ldots, 6\}$ such that the vertex a_i does not have an in-neighbour in $\{a_1, a_2, \ldots, a_6\}$. Observe that a vertex dominating the arc $a_i \rightarrow u$ is either y' or extern, the former begin possible only if $i \geq 4$. We now consider two cases, regarding the value of i.

$i \in \{2, 3\}$. Without loss of generality, let $i = 2$. The vertex a_2 has at least one in-neighbour in T, otherwise T, T_1 and C_{a_2} would be three disjoint cycles of D. We consider two cases regarding whether z dominates a_2.

z dominates a_2. In this case, the triangle za_2u is dominated by three vertices, which must be extern. These three vertices belong to Y, as do the vertices of F. Thus, by Lemma 11(i), all have in-degree at least five in D. Furthermore, among them the vertex u has in-degree at least 10, and z at least 8. We deduce that

$$|A| = 5n = \sum_{v \in V} d_D^-(v) \geq 10 + 8 + 13 \times 5 + (n - 15) \times 3,$$

which implies that n is at least 19, a contradiction.

z does not dominate a_2. So at least one vertex among x, y dominates a_2. By symmetry of the roles played by x and y in what follows, we assume that x dominates a_2. The triangle $T_3 := xa_2u$ is dominated by three vertices, which must be extern. These three vertices belong to Z, and hence to Y. The vertices of F also belong to Y, and every vertex of Y has in-degree at least five in D by Lemma 11(i). Furthermore the in-degree of u is at least 10. Thus we obtain

$$|A| = 5n \geq 10 + 14 \times 5 + (n - 15) \times 3,$$

which yields $n \geq \frac{35}{2}$. As $n \leq 18$, we have $n = 18$. Notice that T is dominated by two vertices distinct from u. So, we infer that $d_D^-(x) + d_D^-(z) \geq 5 + 5 + 2 = 12$. Hence, we obtain

$$|A| = 5n \geq 10 + 12 + 12 \times 5 + (n - 15) \times 3,$$

from which it follows that $n \geq \frac{37}{2}$, a contradiction.

$i \in \{4, 5, 6\}$. Without loss of generality, let $i = 4$. As D does not have three disjoint cycles, $N_D^-(a_4) \cap \{x, y, x', y'\} \neq \emptyset$. We split this case according to the corresponding sub-cases.

x dominates a_4. We set $T_3 := a_4ux$. Among the vertices of F, only y' may dominate T_3. Supposing first that it is not the case, we obtain a contradiction by counting the number of arcs in D. The triangle T_3 is dominated by three extern vertices. These vertices belong

to Z, and thus to Y. Moreover, recall that all the vertices of F also belong to Y, and that every vertex of Y has in-degree five, by Lemma 11(i). Thus, there are at least 15 vertices of in-degree five and, among them, u has in-degree at least 10. Also, the vertex z has in-degree at least 8, because the triangle xa_4z is dominated by three vertices, none of them lying in $\{y, u, a_4, a_5, a_6\}$. Therefore we obtain

$$|A| = 5n \geq 10 + 8 + 13 \times 5 + (n - 15) \times 3,$$

which yields $n \geq 19$, a contradiction.

Hence, the vertex y' dominates the triangle T_3. We seek a contradiction by counting the number of arcs in D. Note that there are at least five arcs between T and T_1, since u dominates T_1, y' dominates x and there is at least one arc from T to T_1 by Corollary 5. So, by Lemma 12, n is at most 17.

We now bound the number of arcs in D. As a_4 has no in-neighbours among the other vertices a_i, there exist two extern vertices dominating the triangle T_3. Recalling that all the vertices of F belong to Y, we obtain $|Y| \geq 14$. By Lemma 11(i), each of these vertices has in-degree at least five in D. Moreover, u has in-degree at least 9, since it has already in-degree at least 7 in F. Also, the in-degree of z is at least 8, because z is dominated by $\{u, a_4, a_5, a_6, y\}$, and by the three vertices dominating the triangle a_4zx, which cannot be any of the preceding ones. Therefore we infer that

$$|A| = 5n \geq 9 + 8 + 12 \times 5 + (n - 14) \times 3,$$

and hence $n \geq \frac{35}{2}$, contradicting the conclusion of the preceding paragraph.

y dominates a_4. Let $T_3 := a_4uy$. This triangle is dominated by three vertices. Among the vertices of F, only y' may dominate it. Suppose first that it is not the case, i.e. T_3 is dominated by three extern vertices, which hence belong to Y. Furthermore, the triangle T is dominated by two vertices different from u. Thus we deduce that $d^-_D(y) + d^-_D(z) \geq 5 + 5 + 2 = 12$. Note also that u has in-degree at least 10. So, recalling that all the vertices of F belong to Y, it follows that

$$|A| = 5n \geq 10 + 12 + 12 \times 5 + (n - 15) \times 3,$$

i.e. $n \geq \frac{37}{2}$, a contradiction. Consequently, we infer that y' dominates T_3. As in the previous case, we note that there are at least five arcs between T and T_1, and thus the Lemma 12 implies that n is at most 17. As T_3 is dominated by two extern vertices, notice that u has in-degree at least 9 (since its in-degree in F is at least 7). Moreover the triangle a_4a_1y' is dominated by three vertices, and none of them belongs to $\{x', a_2, a_3, a_5, a_6, z\}$. Hence, we deduce that both a_1 and y' have in-degree at least 7 in D. Therefore, we obtain

$$|A| = 5n \geq 9 + 7 + 7 + 11 \times 5 + (n - 14) \times 3,$$

so $n \geq 18$, a contradiction.

x' dominates a_4. Then the triangle $T_3 := a_4ux'$ is dominated by three extern vertices. So there are at least 15 vertices of in-degree at least five, and among them u has in-degree at least 10 (since its in-degree in F is at least 7), and x' has in-degree at least 7 (since its in-degree in F is at least 4). Therefore, we deduce that

$$|A| = 5n \geq 10 + 7 + 13 \times 5 + (n - 15) \times 3,$$

which yields $n \geq \frac{37}{2}$, a contradiction.

None of x, y and x' dominates a_4 in D. In this case the vertex y' must dominate a_4. We consider three vertices dominating the triangle $T_3 := a_4a_1y'$. Among the vertices of F, only x and y can dominate T_3, but none of them does since none of them is an in-neighbour of a_4. Thus, T_3 is dominated by three extern vertices. Consequently, Y contains at least 15 vertices, and u, a_1 and y' all have in-degree at least 7. It follows that

$$|A| = 5n \geq 3 \times 7 + 12 \times 5 + (n - 15) \times 3,$$

and hence, $n \geq 18$. As we know that $n \leq 18$, we have $n = 18$. In particular, there are exactly 6 extern vertices. We denote them by r, s, t, r', s' and t', with $\{r, s, t\}$ dominating the triangle T_3.

Now observe that, for every $i \in \{1, 2, 3, 4\}$, $a_i \notin N_D^+(x')$. Moreover, $V(T) \cap N_D^+(x') = \emptyset$ otherwise there would be at least five arcs between T and T', which would imply that $n \leq 17$ by Lemma 12, a contradiction. We assert that the in-degree of x' in D is at least 7. Recalling that u, a_1 and y' also have in-degree at least 7, we would deduce that

$$|A| = 5n = 90 \geq 4 \times 7 + 11 \times 5 + 3 \times 3 = 92,$$

a contradiction. So it only remains to prove the assertion. If $\{a_5, a_6\} \cap N_D^+(x') \neq \emptyset$, we assume without loss of generality that x' dominates a_5. Then, the triangle a_5a_1x' is dominated by three vertices, which cannot be any of u, a_1, a_2, a_3. So x' has at least 7 in-neighbours in D. If $\{a_5, a_6\} \cap N_D^+(x') = \emptyset$, the vertex x' has at least four out-neighbours lying in $\{r, s, t, r', s', t'\}$. So it dominates at least one of r, s and t, say r. The triangle ra_1x' is dominated by three vertices, none of them lying in $\{u, a_1, a_2, a_3\}$. Thus, we again conclude that the vertex x' has in-degree at least 7, which proves the assertion.

The proof of Theorem 2 is complete. □

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