Proceedings of the Last COMBSTRU Workshop

Prague, March 10 – 12, 2006

ed. Martin Pergel

Contents

Preface

The COMBSTRU Workshop was held from March 10 to March 12, 2006 in our newly reconstructed building on Malostranské náměstí 25. This of course contributed to the comfort of the participants as all the activities (including the lunches) could be taken on the same site. Besides, as it was expressed by several participants, the renovated faculty building surely belongs to the most beautiful math and CS departments in the world! The workshop was organized by the EU network (RTN) COMBSTRU jointly with the DIMATIA centre of Charles University.

The workshop was immediately preceeded by colloquium held by Endre Szemeredi. The programme of the workshop followed daily routine with morning and early afternoon discussions and presentations. The workshop was attended by many past and recent trainees from all participating sites of COMBSTRU network as well as by participants of DOCCOURSE 2006 "Harmonic Analysis in Computer Science and Combinatorics". This DOCCOURSE, which is a continuation of similar programme in years 2004 and 2005 (http://kam.mff.cuni.cz/~matousek/doccourse06.html) is a traditional COMBSTRU-activity. This report reflects some of the presentations during the workshop. Perhaps you can digest some of the atmosphere at the workshop from these pages.

This volume was edited by Martin Pergel. All following contributions were supplied by the authors in electronic form. In a few cases slight typographical changes were necessary. We apologize for any possible inaccuracies which might have occurred in the editing process.

We gratefully acknowledge the support of COMBSTRU. We would like to thank to all participating institutions for all their work. In its final year we are witnessing the booming activity of COMBSTRU. This clearly gives to all of us the energy to continue activity of the network in the future.

Jaroslav Nešetřil

COMBSTRU participating sites

Location	**Institute**
Barcelona	Universitat Politècnica de Catalunya
Berlin	Freie Universität, Humboldt Universität and Technische Universität
Bielefeld	Universität Bielefeld
Bordeaux	LABRI, Université Bordeaux I
Budapest	Alfred Rényi Institute of Mathematics, Hungarian Academy of Sciences
Oxford	University of Oxford
Patras	Research Academic Computer Technology Institute
Pisa	Consiglio Nazionale delle Ricerche, IIT
Prague	DIMATIA, Charles University

Variable Length Codes With Marker, Feedback and Localized Errors

R. Ahlswede, C. Deppe, V. Lebedev

A famous problem in coding theory consists in finding good bounds for the maximal size, $M(n,t,q)$, of a t-error correcting code over a q-ary alphabet $Q = \{0,1,\dots,q-1\}$ with blocklength n.
Suppose that having sent letters $x_1,\dots,x_{j-1}$ the encoder knows the letters $y_1,\dots,y_{j-1}$ received before he sends the next letter x_j $(j = 1,2,\dots,n)$. We then have the presence of a noiseless feedback channel. For $q = 2$ this model was considered by Berlekamp [3], who derived striking results for triples of performance $(M,n,t)_f$, that is, the number of messages M, blocklength n and the number of errors t. It is convenient to use the notation relative error $\tau = t/n$ and rate $R = n^{-1}\log M$. We investigate here the q-ary case. Again the Hamming bound for $C_q(\tau)$, the maximal rate achievable for τ and all large n, is a central concept:

$$H_q(\tau) = \begin{cases} 1 - h_q(\tau) - \tau\log_q(q-1) & \text{if } 0 \le \tau \le \frac{q-1}{q} \\ 0 & \text{if } \frac{q-1}{q} < \tau \le 1, \end{cases} \tag{1}$$

where $h_q(\tau) = -\tau\log_q(\tau) - (1-\tau)\log_q(1-\tau)$. We also call $C_q : [0,1] \to \mathbb{R}_+$ the capacity error function (or curve). One readily verifies that for every q

$$C_q(\tau) = 0 \text{ for } \tau \ge \frac{1}{2}. \tag{2}$$

We turn now to another model. Suppose that the **encoder**, who wants to encode message $i \in \{1,2,\dots,M\}$, knows the t-element set $E \subset [n] = \{1,\dots,n\}$ of positions, in which only errors may occur. He then can make the codeword presenting i dependent on $E \in \mathcal{E}_t = \binom{[n]}{t}$, the family of t-element subsets of $[n]$. We call them "a priori error pattern". A family $\{u_E^i : 1 \le i \le M,\ E \in \mathcal{E}_t\}$ of q-ary vectors with n components is an $(M,n,t,q)_l$ code (for localized errors), if for all $E, E' \in \mathcal{E}_t$ and all q-ary vectors $e \in V(E) = \{e = (e_1,\dots,e_n) :\ e_j = 0 \text{ for } j \notin E\}$ and $e' \in V(E')$

$$u_E^i \oplus e \ne u_{E'}^{i'} \oplus e' \text{ for } i \ne i'.$$

We denote now the optimal rate for τ by $C_q^l(\tau)$.
The two models described have ingredients feedback resp. localized errors, which give possibilities for code constructions which are not available without them in the standard model of error correction (c.f. [4] and also for probabilistic channel models ([1], [2]).

Whereas all this work is for block codes, we investigate variable length codes with all lengths bounded from above by n. The end of a word carries the symbol $\square$ and is thus recognizable by the decoder. Very important here is that the lengths carry **sure** data which can be used as a "protocol" information.

For both, the $\square$-model with feedback and the $\square$-model with localized errors, the Hamming bound is the exact capacity curve for $\tau < 1/2$. Somewhat surprisingly, whereas with feedback the capacity curve coincides with the Hamming bound also for $1/2 \leq \tau \leq 1$, in this range for localized errors the capacity curve equals 0.
Also notice that without the marker $\square$ in the range $0 \leq \tau < 1/2$ with feedback the capacity curve is **smaller** than that for localized errors.

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Acyclic Orientation of Drawings

Kevin Buchin[*†] Eyal Ackerman[‡] Christian Knauer[*]
Günter Rote[*]

Given a set of curves in the plane or a topological graph, we ask for an orientation of the curves or edges which induces an acyclic orientation on the corresponding planar map. Depending on the maximum number of crossings on a curve or an edge, we provide algorithms and hardness proofs for this problem.

Let G be a *topological graph*, that is, a graph drawn in the plane such that its vertices are distinct points, and its edge set is a set of Jordan arcs, each connecting two vertices and containing no other vertex. In this work we further assume that G is a *simple* topological graph, i.e., every pair of its edges intersect at most once, either at a common vertex or at a crossing point.

An *orientation* of (the edges of) a graph is an assignment of a direction to every edge in the graph. We say that an orientation is *acyclic* if the resulting directed graph does not contain a directed cycle. Finding an acyclic orientation of a given undirected (abstract) graph can be easily computed in linear time by performing a depth-first search on the graph and then orienting every *backward* edge from the ancestor to the descendent. However, is it always possible to find an orientation of the edges of a topological graph, such that a traveller on that graph will not be able to return to his starting position even if allowed to move from one edge to the other at their crossing point? Rephrasing it in a more formal way, let $M(G)$ be the planar map induced by G. That it, the map obtained by adding the crossing points of G as vertices, and subdividing the edges of G accordingly. Then we ask for an orientation of the edges of G such that the induced directed planar map $M(G)$ is acyclic.

Clearly, if the topological graph is x-*monotone*, that is, every vertical line crosses every edge at most once, then one can orient each edge from its endpoint with the smaller x-coordinate towards its endpoint with the greater x-coordinate. Travelling on the graph under such orientation, one always increases the value of ones x-coordinate and therefore cannot form a directed cycle. However, not every topological graph is acyclic-orientable as Figure 1 demonstrates.

Note that the degree of every vertex in this example is one. This gives rise for considering the orientation problem in the special case the degree of each

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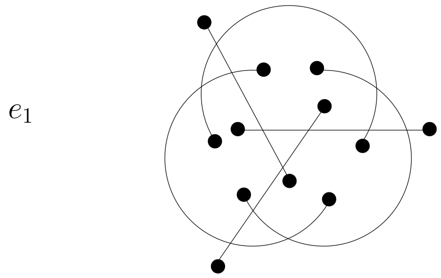

Figure 1: A non-orientable topological graph

vertex is one, or in other words, when one looks for an acyclic orientation of a set of *curves* embedded in the plane.

It turns out that determining whether a topological graph (resp., a set of curves) has an acyclic orientation depends crucially on the maximum number of times an edge in the graph (resp., a curve) can be crossed. Given a (simple) topological graph G on n vertices, such that each edge in G is crossed at most once, we show that one can find an acyclic orientation of G in $O(n)$ time. When four crossing per edge are allowed, deciding whether there exists an acyclic orientation becomes NP-complete. For a set of n curves in which each pair of curves intersects at most once and every curve is crossed at most k times, we describe an $O(n)$-time orientation algorithm for the case $k \leq 3$. When $k \geq 5$ finding an acyclic orientation of the set of curves is NP-complete.

On the Computability of the Fréchet Distance Between Triangulated Surfaces

Maike Buchin[*]

The Fréchet distance is a distance measure for comparing geometric shapes represented by parameterized curves or two- or higher-dimensional surfaces. Whereas efficient algorithms are known for computing the Fréchet distance between polygonal curves, the same problem for triangulated surfaces is NP-hard. Furthermore, it is not known whether it is computable at all. In this talk we discuss two partial answers to this open problem: the Fréchet distance between triangulated surfaces is semi-computable and for simple polygons the Fréchet distance is polynomial time computable.

Suitable distance measures for comparing the similarity of shapes are an important issue in application areas like computer vision and pattern recognition. A distance measure that is often used is the Hausdorff distance which is a distance measure for point sets. However, if shapes are modeled by curves or surfaces, there are examples of objects having little resemblance but a small Hausdorff distance.

In these cases the Fréchet distance is more appropriate, which is a metric for parameterized geometric objects. The idea of the Fréchet distance is to take into account the "flow" of the curve or surface given by its parameterization.

A popular illustration of the Fréchet distance between two curves is the following. Suppose a man is walking his dog on a leash. The man is walking on one curve and the dog on the other. Both may stop but not walk backwards. Then the Fréchet distance is the shortest length of leash allowing them to walk on the two curves from start to end. Formally the Fréchet distance is defined as follows. Let f, g be parameterizations of two curves or surfaces, i. e., continuous functions $f, g : [0, 1]^k \to \mathbb{R}^d$ where $d \geq k$, $k = 1$ for curves and $k = 2$ for two–dimensional surfaces. Then their Fréchet distance is

$$\delta_F(f, g) := \inf_{\sigma:[0,1]^k \to [0,1]^k} \sup_{t \in [0,1]^k} ||f(t) - g(\sigma(t))||.$$

where the reparameterization σ ranges over all orientation preserving homeomorphisms. As underlying norm $||.||$ we assume the Euclidean norm, but any other computable norm may be considered as well.

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Note that the Fréchet distance is defined for parameterized shapes, however usually non-parameterized continuous shapes can be meaningfully parameterized by a natural parameterization based on the geometric description of the object, such as arc length for the case of curves.

For polygonal curves Alt and Godau have shown that the Fréchet distance is computable in polynomial time. For two–dimensional surfaces, however, the computation of the Fréchet distance is much harder. In fact, Godau showed that computing the Fréchet distance between triangulated surfaces even in two–dimensional space is NP-hard. The computationally hard part of computing the Fréchet distance for higher dimensions is that, according to the definition, the infimum over all homeomorphisms of the parameter space has to be taken. For curves these are the orientation-preserving homeomorphisms on the unit interval which can be characterized as continuous, onto, monotone increasing functions. For higher dimensions such a characterization does not exist and the homeomorphisms can be much "wilder".

In this talk, we present two partial results concerning the computability. First, we show that the Fréchet distance between triangulated surfaces is upper semi-computable, i. e. there is a non-halting Turing machine which produces a monotone decreasing sequence of rationals converging to the result. It follows that the decision problem whether the Fréchet distance between two given surfaces lies below some specified value a is recursively enumerable.

For showing the semi-computability we approximate the homeomorphisms by discrete maps which are easier to handle. We do this by first approximating arbitrary homeomorphisms by piecewise linear homeomorphisms which is a known result from topology. These piecewise linear homeomorphisms are then approximated by homeomorphisms which are compatible with certain subdivisions of the original triangulations of the parameter spaces. Finally, as we are considering arbitrary fine subdivisions it suffices to compute the distances at the finitely many vertices of a fixed subdivision.

The second answer concerning the computability of the Fréchet distance between surfaces that we discuss is the polynomial time computability of the Fréchet distance between simple polygons. A simple polygon is the area enclosed by a non-self-intersecting closed polygonal curve in the plane. Simple polygons are a very restricted but also important class of triangulated surfaces that arise often in two–dimensional applications. When considering the Fréchet distance between simple polygons, the first question that comes to mind is: Is the Fréchet distance between polygons different from the Fréchet distance between their boundary curves? For the special case of convex polygons the Fréchet distance between the polygons equals the Fréchet distance between their boundary curves, but in general the Fréchet distance between two polygons may be arbitrarily much larger than the Fréchet distance between their boundary curves.

For showing that the Fréchet distance between simple polygons is polynomial time computable, we show that it suffices to consider homeomorphisms that map an arbitrary triangulation of one polygon to the other polygon such that diagonals of the triangulation are mapped to shortest paths in the other polygon. This yields a class of homeomorphisms that we can handle by extending the algorithm for curves to include also the diagonals of a triangulation.

Concluding we can say that the Fréchet distance is a well-suited distance measure for curves and surfaces. For triangulated surfaces in general it is still hard to compute but for simple polygons we have given a polynomial time algorithm. And by showing the semi-computability for triangulated surfaces we have gained insight on the problem. In the future we hope to find polynomial time or approximation algorithms for more classes of surfaces, so that the Fréchet distance can be employed as a distance measure for surfaces in practice.

Efficient Computation of Nash Equilibria for Very Sparse Win-Lose Bimatrix Games

Bruno Codenotti * *Mauro Leoncini* † *Giovanni Resta**

In 1951 Nash proved that any n-player game has an equilibrium in the mixed strategies [8]. The proof was based on a fixed point argument, and left open the associated computational question of finding such an equilibrium.

Abbott, Kane, and Valiant [1] proved that finding a Nash equilibrium for win-lose bimatrix games, i.e., 2-player games where the players' payoffs are zero and one, is as hard as for general bimatrix games.

In 1994 Papadimitriou introduced a complexity class, $PPAD$, which captures a wealth of equilibrium problems, e.g., the market equilibrium problem as well as Nash equilibria for n-player games [9]. Problems complete for this class include a (suitably defined) computational version of the Brouwer Fixed Point Theorem.

In 2005 a flurry of results appeared, where first the $PPAD$-completeness of 4-player games [6], then of 3-player games [2, 7], and finally of 2-player games [3] were proven. In particular, the latter hardness result by Chen and Deng came as a sort of surprise, since the 2-player case was conjectured to be computationally tractable. Combined with the result by Abbott, Kane, and Valiant [1], it also implies the $PPAD$-completeness of win-lose bimatrix games.

In this paper we describe a linear time algorithm which computes a Nash equilibrium for win-lose bimatrix games where the number of winning positions per strategy of each of the players is at most two.

The algorithm acts on the directed graph that represents the zero-one pattern of the payoff matrices describing the game. It is based upon the efficient detection of certain subgraphs which enable us to determine the support of a Nash equilibrium.

Following [5], we cast the problem of computing an equilibrium for win-lose games in terms of finding a *good assignment* in a directed graph.

The restriction on the zero-one pattern induces very sparse directed graphs. We show how to efficiently detect suitable subgraphs of these sparse graphs, which lead to the discovery of the support of a Nash equilibrium, and to the actual determination of the equilibrium strategies. The full paper is available at [4].

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On Hamilton Cycles in Random Intersection Graphs*

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Random Intersection Graphs is a relatively new model of random graphs introduced in [3], in which each of n vertices randomly and independently chooses some elements from a universal set, of cardinality m. Each element is chosen with probability p. Two vertices are joined by an edge iff their chosen element sets intersect.

This talk is based on [1] and [4] where we consider the existence and efficient construction of Hamilton Cycles in random intersection graphs. We begin by presenting a result for the case $m = n^{\alpha}, \alpha > 1$, that allows us to apply (with the same probability of success) any algorithm that finds a Hamilton cycle with high probability in a $G_{n,k}$ graph (i.e. a graph chosen equiprobably form the space of all graphs with k edges). This can also serve as an existential result.

We continue by presenting tighter lower bounds $p_0(n, m)$, on the value of p, as a function of n and m, above which the graph $G_{n,m,p}$ is almost certainly Hamiltonian. These bounds are tight in the sense that when p is asymptotically smaller than $p_0(n, m)$ then $G_{n,m,p}$ almost surely has a vertex of degree less than 2. Our proof involves new, nontrivial, coupling techniques that allow us to circumvent the edge dependencies in the random intersection model. Interestingly, Hamiltonicity appears well below the general thresholds, of [2], at which $G_{n,m,p}$ looks like a usual random graph. Thus bounds are much stronger than the trivial bounds implied by those thresholds. Our results strongly support the existence of a threshold for Hamiltonicity in $G_{n,m,p}$.

Finally, we present two algorithms for finding Hamilton cycles in random intersection graphs. The first runs in **expected polynomial time** for the case $p =$ constant and $m \leq \alpha\sqrt{\frac{n}{\log n}}$ for some constant α. The second basically suggests that the greedy approach still works well even in the case $m = o(\frac{n}{\log n})$ and p just above the connectivity threshold of $G_{n,m,p}$.

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Strong Oriented Coloring of 2-Outerplanar Graphs

Louis Esperet[*] *and Pascal Ochem*[*]

1 Introduction

A homomorphism from an oriented graph G to an oriented graph H is a mapping φ from $V(G)$ to $V(H)$ which preserves the arcs, that is $(x, y) \in E(G) \Longrightarrow (\varphi(x), \varphi(y)) \in E(H)$. We say that H is a *target graph* of G if there exists a homomorphism from G to H. The oriented chromatic number $\chi_o(G)$ of an oriented graph G is defined as the minimum order of a target graph of G. The oriented chromatic number $\chi_o(G)$ of an undirected graph G is then defined as the maximum oriented chromatic number of its orientations. Nešetřil and Raspaud introduced in [4] the *strong oriented chromatic number* of an oriented graph G (denoted by $\chi_s(G)$), which definition differs from that of $\chi_o(G)$ by requiring that the target graph is an oriented Cayley graph. A graph G is 2-outerplanar if it has a planar embedding such that the subgraph obtained by removing the vertices of the external face is outerplanar.

2 Structural properties of 2-outerplanar graphs

Definition 1. *A 2-outerplanar graph embedded in the plane is said to be a* block *if its external face is an induced cycle.*

Lemma 1. *Let G be an outerplanar graph. G contains either a 1-vertex, two adjacent 2-vertices, a 2-vertex adjacent to a 3-vertex as depicted in Figure 1.a, or two 2-vertices adjacent to a 4-vertex as depicted in Figure 1.b.*

Proof. By induction on the structure of outerplanar graphs. □

We now use Lemma 1 to prove a key structural theorem on 2-outerplanar graphs admitting a block embedding in the plane.

Theorem 2. *Let G be a 2-outerplanar graph admitting a* block *embedding in the plane. G contains either a* $^{\leq}$3*-vertex, two adjacent 4-vertices, or the configuration depicted in Figure 2.*

Proof. Follows from Lemma 1. □

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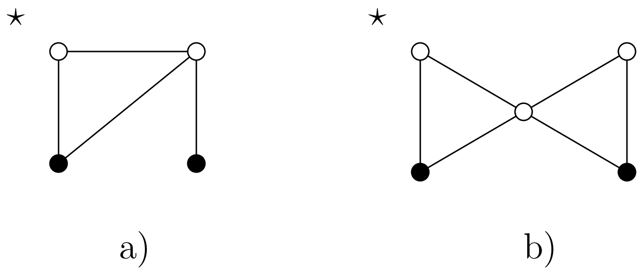

Figure 1: Unavoidable configurations in an outerplanar graph without two adjacent 2-vertices. The star symbol indicates the external face.

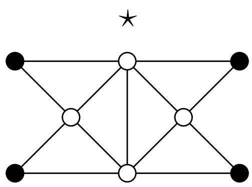

Figure 2: Unavoidable configuration in a 2-outerplanar block containing neither a $^{\leq}3$-vertex nor two adjacent 4-vertices.

3 Strong oriented coloring of 2-outerplanar graphs

Theorem 3. *If G is a 2-outerplanar graph, then $\chi_s(G) \leq 67$.*

For a prime power $q \equiv 3 \pmod 4$, the vertices of the Paley tournament QR_q are the elements of $\mathbb{F}_q$ and (i, j) is an arc in QR_q if and only if $j - i$ is a non-zero quadratic residue of $\mathbb{F}_q$. An *orientation vector* of size k is a sequence $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ in $\{0, 1\}^k$. Let G be an oriented graph and $X = (x_1, x_2, \ldots, x_k)$ be a sequence of pairwise distinct vertices of G. A vertex y of G is said to be an *α-successor of X* if for every i, $1 \leq i \leq k$, we have $\alpha_i = 1 \Rightarrow (x_i, y) \in E(G)$ and $\alpha_i = 0 \Rightarrow (y, x_i) \in E(G)$. The graph G satisfies property $S_{k,n}$ if for every sequence $X = (s_1, s_2, \ldots, s_k)$ of k pairwise distinct vertices of G, and for every orientation vector α of size k, there exist at least n vertices in $V(G)$ which are α-successors of X.

A computer check proves the following lemma:

Lemma 4. *The tournament QR_{67} satisfies properties $S_{2,16}$, $S_{3,6}$ and $S_{4,1}$.*

We use the method of reducible configurations to show that every 2-outerplanar graph is QR_{67}-colorable. Let $w(G) = |V(G)| + |E(G)|$. We consider a 2-outerplanar graph G having no homomorphism to QR_{67} such that $w(G)$ is minimum.

Lemma 5. *G is 2-connected and does not contain a cut consisting in two adjacent vertices.*

Notice that Lemma 5 implies that every 2-outerplanar embedding of G is a block.

Lemma 6.

1. *The graph G does not contain any $^{\leq}3$-vertex.*

2. *The graph G does not contain two adjacent 4-vertices.*

3. *The graph G does not contain the configuration depicted in Figure 2.*

Proof.

1. By contradiction, using properties $S_{2,1}$ and $S_{3,1}$.

2. and 3. Using properties $S_{3,6}$ and $S_{4,1}$. □

By Lemma 5 G is a block. Using Theorem 2, G must contain one of the configurations that are forbidden by Lemma 6. This contradiction completes the proof of Theorem 3.

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Orthogonal Surfaces: Combinatorics and Geometry

Stefan Felsner[*]

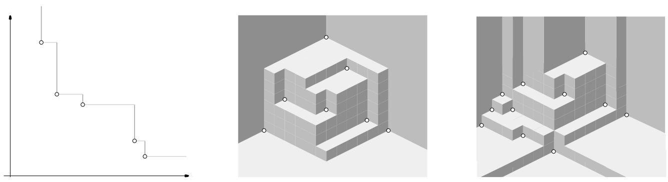
Let $\mathbb{R}^d$ be equipped with the dominance order:

$$x \le y \quad \iff \quad x_i \le y_i \quad \text{for} \quad i = 1, .., d$$

Let $V \subset \mathbb{R}^d$ be a finite antichain in the dominance order. The orthogonal surface $\mathcal{S}_V$ generated by V is the boundary of the filter

$$I_V^{\geq} = \{y \in \mathbb{R}^d \,:\, \exists x \in V \text{ with } y \ge x\}.$$

Example.



- The left figure shows an orthogonal 1-surface, i.e., an orthogonal surface in two dimensions.
- The middle figure shows a *suspended* and *generic* orthogonal 2-surface in three dimensions. Suspended means that there are special suspension vectors $s_i = (0, .., 0, M, 0, ..0) \in V$ such that $0 < x_j^i < M$ for all the other elements of V. Generic means that the non-suspension vectors in V have pairwise different coordinates.
- The right figure shows a 2-surface which shows all kinds of 'unfriendly' features.

Orthogonal surfaces are related to various mathematical fields:

- Study of discrete production sets in mathematical economics, (Scarf).
- Resolutions of monomial ideals, (Miller, Sturmfels)
- Connections with order dimension.
- Planar graphs and Schnyder woods.

The most remarkable result in the theory of orthogonal surfaces goes back to Scarf [3].

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Theorem 7 (Scarf's Theorem)**.** *Generic suspended orthogonal surfaces in* $\mathbb{R}^d$ *induce simplicial complexes which are face complexes of simplicial* d*-polytopes (minus one facet).*

We review a proof of this theorem in the 3-dimensional case. This proof naturally leads to the notion of a Schnyder wood and to Schnyder's characterization of planar graphs via order dimension, [4].

Theorem 8 (Schnyder's Theorem)**.** *A graph is planar if and only if the dimension of its incidence order is at most 3.*

Order dimension is also useful as a tool to classify simplicial d-polytopes as not realizable on an orthogonal surface. As an example: Trotter has shown that the order dimension of K_{12} is four but the dimension of K_{13} is five, it follows that neighbourly 4-polytopes with more than 13 vertices are not realizable on an orthogonal surface in $\mathbb{R}^4$.

The extremal function for the number of edges of a graph of dimension d was studied by Agnarsson, Felsner and Trotter [1].

Theorem 9. *The number of edges of a graph of dimension 4 is at most* $\frac{3}{8}n^2 + o(n^2)$*.*

We consider rectangle graphs whose edges are defined by pairs of points in diagonally opposite corners of empty axis-aligned rectangles. The maximum number of edges of such a graph on n points is shown to be $\lfloor \frac{1}{4}n^2 + n - 2 \rfloor$. This number also has other interpretations:

- It is the maximum number of edges of a graph of dimension $[3 \updownarrow\updownarrow 4]$, i.e., of a graph with a realizer of the form $\pi_1, \pi_2, \overline{\pi_1}, \overline{\pi_2}$.
- It is the number of 1-faces in a special Scarf complex.

The last of these interpretations allows to deduce the maximum number of empty axis-aligned rectangles spanned by 4-element subsets of a set of n points. Moreover, it follows that the extremal point sets for the two problems coincide.

We investigate the maximum number of of edges of a graph of dimension $[3 \updownarrow 4]$, i.e., of a graph with a realizer of the form $\pi_1, \pi_2, \pi_3, \overline{\pi_3}$. This maximum is shown to be at most $\frac{1}{4}n^2 + 3n - 1$.

Box graphs are defined as the 3-dimensional analog of rectangle graphs. The maximum number of edges of such a graph on n points is shown to be $\frac{7}{16}n^2 + o(n^2)$.

These results are from [2].

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The Number of 3-Orientations of a Triangulation

Stefan Felsner[*] *Florian Zickfeld*[*]

Schnyder woods have been introduced by W. Schnyder in [4] and [5], and they are a powerful tool in the theory of planar triangulations. Schnyder woods are for example useful in graph drawing and in the dimension theory of graphs. We refer to [2] for a comprehensive introduction to the topic. In this extended abstract we are concerned with estimating the number of Schnyder woods of a triangulation. We consider only plane triangulations and will omit to repeat this in the sequel.

Definition 2. *Let T be a triangulation with outer face $\{a_1, a_2, a_3\}$. A Schnyder wood of T is an orientation and coloring of the inner edges of T with colors $1, 2, 3$ such that*

W1 *Each inner vertex v has three outgoing edges, which are colored $1, 2, 3$ in clockwise order around v. The edges incoming between the outgoing edges colored i, $i+1$ have color $i-1$, where we use cyclic arithmetics on $\{1, 2, 3\}$.*

W2 *All inner egdes at a_i are directed towards a_i and have color i.*

The set of triangulations, which have a unique Schnyder wood, coincides with the class of *stacked* triangulations. For the maximum number of Schnyder woods on a triangulation we obtain the following bounds.

Theorem 10. *Let T be a plane triangulation with n vertices and $\mathcal{S}(T)$ the set of Schnyder woods of T. Then,*

$$|\mathcal{S}(T)| \leq 3.87^n$$

and there exist infinitely many triangulations with

$$|\mathcal{S}(T)| \geq 2.27^n.$$

The following results are of interest in this context.

Theorem 11 ([3])**.** *Let $G = (V, E)$ be a plane graph and X an orientation of its edges. Let $\alpha : V \to \mathbb{N}$ and call X an α-orientation if $d_X^+(v) = \alpha(v)$, $\forall v \in V$, where $d_X^+(v)$ denotes the outdegree of X at v. Then, the set of α-orientations of G is a distributive lattice.*

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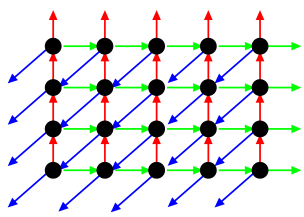
Theorem 12 ([1])**.** *Let T be a plane triangulation. Define a function α_T by $\alpha_T(v) := 0$ if v lies on the outer face, $\alpha_T(v) := 3$ otherwise. Then, there is a bijection between the Schnyder woods and the α_T-orientations of T.*

Theorem 12 says, that just the edge orientations are necessary to define a Schnyder wood, while the edge colors can be deduced. We will use this in the sequel and refer to an α_T-orientation simply as a 3-orientation.

We now outline briefly by which means we obtain the results in Theorem 10.

Upper bound. A trivial upper bound for the number of 3-orientations of $T = (V, E)$ is $2^{|E|} = 2^{3n-6}$. We observe that if $A \subset E$ is cycle-free, then there is at most one way to complete an orientation of $E \setminus A$ to a 3-orientation of E. This is because at a vertex v with only one unoriented edge e, the already oriented edges and the degree requirement determine the orientation of e. By choosing A to be a spanning tree we obtain the bound $2^{2n-5} < 4^n$.

To obtain the upper bound of 3.87^n a first step is to show that we can restrict our attention to 4-connected triangulations. A 4-connected triangulation T has a Hamilton cycle H and by the Four Color Theorem an independent set of size at least $n/4$. If we orient $E \setminus H$, then there are at most two ways to complete this to a 3-orientation of T. One needs to check, that for a vertex v at most a fraction of $7/8$ of the $2^{d(v)-2}$ possible orientations of $E \setminus H$ at v can be completed to a 3-orientation. For the $v \in I$ the events of an orientation being locally completable at v are independent, and we infer the bound of $(7/8)^{n/4} \cdot 4^n \leq 3.87^n$.



Lower Bound. We present a family $T^*_{i,j}$ which provides the lower bound from Theorem 10, the figure shows $T^*_{4,5}$. The triangulation $T^*_{i,i}$ has $i^2 + 3$ vertices and $(i-1)^2$ edge disjoint directed cycles. From the observation, that reversing a directed cycle yields another 3-orientation we deduce a lower bound of $2^{(i-1)^2} \geq 2^{(1-\epsilon)n}$, for $\epsilon > 0$ and i sufficiently large. The bound 2.27^n is obtained by a more thorough analysis of $T^*_{7,j}$. This analysis follows a transfer matrix approach and the lower bound is obtained from the dominant eigenvalue of the transfer matrix.

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l-Chain Profile Vectors

Dniel Gerbner* Balzs Patks†

Basic problems in extremal set theory consider a class $\mathcal{A}$ of families of subsets of an n-element set all having some fixed property (Sperner, intersecting, complement free, etc.) and ask for the family with largest size. A natural generalization of this problem if in the summation we allow *weights* different from 1, but depending only on the size of the subset in question.

The tool that helps to deal with this generalized problem is the so-called *profile vector* of a family $\mathcal{F}$ on an n-element base set. It is a vector $p(\mathcal{F}) \in \mathbb{R}^{n+1}$ the components of which are defined by $p(\mathcal{F})_i = |\{F \in \mathcal{F} : |F| = i\}|$.

Given a class of families $\mathbf{A}$, let $\mu(\mathbf{A})$ denote the set of the profile vectors of families in $\mathbf{A}$ (i.e. $\{p(\mathcal{F}) : \mathcal{F} \in \mathbf{A}\}$). A basic fact of linear programming says that the maximum weight can be attained only at an *extreme point* of the *convex hull* of $\mu(\mathbf{A})$, which we call the *profile polytope* of $\mathbf{A}$ and we denote it by $\langle\mu(\mathbf{A})\rangle$. And if all w_is are non-negative, then the maximum is taken at an *essential extreme point* (i.e. an extreme point that is maximal with respect to the component-wise ordering).The first result in this area (implicitly, without using the notion of the profile polytope) is due to Katona, the systematic investigation of profile polytopes was started by P.L. Erdős, P. Frankl, G.O.H. Katona.

However, there are problems dealing with other kind of weight functions, and problems not dealing with sets of some families, but subfamilies of families. A natural question is the following: let $l \leq k$ be two integers, how many l-chains (a sequence of sets of length l in which every set contains the previous one) can be contained in a family without a $k+1$-chain.

To deal with the above problem we introduce the notion of *l-chain profile vector* of a family $\mathcal{F}$. This has $\binom{n+1}{l}$ components, and the αth component p_α, where $\alpha = (\alpha_1, ..., \alpha_l)$ with $0 \leq \alpha_1 < \alpha_2 < ... < \alpha_l \leq n$, denotes the number of l-chains in $\mathcal{F}$ in which the smallest set has size α_1, the second smallest has size α_2, and so on. Note that for $l = 1$ this is just the original notion of the profile vector.

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Let $\mu_l(\mathbf{A})$ denote the set of all l-chain profile vectors of families in $\mathbf{A}$, $\mathcal{E}_l(\mathbf{A})$ the extreme points of $\langle \mu_l(\mathbf{A})\rangle$ and $E_l(\mathbf{A})$ the families from $\mathbf{A}$ with l-chain profile in $\mathcal{E}_l(\mathbf{A})$. Let $\mathcal{E}_l^*(\mathbf{A})$ denote the essential extreme points and $E_l^*(\mathbf{A})$ the corresponding families. We need the following notation:

$$\binom{n}{\alpha} = \prod_{i=1}^{l-1} \binom{n-\alpha_{i-1}}{\alpha_{i+1}-\alpha_i} = \frac{n!}{\alpha_1!(\alpha_2-\alpha_1)!...(\alpha_l-\alpha_{l-1})!(n-\alpha_l)!}$$

where $\alpha_0 = 0$ and $0! = 1$ as usual. Note that $\binom{n}{\alpha}$ is the number of l-chains that can be formed from subsets of an n-element set in such a way that the smallest set has size α_1, the second smallest has size α_2 and so on.

The main tool that helps to determine the extreme points of the profile polytope is the reduction to the circle. We observed that the method works for the l-chain case as well, and - what seems to us more important - in some cases it is enough to reduce the original problem to the chain instead of the circle.

Definition: For a family $\mathcal{F}$ on a base set X and a maximal chain $\mathcal{C}$ in X let $\mathcal{F}(\mathcal{C}) = \{F \in \mathcal{F} \cap \mathcal{C}\}$ and for a class of families $\mathbf{A}$ let $\mathbf{A}(\mathcal{C}) = \{\mathcal{F}(\mathcal{C}) : \mathcal{F} \in \mathbf{A}\}$.

Let $T_{\mathcal{C}}^l$ denote the following operator acting on the $\binom{n+1}{l}$-dimensional $\mathbb{R}$-space

$$T_{\mathcal{C}}^l : e \mapsto T_{\mathcal{C}}^l(e) \qquad \text{where} \qquad T_{\mathcal{C}}^l(e)_\alpha = \binom{n}{\alpha} e_\alpha.$$

Theorem. For any set of families $\mathbf{A} \subseteq 2^{2^X}$ if the extreme points $e_1, e_2, ..., e_m$ of $\langle \mu_l(\mathbf{A}(\mathcal{C}))\rangle$ do not depend on the choice of $\mathcal{C}$, then

$$\langle \mu_l(\mathbf{A})\rangle \subseteq \langle \{T_{\mathcal{C}}^l(e_1), ..., T_{\mathcal{C}}^l(e_m)\}\rangle.$$

Corollary. For any $l \leq k$ the extreme points of the l-chain profile polytope of k-Sperner families are the following: the all zero vector $\mathbf{0}$ and for all $l \leq z \leq k$ and $\beta = \{\beta_1, ..., \beta_z\}$ with $0 \leq \beta_1 < ... < \beta_z \leq n$ the vectors v_β

$$(v_\beta)_\alpha = \begin{cases} \binom{n}{\alpha} & \text{if } \alpha \subseteq \beta \\ 0 & \text{otherwise.} \end{cases}$$

Much easier to determine the extreme points of the l-chain profile polytope when $l = 1$. Therefor it is practical to reduce the problem to that case.

Definition: The *upset* of $\mathcal{F}$ is $\mathcal{U}(\mathcal{F}) = \{G \subseteq X : \exists F \in \mathcal{F}(F \subseteq G)\}$. A set $\mathbf{A}$ of families is *upward closed* if $\mathcal{F} \in \mathbf{A}$ implies $\mathcal{U}(\mathcal{F}) \in \mathbf{A}$.

Theorem. For any upward or downward closed set of families $\mathbf{A}$ and for any $l \geq 1$

$$\mathcal{E}_l^*(\mathbf{A}) \subseteq \mu_l(E_1^*(\mathbf{A})).$$

Clearly the class of t-intersecting families is upward closed. The profile polytope of this class has been already determined in the case $t = 1$.

Profile Vectors in the Lattice of Subspaces

Dániel Gerbner * <u>*Balázs Patkós*</u>†

The profile vector $f(\mathcal{F}) \in \mathbb{R}^{n+1}$ of a family $\mathcal{F} \subseteq 2^{[n]}$ is defined by $f(\mathcal{F})_i = |\{F \in \mathcal{F} : |F| = i\}|$, and for a set of families $\mathbb{A} \subseteq 2^{2^{[n]}}$ its profile polytope is the convex hull (in $\mathbb{R}^{n+1}$) of $\mu(\mathbb{A}) = \{f(\mathcal{F}) : \mathcal{F} \in \mathbb{A}\}$.

These notions have been studied for more than 20 years, the reason for which is that they generalize extremal problems for families of subsets. If $w : [n] \to \mathbb{R}$ is a *weight-function*, than the maximum of $\sum_i w(i) f(\mathcal{F})_i$ (where $\mathcal{F}$ ranges through all families of some set $\mathbb{A}$) is taken at an *extreme point* of the profile polytope. (If all weights are non-negative, then at an *essential extreme point*, i.e. an extreme point maximal with respect to the component-wise ordering.)

Considering the constant one weight (in this case the above sum is just the size of the family) we get the special case of finding the largest family of the set. Putting $w(k) = 1$ and $w(j) = 0, j \neq k$, we get the case of determining the largest k-uniform family in the set. If $\mathbb{A}$ is the set of Sperner-families and $w(i) = \frac{1}{\binom{n}{i}}$, then we get the famous LYM-inequality.

Profile vectors can be introduced not only in the Boolean poset, but for example in the lattice of subspaces of an n-dimensional vectorspace over $GF(q)$. For a family $\mathcal{U}$ of subspaces let its profile $f(\mathcal{U}) \in \mathbb{R}^{n+1}$ be defined by $f(\mathcal{U})_i = |\{U \in \mathcal{U} : \dim U = i\}|$.

To determine the profile polytope of intersecting families of subspaces (a family $\mathcal{U}$ of subspaces is called intersecting if for any $U_1, U_2 \in \mathcal{U}$ we have $\dim(U_1 \cap U_2) \geq 1$) we follow the so-called *method of inequalities*. Briefly it consists of the following steps:

- establish as many linear inequalities valid for the profile of any intersecting family (each inequality determine a halfspace, therefore the profiles must lie in the intersection of all halfspaces determined by the inequalities),
- determine the extreme points of the polytope determined by the above halfspaces,

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- for all of the above extreme points find an intersecting family having this extreme point as profile vector.

The last step gives that the extreme points of the polytope determined by the halfspaces are the extreme points of the profile polytope that we are looking for.

By earlier results of Hsieh and Greene and Kleitman we know the following inequalities on the profiles of intersecting families:

- $0 \leq f_i \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}, \quad 0 \leq i \leq n/2$
- $0 \leq f_i \leq \begin{bmatrix} n \\ i \end{bmatrix}, \quad n/2 < i \leq n$

Investigating intersecting families of k and d-dimensional subspaces where $1 \leq kn/2$, $n/2 < d$ and $k + d \leq n$ (using the idea of the proof of Hsieh and some further ideas) we are able to prove

Theorem 1. *For the profile vector f of any family $\mathcal{F}$ of intersecting subspaces of an n-dimensional vectorspace V, and for any $k < n/2$ and $n/2 < d \leq n - k$, the following holds*

$$c_{k,d} f_k + f_d \leq \begin{bmatrix} n \\ d \end{bmatrix},$$

where $c_{k,d} = q^d \frac{\begin{bmatrix} n-k \\ d \end{bmatrix}}{\begin{bmatrix} n-d-1 \\ k-1 \end{bmatrix}}$, and equality holds if $f_k = 0, f_d = \begin{bmatrix} n \\ d \end{bmatrix}$ or if $f_k = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}, f_d = \begin{bmatrix} n-1 \\ d-1 \end{bmatrix}$.

These new inequalities (together with the old ones) are sufficient to deduce our main result.

Theorem 2. *The essential extreme points of the profile polytope of the set of intersecting families are the vectors v_i $1 \leq i \leq n/2$ for even n and there is an additional essential extreme point v^+ for odd n, where*

$$(v_i)_j = \begin{cases} 0 & \text{if } \ 0 \leq j < i \\ \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} & \text{if } \ i \leq j \leq n-i \\ \begin{bmatrix} n \\ j \end{bmatrix} & \text{if } \ j > n-i. \end{cases} \tag{0.1}$$

and

$$(v^+)_j = \begin{cases} 0 & \text{if } \ 0 \leq j < n/2 \\ \begin{bmatrix} n \\ j \end{bmatrix} & \text{if } \ j > n/2. \end{cases} \tag{0.2}$$

On Vertex Partitions and the Colin de Verdière Parameter

D. Gonçalves[*]

In 1990, Colin de Verdire [1, 2] introduced an interesting new graph parameter $\mu(G)$ for any graph G. The parameter was motivated by the study of the maximum multiplicity of the second eigenvalue of certain Schrdinger operators. The parameter $\mu(G)$ can be described fully in terms of properties of matrices related to G. We do not provide its definition in this abstract (see [4] for a survey on μ). We just mention that this parameter is minor monotone and that it gives a new characterization of well known minor closed families of graphs. Indeed, for any graph G :

- $\mu(G) = 0$ iff G has at most two vertices and no edges.
- $\mu(G) \leq 1$ iff G is a forest of paths (*i.e.* K_3 and $K_{3,1}$ minor-free).
- $\mu(G) \leq 2$ iff G is an outerplanar graph (*i.e.* K_4 and $K_{3,2}$ minor-free).
- $\mu(G) \leq 3$ iff G is a planar graph [6] (*i.e.* K_5 and $K_{3,3}$ minor-free).
- $\mu(G) \leq 4$ iff G is a linkless embeddable graph in $\mathbb{R}^3$ [5, 3] (*i.e.* without minor in the Petersen family, a set of seven graphs including K_6 and the Petersen graph).

So using μ, topological properties of a graph G can be characterized by spectral properties of matrices associated with G. Conversely the kernel of these matrices related to G can be used to construct nice embeddings of G in $\mathbb{R}^{\mu(G)}$.

Here we investigate some links between this parameter and vertex partitionning problems. A stable graph G with more than two vertices is such that $\mu(G) = 1$. Since stable graphs are the K_2 minor-free graphs, we would like that $\mu(G) = 0$ for stable graphs. So we define a slightly different parameter μ' : if G is a stable let $\mu'(G) = 0$, else $\mu'(G) = \mu(G)$. Since graphs with $\mu(G) \leq 4$ are $(\mu(G)+1)$-colorable. Colin de Verdire made the following conjecture.

Conjecture 1 (by Colin de Verdire)**.** *For any graph G, $\chi(G) \leq \mu(G) + 1$.*

Since any graph G is $K_{\mu(G)+2}$ minor-free, this conjecture is implied by the following well known conjecture.

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Conjecture 2 (by Hadwiger). *A graph G without K_k minor is $(k-1)$-colorable.*

Actually, a k-coloring of a graph G is a k-partition of its vertex set $V(G) = V_1 \cup \cdots \cup V_k$ such that each subset induces a stable graph $G[V_i]$. So Conjecture 1 can be reformulated as follow : "Any graph G with $\mu(G) = k-1$ has a vertex k-partition $V_1, \ldots V_k$ into stable graphs $G[V_i]$ (*i.e.* $\mu'(G[V_i]) = 0$)". We observe that outerplanar graphs have a vertex partition into 2 forests of paths and that planar graphs have vertex partitions into 2 outerplanar graphs or into 3 forests of paths. So we generalizes Conjecture 1.

Conjecture 3. *For any graph G and any $k \in [1 \ldots \mu'(G)+1]$, the graph G has a vertex k-partition $V_1, \ldots V_k$ into graphs $G[V_i]$ such that $\mu'(G[V_i]) \leq \mu'(G) + 1 - k$.*

When $k = 1$ it is trivially true and when $k = \mu'(G) + 1$ it corresponds to Conjecture 1. We proved that it is also true for $k = 2$ or 3.

Theorem 1. *Any graph G has a vertex 2-partition (resp. 3-partition) into graphs $G[V_i]$ such that $\mu'(G[V_i]) \leq \mu'(G) - 1$ (resp. $\mu'(G[V_i]) \leq \mu'(G) - 2$).*

The proof of this theorem is similar to the proof of the fact that forests of paths (resp. outerplanar graphs) are 2-colorable (resp. 3-colorable). Unfortunately, it seems difficult to use the proof of the 4 Color Theorem to find a proof for the next step, $k = 4$. To prove this case ($k = 4$) we could use a different technique. Given two graphs H and G let $H \times G$ be the graph with vertex set $V(H) \times V(G)$ and such that $(u, v)(u', v')$ is an edge of $H \times G$ iff $uu' \in E(H)$ or if $u = u'$ and $vv' \in E(G)$. We do the following conjecture :

Conjecture 4. *For any graph X and any $k \in [0 \ldots \mu'(X)]$ there exist two graphs H and G, with $\mu'(H) \leq k$ and $\mu'(G) \leq \mu'(X) - k$, such that the graph X is a subgraph of $H \times G$.*

This is true for $k \leq 3$, and we note that if Conjectures 1 and Conjecture 4 are true, then Conjecture 3 is true.

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The $\boldsymbol{t}$-Improper Chromatic Number of Random Graphs

Ross J. Kang and Colin J. H. McDiarmid

We consider the t-improper chromatic numbers of the Erdös-Rényi random graph. As usual, $G_{n,p}$ denotes a random graph with vertex set $\{1, \ldots, n\}$ in which the edges are chosen independently at random with probability p. We shall assume here that p is a constant, with $0 < p < 1$. A t-dependent set in a graph G is a vertex subset which induces a subgraph of maximum degree at most t. The t-improper chromatic number $\chi^t(G)$ is the smallest number of colours needed in a t-improper colouring — a colouring of the vertices in which colour classes are t-dependent sets.

The t-improper chromatic number was introduced twenty years ago independently by Andrews and Jacobson [1], Harary and Fraughnaugh (née Jones) [6, 7], and Cowen *et al* [3]. In the first paper, the authors considered various general lower bounds for the t-improper chromatic number; in the second, the authors studied χ^t as part of the more general setting of generalised chromatic numbers; in the third, the authors established best upper bounds on χ^t for planar graphs to generalise the Four Colour Theorem. Several papers on the topic have since appeared; in particular, two papers, by Eaton and Hull [4] and Škrekovski [10], have extended the program of Cowen *et al* to a list colouring variant of χ^t and both pose the question: is every planar graph 1-improper 4-choosable?

Clearly, when $t = 0$, we are simply considering the ordinary notion of chromatic number for random graphs, and this topic is well studied. Let $\gamma = 2/\log\frac{1}{1-p}$. In 1975 Grimmett and McDiarmid [5] proved that for any $\varepsilon > 0$ the expected number of j-colourings of $G_{n,p}$ tends to 0 if $j \le (1-\varepsilon)\frac{n}{\gamma \log n}$ and tends to ∞ if $j \ge (1+\varepsilon)\frac{n}{\gamma \log n}$, showing that $\chi(G_{n,p}) \ge (1-\varepsilon)\frac{n}{\gamma n}$ asymptotically almost surely (a.a.s.), and suggesting that

$$\chi(G_{n,p}) \sim \frac{n}{\gamma \log n} \quad \text{a.a.s.}$$

A decade or so later Bollobás [2], and Matula and Kučera [9], showed that this is indeed correct.

Now clearly $\chi^t(G_{n,p}) \le \chi(G_{n,p})$. Also, it is easy to see that $\chi^t(G) \ge \chi(G)/(t+1)$, since each colour class of a t-improper colouring can be properly coloured with at most $t+1$ colours. Thus

$$\frac{\chi(G_{n,p})}{t+1} \le \chi^t(G_{n,p}) \le \chi(G_{n,p}).$$

It turns out that $\chi^t(G_{n,p})$ is close to $\chi(G_{n,p})$ as long as $t(n) = o(\log n)$. This is in contrast to the behaviour of random geometric graphs, where χ^t is close to $\chi/(t+1)$ for t smaller than the expected average degree – see [8]. More fully, we have:

Theorem 13. *(a) if $t(n) = o(\log n)$, then $\chi^t(G_{n,p}) \sim \frac{n}{\gamma \log n}$ a.a.s.;*

(b) if $t(n) = \Theta(\log n)$, then $\chi^t(G_{n,p}) = \Theta(\frac{np}{t})$ a.a.s.;

(c) if $t(n) = \omega(\log n)$ and $t(n) = o(n)$, then $\chi^t(G_{n,p}) \sim \frac{np}{t}$ a.a.s.;

(d) if $t(n)$ satisfies $\frac{np}{t} \to x$, where $0 < x < \infty$ and x is not integral, then $\chi^t(G_{n,p}) = \lceil x \rceil$ a.a.s.

Now let us examine more closely case (b), where the above theorem is imprecise. Suppose that $t(n) \sim \tau \log n$ for some fixed $\tau > 0$. We identify a constant $\kappa > 0$ (depending on p and τ) such that the expected number of t-improper j-colourings of $G_{n,p}$ tends to 0 if $j \leq (1-\varepsilon)\frac{n}{\kappa \log n}$ and to ∞ if $j \geq (1+\varepsilon)\frac{n}{\kappa \log n}$. Note that this result is analogous to (and extends) the result which we mentioned above on the expected numbers of ordinary colourings. It is natural now to conjecture that $\chi^t(G_{n,p}) \sim \frac{n}{\kappa \log n}$ a.a.s.

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Non-Generic Orthogonal Surfaces

Sarah Kappes[*]

The *dominance order* on $\mathbb{R}^d$ is defined by

$$x \leq y \Leftrightarrow x_i \leq y_i \text{ for all } i = 1, \dots, d$$

The *join* of a finite set of points $U \subset \mathbb{R}^d$ is the coordinate-wise maximum of all points in U.

Let $V \subset \mathbb{N}^d$ be a finite set of pairwise incomparable points containing d suspensions $s^1, \dots, s^d$, where $s^i_j = 0$ for $i \neq j$ and s^i_i is maximal within V.

The *orthogonal surface* S_V generated by V is the boundary of the set $\{\alpha : v \leq \alpha \text{ for some } v \in V\}$. The elements of V are the local minima of the surface.

We are interested in the combinatorial structure of S_V. This is captured in the *cp-order*, the set of *corners* of S_V equipped with the dominance order. All corners are joins of subsets of V, the reverse is not true in general.

The surface S_V is *generic* if no two elements of V share a non-zero coordinate. In this case, the corners are exactly those joins of sets $U \subset V$ such that $join(U) \in S_V$. These sets form a simplicial complex. By the theorem of Scarf, the cp-order of a generic orthogonal surface in dimension d can be extended to the face-lattice of a simplicial d-polytope. Figure 1 shows a 3-dimensional generic surface and the plane triangulation generated by it.

In general, the cp-order does not even satisfy the most basic properties needed to define a face-complex. Even the restriction to *non-degenerate* and *rigid* surfaces does not suffice to guarantee a polytopal structure, but we can show that the cp-orders of rigid surfaces have some other nice properties.

The cp-order of a non-degenerate surfaces allows *orthogonal matchings* on the set of characteristic points. The ith orthogonal matching M_i is defined by $(c, d) \in M_i \iff c_j = d_j$ for all $j \neq i$ and $c_i < d_i$. This matching is almost perfect, only one minimum remains unmatched. Figure 2 shows an orthogonal matching on a rigid orthogonal surface of dimension 3 and on the Hasse-diagram of its cp-order.

For rigid surfaces, the orthogonal matchings satisfy an additional acyclicity property, this leads to the following proposition:

Proposition 1. *An orthogonal matching on the cp-order of a non-degenerate, rigid orthogonal surface is a Morse-matching.*

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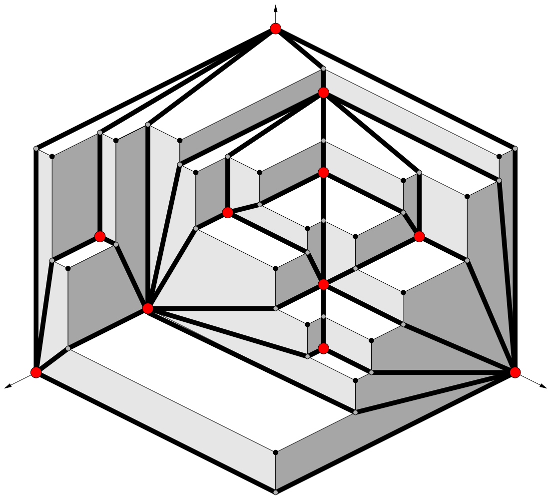

Figure 1: A 3-dimensional generic surface generates a plane triangulation

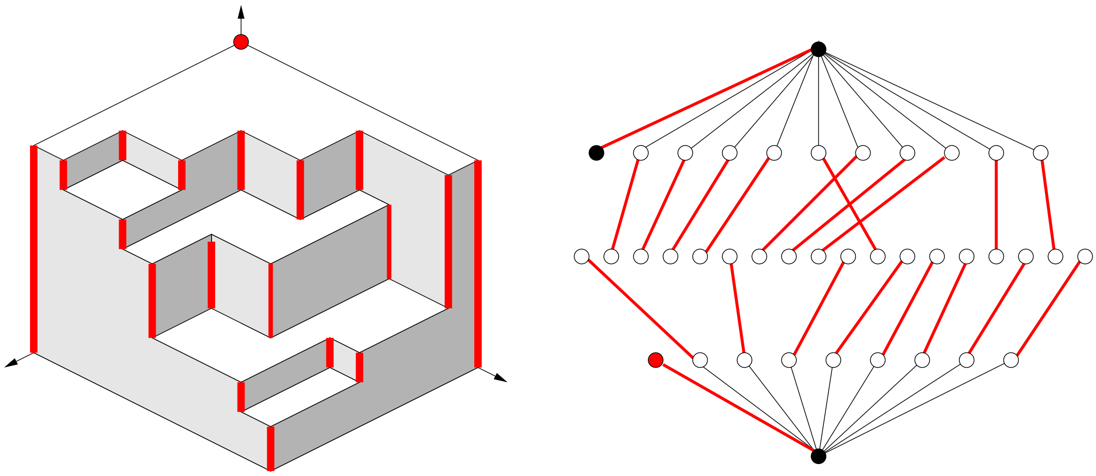

Figure 2: An orthogonal matching

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Forbidden Submatrices in 0-1 Matrices

Balázs Keszegh[*] *Gábor Tardos*[†]

We say that a 0-1 matrix (or pattern) P is contained in the 0-1 matrix A if it can be obtained from a submatrix of it by changing extra 1 entries to 0. We investigate the extremal function $ex(n, P)$ which is the maximum number of 1 entries in an n by n matrix not containing P. Note that this is an ordered variant of the classical Turán extremal graph theory for bipartite graphs. This problem was first studied in [1], [2]. The case of patterns with at most four 1 entries was asymptotically solved by [3], [5].

Theorem 1. *Let A be a pattern which has two 1 entries in its first column in row i and $i+1$ for a given i. Let A' be the pattern obtained from A by adding two new rows between the ith and the $(i+1)$th row and a new column before the first column with exactly two 1 entries in the intersection of the new column and rows. Then $ex(n, A') = O(ex(n, A))$.*

Corollary 1. $ex(n, L_2) = O(n)$.
(in the figure we represent 1 entries with dots and 0 entries with blanks)

$$L_2 = \begin{pmatrix} & \bullet & & \bullet \\ \bullet & & & \\ \bullet & & & \\ & & \bullet & \end{pmatrix}$$

If the pattern P has higher than linear extremal function and by deleting any 1 entry from it we obtain a pattern with linear extremal function, then we call it *minimal non-linear*. We proved that the pattern H_0 is such and so far, this is the only pattern with more than four 1 entries known to be such.

Theorem 2. $ex(n, H_0) = \Theta(n \log n)$.

$$H_0 = \begin{pmatrix} & \bullet & \bullet & \\ & & & \bullet \\ & & & \bullet \\ \bullet & & & \end{pmatrix}$$

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It is not known how many minimal non-linear patterns are.

Conjecture 1. *There are infinitely many minimal non-linear patterns.*

We solve the analogous question on quasi-linear patterns, where we call a pattern quasi-linear if it is bounded by n times an exponencial function of $\alpha(n)$, where $\alpha(n)$ is the extremely slowly growing inverse Ackermann function.

Theorem 3. *There exist infinitely many pairwise different minimal non-quasilinear patterns.*

The following theorem was proved in [4]:

Theorem 4. *(Marcus, Tardos)*
For all permutation matrices P we have $ex(n,P) = O(n)$.

The conjectures below are in strengthening order and the second would already imply Conjecture 1.

Conjecture 2.

1. $ex(n,G) = O(n)$.

$$G = \begin{pmatrix} & \bullet & \bullet & & \\ & & & & \bullet \\ & & & \bullet & \\ \bullet & & & & \end{pmatrix}$$

2. *For any permutation pattern by doubling the column containing the 1 entry in its first row we obtain a pattern with linear extremal function.*

3. *By doubling one column of a permutation pattern we obtain a pattern with linear extremal function.*

4. *By doubling every column of a permutation pattern we obtain a pattern with linear extremal function.*

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Bordeaux 3-Color Conjecture and 3-Choosability

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A *proper vertex coloring* of a graph G is an assignment c of integers (or labels) to the vertices of G such that $c(u) \neq c(v)$ if the vertices u and v are adjacent in G. A graph G is list L-colorable if for a given list assignment $L = \{L(v) : v \in V(G)\}$ there exists a proper coloring c of the vertices such that $\forall v \in V(G), c(v) \in L(v)$. If G is list L-colorable for every list assignment with $|L(v)| \geq k$ for all $v \in V(G)$, then G is k-choosable.

All 2-choosable graphs were characterized completely in [ERT79]. Thomassen [Tho94] proved that every planar graph is 5-choosable, whereas Voigt [Voi93] presented an example of a planar graph which is not 4-choosable. Thus it remains to determine whether a given planar graph is 3- or 4-choosable. In [Gut96], Gutner proved that these problems are NP-hard. Therefore, many authors tried to find sufficient conditions for a planar graph to be 3- or 4-choosable. Alon and Tarsi [AT92] proved that every planar bipartite graph is 3-choosable. Thomassen [Tho95] proved that every planar graph of girth 5 is 3-choosable. More recently, some new sufficient conditions for a planar graph to be 3-choosable have been given. These conditions include: having no 3-,5- and 6-cycles [LSS05]; having no 4-,5-,6- and 9-cycles [ZW05]; or having no 4-,5-,7- and 9-cycles [ZW04]. The reader is referred to [LXL99, LSX01, WL02b, WL02a] for results about the 4-choosability of planar graphs.

In this paper, we investigate the 3-choosability of planar graphs with sparse triangles and without cycles of special length. We prove that :

1. Every planar graph without 4- and 5-cycles, and without triangles at distance less than 4 is 3-choosable.

2. Every planar graph without 4-, 5- and 6-cycles, and without triangles at distance less than 3 is 3-choosable.

Moreover we show that :

3. There exists a non-3-choosable planar graph without 4-cycles, 5-cycles and intersecting triangles.

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In 1969, Havel asked if there existed a constant C such that every planar graph with the minimal distance between triangles at least C is 3-colorable [Hav69]. This problem remains widely open. Borodin and Raspaud [BR03] proved that every planar graph with neither 3-cycles at distance less than 4 nor 5-cycles is 3-colorable. Moreover, they made the following conjecture:

Conjecture 1 (Bordeaux 3-color Conjecture)**.** *Every planar graph without intersecting 3-cycles and without 5-cycles is 3-colorable.*

The result 3. shows that like Grtzsch's Theorem and Steinberg's Conjecture, the Bordeaux 3-color Conjecture can not be extended to the list coloring situation.

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Oriented Chromatic Index of Oriented Graphs

Pascal Ochem *Alexandre Pinlou* *Éric Sopena*[*]

Introduction The notion of oriented vertex-coloring was introduced by Courcelle [3] as follows: an *oriented k-vertex-coloring* of an oriented graph G is a homomorphism φ from G to H, where H is an oriented graph of order k. The *oriented chromatic number* of G, denoted by $\chi_o(G)$, is then defined as the smallest order of an oriented graph H such that G admits a homomorphism to H.

Oriented vertex-colorings have been studied by several authors in the last decade and the problem of bounding the oriented chromatic number has been investigated for various graph classes (see [2, 9, 10] for an overview).

One can define *oriented arc-colorings* of oriented graphs in a natural way by saying that, as in the undirected case, an oriented arc-coloring of an oriented graph G is an oriented vertex-coloring of its line digraph $LD(G)$. We say that an oriented graph G is *H-arc-colorable* if there exists a homomorphism φ from $LD(G)$ to H and φ is then an *H-arc-coloring*. The *oriented chromatic index* of G, denoted by $\chi'_o(G)$, is defined as the smallest order of an oriented graph H such that $LD(G)$ admits a homorphism to H.

The full proofs of the results presented in this abstract are available as internal reports [7, 8].

The first easy result concerning oriented arc-coloring relates the oriented chromatic index to the oriented chromatic number:

Theorem 1. *Let G be an oriented graph. Then $\chi'_o(G) \leq \chi_o(G)$.*

Therefore, all upper bounds for the oriented chromatic number are also valid for the oriented chromatic index.

Oriented chromatic index and acyclic chromatic number Raspaud and Sopena [9] proved that every oriented graph whose underlying undirected graph has acyclic chromatic number at most k has oriented chromatic number at most $k \cdot 2^{k-1}$ and recently, Ochem [6] proved that this bound is tight. By Theorem 1, every oriented graph with acyclic chromatic number

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k has oriented chromatic index at most $k \cdot 2^{k-1}$. We get a new bound which is quadratic in terms of the acyclic chromatic number:

Theorem 2. *Every oriented graph whose underlying undirected graph has acyclic chromatic number at most k has oriented chromatic index at most $2k(k-1) - \lfloor \frac{k}{2} \rfloor$.*

Oriented graphs whose underlying undirected graph has acyclic chromatic number at most k can be decomposed in $\binom{k}{2}$ forests. Then, we prove that we can obtain an oriented arc-coloring of such graphs using three colors for $\lfloor \frac{k}{2} \rfloor$ forests and four colors for the remaining ones.

A celebrated result of Borodin [1] states that every planar graph has acyclic chromatic number at most five. Theorem 2 and Borodin's result give the following upper bound:

Corollary 2. *Let G be a planar graph. Then $\chi'_o(G) \leq 38$.*

Graphs with bounded degree Kostochka *et al.* [5] proved that every oriented graph with maximum degree Δ has oriented chromatic number at most $2\Delta^2 2^\Delta$. Therefore, for such a graph G we also have $\chi'_o(G) \leq 2\Delta^2 2^\Delta$. We improve this bound and show the following:

Theorem 3. *Let G be an oriented graph with maximum degree Δ. Then, $\chi'_o(G) \leq 2\Delta^2$.*

For an oriented graph G, we relate its oriented chromatic index and the chromatic number of G^2 and prove that $\chi'_o(G) \leq 2(\chi(G^2) - 1)$. Then, the bound of Theorem 3 directly follows from this fact.

The following result improves the bound implied by Theorem 3 for subcubic graphs.

Theorem 4. *[8] Let G be a subcubic graph. Then $\chi'_o(G) \leq 7$.*

NP-completeness Klostermeyer and MacGillivray [4] have shown that given an oriented graph G, deciding whether $\chi_o(G) \leq k$ is polynomial time if $k \leq 3$ and is NP-complete if $k \geq 4$.

We determine the complexity of deciding whether the oriented chromatic index of a given oriented graph is at most a fixed positive integer and we obtain the following:

Theorem 5. *Given an oriented graph G, deciding whether $\chi'_o(G) \leq k$ is polynomial time if $k \leq 3$ and NP-complete if $k \geq 4$.*

The case $k \leq 3$ is a direct consequence from Klostermeyer and MacGillivray's result. We show that the case $k \geq 4$ is NP-complete using a reduction from 3-COLORABILITY.

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Injective Coloring of Graphs

*André Raspaud**

For a graph $G = (V(G), E(G))$, a vertex k-colouring is a mapping $c : V(G) \longrightarrow [k]$, with $[k] = \{0, 1, \ldots, k-1\}$. We say that a colouring of a graph is *injective* if its restriction to the neighbourhood of any vertex is injective. The *injective chromatic number* $\chi_i(G)$ of a graph G is the least k such that there is an injective k-colouring.
We will use the *maximum average degree* of a graph to give bounds of its injective chromatic number:

Definition 3. $mad(G) = max\{\frac{2|E(H)|}{|V(H)|} : H \textit{ is a subgraph of } G\}$.

Proposition 1. *If G is a planar graph with girth at least g then* $mad(G) < 2 + \frac{4}{g-2}$

We will denote by $\Delta(G)$ the maximum degree of a graph G.

In this talk we present two results obtained recently.

Theorem 14 (A. Doyon, G. Hahn, A.R. '05)**.** *Let G be a graph*

1. *If $mad(G) < \frac{14}{5}$ then $\chi_i(G) \leq \Delta(G) + 3$.*
2. *If $mad(G) < 3$ then $\chi_i(G) \leq \Delta(G) + 4$.*
3. *If $mad(G) < \frac{10}{3}$ then $\chi_i(G) \leq \Delta(G) + 8$.*

Corollary 15. *Let G be a planar graph*

1. *If $g(G) \geq 7$ then $\chi_i(G) \leq \Delta(G) + 3$*
2. *If $g(G) \geq 6$ then $\chi_i(G) \leq \Delta(G) + 4$*
3. *If $g(G) \geq 5$ then $\chi_i(G) \leq \Delta(G) + 8$*

Theorem 16 (G. Hahn, A.R., W. Wang '05)**.** *Let G be a K_4-minor free graph. Then $\chi_i(G) \leq \lceil \frac{3}{2}\Delta(G) \rceil$.*

Moreover we propose the following conjecture:

Conjecture 2 (G. Hahn, A.R., W. Wang '05)**.** *Let G be a planar graph with maximum degree $\Delta(G)$. Then $\chi_i(G) \leq \lceil \frac{3}{2}\Delta(G) \rceil$.*

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Variations on H-Coloring*

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Given two graphs G and H, an *homomorphism* from G to H is any function mapping the vertices in G to vertices in H, in such a way that the image of an edge is also an edge. In the case that H is fixed, such a homomorphism is called an H-coloring of G. For a given graph H, the *H-coloring problem* asks whether there exists an H-coloring of the input graph G. The more general version in which a list of allowed colors (vertices of H) is given for each vertex is known as the list H-coloring. See [8] for more variations on the problem and complexity results.

We consider the variation of H-coloring in which the number of preimages of some vertices in the target graph H is restricted through a partial weight assignment (C, K), assignig weight $K(c)$ to any vertex $c \in C \subset V(H)$. The *restrictive H-coloring* problem has as input a graph G and a partial weight assignment (C, K) and ask for the existend of an H-coloring χ of G in which, for any $c \in C$, $|\{v \mid \chi(v) = c\}| = K(c)$. We consider also the list version of the restrictive H-coloring and the corresponding counting problems: The restrictive $\#H$-coloring, the restrictive list $\#H$-coloring, the restrictive $\#H$-coloring, and the restrictive list $\#H$-coloring. We survey complexity classification of those problems given in [2].

The parameterized version of the *restrictive H-coloring* problems in which the partial weight asignment is fixed independently of the input is known as the *(H, C, K)-coloring* and was introduced in [1]. We survey the complexity of the (H, C, K)-coloring presented in [5] and the efficient fixed parameter algorithms known for some particular classes of partially weighted graphs [3, 4]. One of the fundamental tools for designing efficient fixed-parameter algorithms for decision problems is the so called *reduction to problem kernel*. The method consists in the polynomial time (in n and k) self-reduction that transforms a problem input (S, K) to another instance (S', K'), the kernel, such that the size of the new instance depends *only* on some function of k. We refer the reader to [6, 9] for discussions on fixed parameter tractability and the ways of constructing kernels. Our algorithms use this method as basic tool.

In recent years, there has been a big effort focused on developing a theory for the intractability of parameterized counting problems [7, 10]. However,

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so far, no progress has been noticed in the development of algorithmic techniques and, in particular, tools parallel to the reduction to problem kernel for counting problems. We introduce a new algorithmic tool, that we call the *compactor enumeration*. Instead of isolating a kernel to which the counting problem can be reduced to, we define a general type of set, called the *compactor*, which contains a *certificate* for each class in a suitable partition of the solution space of (S, K).

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Some New Classification Results for Linear Binary Codes With Minimum Distance 5 and 6

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Let F_2^n be the n-dimensional vector space over the Galois field $F_2 = GF(2)$. The Hamming distance between two vectors of F_2^n is defined to be the number of coordinates in which they differ. A linear binary $[n,k,d]$-code is a k-dimensional linear subspace of F_2^n with minimum Hamming distance d. Generator matrix G of linear binary $[n,k,d]$-code C is any matrix of rank k with rows from C. *Dual code* is $C^\perp = \{v \in F_2^n \mid (u,v) = 0, \text{ for all } u \in C\}$, where $(u,v) = \sum_{i=1}^n u_i v_i \in F_2$ for $u = (u_1,\ldots,u_n), v = (v_1,\ldots,v_n) \in F_2^n$ is the inner product in F_2^n. *Residual code* $Res(C,c)$ with respect to codeword c is the code generated by the restriction of generator matrix G to the columns where c has a zero entry.

A central problem in coding theory is that of optimizing one of the parameters n, k and d for given values of the other two. Two versions are:

- Find $d_2(n,k)$, the largest value of d for which an $[n,k,d]_q$-code exists.
- Find $k_2(n,d)$, the largest value of k for which an $[n,k,d]_q$-code exists.

Another important problem is: Characterize all nonequivalent binary $[n, k_2(n,d), d]$ codes with given values of n and d.

We investigate linear binary codes with minimum distance 5 and 6. Our main problem is: Are linear binary codes with parameters $[34, 24, 5]$?

Bounds for $d_2(n,k)$ were presented in [1]. The exact values of $k_2(n,d)$ are known for $d \leq 4$ and for $d = 5$, $n \leq 33$. This is the reason to consider the problem for existence or nonexistence of linear binary codes with parameters $[34, 24, 5]$.

The bounds for binary codes with minimum distance 5 and 6 are strongly related because of the parity check bits in binary case. Some results for $d = 5$ and $d = 6$ have been presented in [2], [3], [4], [5], [6], [7], etc. A linear binary $[33, 23, 5]$ code was found in [2].

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In this research, we use some theoretical and software tools. One of our tools is $Q - Extension$. The main problem which we solve in some cases with this program is the problem to construct all inequivalent linear codes with length n, dimension k, and minimum distance d, using a generator matrix of residual code. But $Q - Extension$ computes the dual distance after the construction of the code and does not work efficiently for codes with fixed dual distance. In our research for some specific cases we use another algorithm for construction of codes - $CCFDD$ (constructing codes with fixed dual distance).

We have three basic results:

1. A linear binary code with parameters $[34, 24, 5]$ does not exist and $k_2(34, 5) = 23$.

2. There are exact four nonequivalent linear binary codes with parameters $[33, 23, 5]$.

3. There exists a unique linear binary code with parameters $[34, 23, 6]$.

Also, we characterize some other linear binary codes with minimum distance 5 and 6.

n, k	**number**	**n, k**	**number**	**n, k**	**number**	**n, k**	**number**
10,3	2	16,8	1	22,13	128	28,18	≥ 499
11,4	1	17,9	1 [6]	23,14	1 [7]	29,19	≥ 74360
12,4	12	18,9	1558	24,14	≥ 206600	30,20	≥ 561458
13,5	15	19,10	16062	25,15	≥ 135500	31,21	5146
14,6	11	20,11	13924	26,16	≥ 31606	32,22	62
15,7	6 [3]	21,12	2373	27,17	≥ 8655	33,23	4

Table 1: Classification results for optimal codes with $d = 5$, $n < 34$.

n, k	**number**	**n, k**	**number**	**n, k**	**number**	**n, k**	**number**
11,3	1	17,8	1 [6]	23,13	8	29,18	≥ 498
12,4	1	18,9	1 [6]	24,14	1 [7]	30,19	≥ 49696
13,4	6	19,9	1700	25,14	≥ 47384	31,20	≥ 45461
14,5	6 [3]	20,10	1308	26,15	≥ 116642	32,21	≥ 200
15,6	5 [3]	21,11	737	27,16	≥ 31605	33,22	7
16,7	3 [5]	22,12	128	28,17	≥ 8654	34,23	1

Table 2: Classification results for optimal codes with $d = 6$, $n < 35$.

To obtain classification results for codes with dimension at most 8 we use $Q - Extension$. For dimensions greater than 8 we use $CCFDD$. In general, if C is a linear binary $[n, n-10, 5]$ (or $[n, n-9, 5]$) code, we consider its dual $[n, 10, d]$ ($[n, 9, d]$) code. We summarize all obtained results for codes with minimum distance 5 and 6 in Table 1 and Table 2. To construct the codes with minimum distance 6, we use the generator matrices of the codes in Table 1.

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