The Loebl–Komlós–Sós conjecture for trees of diameter 5 and for certain caterpillars

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Abstract

Loebl, Komlós, and Sós conjectured that if at least half the vertices of a graph G have degree at least some $k \in \mathbb{N}$, then every tree with at most k edges is a subgraph of G.

We prove the conjecture for all trees of diameter at most 5 and for a class of caterpillars. Our result implies a bound on the Ramsey number $r(T, T')$ of trees T, T' from the above classes.

1 Introduction

Loebl conjectured (see [5]) that if G is a graph of order n, and at least $n/2$ vertices of G have degree at least $n/2$, then every tree with at most $n/2$ edges is a subgraph of G. Komlós and Sós generalised his conjecture to the following.

Conjecture 1 (Loebl–Komlós–Sós conjecture [5]). *Let $k, n \in \mathbb{N}$, and let G be a graph of order n so that at least $n/2$ vertices of G have degree at least k. Then every tree with at most k edges is a subgraph of G.*

The conjecture is asymptotically correct, at least if $k \in \Theta(n)$: In [8], the authors of this paper prove an approximate version of the Loebl–Komlós–Sós conjecture for large n, and k linear in n. In Loebl's original form, an approximate version has been shown by Ajtai, Komlós and Szemerédi [1], and recently, Zhao [10] has shown the exact version.
The bounds from the conjecture could not be significantly lower. It is easy to see that we need at least one vertex of degree at least k in G. On the other

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hand, the amount of vertices of large degree that is required in Conjecture 1 is necessary. We shall discuss the bounds in more detail in Section 3.

Conjecture 1 trivially holds for stars. In order to see the conjecture for trees that consist of two stars with adjacent centres, it is enough to realise that G must have two adjacent vertices of degree at least k. Indeed, otherwise one easily reaches a contradiction by double-counting the number of edges between the set $L \subseteq V(G)$ of vertices of degree at least k, and the set $S := V(G) \setminus L$.

Hence, the Loebl–Komlós–Sós conjecture is true for all trees of diameter at most 3. Barr and Johansson [2], and independently Sun [9], proved the conjecture for all trees of diameter 4. Our main result is a proof of Conjecture 1 for all trees of diameter at most 5.

Theorem 2. *Let $k, n \in \mathbb{N}$, and let G be a graph of order n so that at least $n/2$ vertices of G have degree at least k. Then every tree of diameter at most 5 and with at most k edges is a subgraph of G.*

Paths and path-like trees constitute another class of trees for which Conjecture 1 has been studied. Bazgan, Li, and Woźniak [3] proved the conjecture for paths and for all trees that can be obtained from a path and a star by identifying one of the vertices of the path with the centre of the star.

We extend their result to a larger class of trees, allowing for two stars instead of one, under certain restrictions. Let $\mathcal{T}(k, \ell, c)$ be the class of all trees with k edges which can be obtained from a path P of length $k - \ell$, and two stars S_1 and S_2 by identifying the centres of the S_i with two vertices that lie at distance c from each other on P.

Theorem 3. *Let $k, \ell, c, n \in \mathbb{N}$ such that $\ell \geq c$. Let $T \in \mathcal{T}(k, \ell, c)$, and let G be a graph of order n so that at least $n/2$ vertices of G have degree at least k. If c is even, or $\ell + c \geq \lfloor n/2 \rfloor$ (or both), then T is a subgraph of G.*

If true, Conjecture 1 has an interesting application in Ramsey theory, as has been first observed in [5]. The Ramsey number $r(T_{k+1}, T_{m+1})$ of two trees T_{k+1}, T_{m+1} with k, resp. m edges is defined as the minimal integer n so that any colouring of the edges of the complete graph K^n of order n with two colours, say red and blue, yields either a red copy of T_{k+1}, or a blue copy of T_{m+1} (or both).

Observe that in any such colouring, either the red subgraph of K^n has at least $n/2$ vertices of degree at least k, or the blue subgraph has at least $n/2$ vertices of degree at least $n - k$. Hence, if Conjecture 1 holds for all k and n, then $r(T_{k+1}, T_{m+1}) \leq k + m$ for all $k, m \in \mathbb{N}$.

The bound $k + m$ is asymptotically true: the authors of this article prove in [8] that $r(T_{k+1}, T_{m+1}) \leq k + m + o(k + m)$, provided that $k, m \in \Theta(n)$.

Although particular classes of trees (such as paths [6]) have smaller Ramsey numbers, the bound $k+m$ would be tight in the class of all trees. In fact, the Ramsey number of two stars with k, resp. m edges, is $k+m-1$, if both k and m are even, and $k+m$ otherwise [7].
Our results on Conjecture 1 allow us to bound the Ramsey numbers of further classes of trees. Theorem 2 and Theorem 3 have the following corollary.

Corollary 4. *Let T_1, T_2 be trees with k resp. m edges such that, for $i = 1, 2$, either T_i is as in Theorem 3 or has diameter at most 5 (or both). Then $r(T_1, T_2) \leq k+m$.*

2 Notation

Throughout the paper, $\mathbb{N} = \mathbb{N}_+$.
Our graph-theoretic notation follows [4], let us here review the main definitions needed. A graph G has vertex set $V(G)$ and edge set $E(G)$. As we will not distinguish between isomorphic graphs we consider a graph H to be a subgraph of G, if there exists an injective mapping from $V(H)$ to $V(G)$ which preserves adjacencies. We shall then write $H \subseteq G$, and call any mapping as above an *embedding* of $V(H)$ in $V(G)$.
The *neighbourhood* of a vertex v is $N(v)$, and the neighbourhood of a set $X \subseteq V(G)$ is $N(X) := \bigcup_{v \in X} N(v) \setminus X$. We set $\deg_X(v) := |N(v) \cap X|$ and $\deg(v) := \deg_{V(G)}(v)$.
The *length* of a path is the number of its edges. For a path P and two vertices $x, y \in V(P)$, let xPy denote the subpath of P which starts in x and ends in y. We define xP and Py analogously. The *distance* between two vertices is the length of the shortest path connecting them. The *diameter* of G is the longest distance between any two vertices of G.

3 Discussion of the bounds

Let us now discuss the bounds in Conjecture 1. On one hand, as T could be a star, it is clear that we need that G has a vertex of degree at least k.
On the other hand, we also need a certain amount of vertices of large degree. In fact, the amount $n/2$ we require cannot be lowered by a factor of $(k-1)/(k+1)$. We shall show now that if we require only $\frac{k-1}{k+1}n/2 = n/2 - n/(k+1)$ vertices to have degree at least k, the conjecture becomes false whenever $k+1$ is even and divides n.
To see this, construct a graph G on n vertices as follows. Divide $V(G)$ into $2n/(k+1)$ sets A_i, B_i, so that $|A_i| = (k-1)/2$, and $|B_i| = (k+3)/2$, for

$i = 1, \ldots, n/(k+1)$. Insert all edges inside each A_i, and insert all edges between each pair A_i, B_i. Now, consider the tree T we obtain from a star with $(k+1)/2$ edges by subdividing each edge but one. Clearly, T is not a subgraph of G.
A similar construction shows that we need more than $\frac{n}{2} - \frac{2n}{k+1}$ vertices of large degree, when $k+1$ is odd and divides n, and furthermore, by adding some isolated vertices, our example can be modified for arbitrary k. This shows that at least $n/2 - 2\lfloor n/(k+1) \rfloor - (n \mod (k+1))$ vertices of large degree are needed. Hence, when $\max\{n/k, n \mod k\} \in o(n)$, the bound $n/2$ is asymptotically best possible.

4 Trees of small diameter

In this section, we prove Theorem 2. We shall prove the theorem by contradiction. So, assume that there are $k, n \in \mathbb{N}$, and a graph G with $|V(G)| = n$, such that at least $n/2$ vertices of G have degree at least k. Furthermore, suppose that T is a tree of diameter at most 5 with $|E(T)| \leq k$ such that $T \not\subseteq G$.
We may assume that among all such counterexamples G for T, we have chosen G edge-minimal. In other words, we assume that the deletion of any edge of G results in a graph which has less than $n/2$ vertices of degree k.
Denote by L the set of those vertices of G that have degree at least k, and set $S := V(G) \setminus L$. Observe that, by our edge-minimal choice of G, we know that S is independent. Also, we may assume that S is not empty.
Clearly, our assumption that $T \not\subseteq G$ implies that for each set M of leaves of T it holds that

$$\text{there is no embedding } \varphi \text{ of } V(T) \setminus M \text{ in } V(G) \text{ so that } \varphi(N(M)) \subseteq L. \quad (1)$$

In what follows, we shall often use the fact that both the degree of a vertex and the cardinality of a set of vertices are natural numbers. In particular, assume that $U, X \subseteq V(G)$, $u \in V(G)$, and $x \in \mathbb{Q}$. Then the following implication holds.

$$\textit{If } |U| < x + 1 \textit{ and } \deg_X(u) \geq x, \textit{ then } |U| \leq \deg_X(u). \quad (2)$$

Let us now define a useful partition of $V(G)$. Set

$$A := \{v \in L : \deg_L(v) < \frac{k}{2}\},$$
$$B := L \setminus A,$$

$$C := \{v \in S : \deg(v) = \deg_L(v) \geq \frac{k}{2}\}, \text{ and}$$
$$D := S \setminus C.$$

Let $r_1 r_2 \in E(T)$ be such that each vertex of T has distance at most 2 to at least one of r_1, r_2. Set

$$V_1 := N(r_1) \setminus \{r_2\}, \qquad V_2 := N(r_2) \setminus \{r_1\},$$
$$W_1 := N(V_1) \setminus \{r_1\}, \qquad W_2 := N(V_2) \setminus \{r_2\}.$$

Furthermore, set

$$V_1' := N(W_1) \quad \text{and} \quad V_2' := N(W_2).$$

Observe that $|V_1 \cup V_2 \cup W_1 \cup W_2| < k$. So, without loss of generality (since we can otherwise interchange the roles of r_1 and r_2), we may assume that

$$|V_2 \cup W_1| < \frac{k}{2}. \tag{3}$$

Since $|V_1'| \leq |W_1|$, this implies that

$$|V_1' \cup V_2| < \frac{k}{2}. \tag{4}$$

Now, assume that there is an edge $uv \in E(G)$ with $u, v \in B$. We shall conduct this assumption to a contradiction to (1) by proving that then we can define an embedding φ so that $\varphi(V_1' \cup V_2 \cup \{r_1, r_2\}) \subseteq L$.
Define the embedding φ as follows. Set $\varphi(r_1) := u$, and set $\varphi(r_2) := v$. Map V_1' to a subset of $N(u) \cap L$, and V_2 to a subset of $N(v) \cap L$. This is possible, as (2) and (4) imply that $|V_1' \cup V_2| + 1 \leq \deg_L(v)$.
We have thus reached the desired contradiction to (1). This proves that

$$B \textit{ is independent.} \tag{5}$$

Set

$$N := N(B) \cap L \subseteq A.$$

We claim that each vertex $v \in N$ has degree

$$\deg_B(v) < \frac{k}{4}. \tag{6}$$

Then, (5) and (6) together imply that

$$|B|\frac{k}{2} \leq e(N, B) \leq |N|\frac{k}{4},$$

and hence,

$$|N| \geq 2|B|. \tag{7}$$

In order to see (6), suppose otherwise, i. e., suppose that there is a vertex $v \in N$ with $\deg_B(v) \geq \frac{k}{4}$. Observe that by (4), $|V_1' \cup V_2'| < \frac{k}{2}$ and hence we may assume that at least one of $|V_1'|$, $|V_2'|$, say $|V_1|$, is smaller than $\frac{k}{4}$. The case when $|V_2'| < \frac{k}{4}$ is done analogously.

We define an embedding φ of $V_1' \cup V_2' \cup \{r_1, r_2\}$ in $V(G)$ as follows. Set $\varphi(r_1) := v$ and map $V_1' \cup \{r_2\}$ to $N(v) \cap B$. This is possible, because $|V_1'| + 1 < \frac{k}{4} + 1$, and thus, by (2), $|V_1' \cup \{r_2\}| \leq \deg_B(v)$.

Next, map V_2' to $N(u) \cap L$, where $u := \varphi(r_2)$. This is safe, as (5) implies that

$$N(u) \cap L \cap \varphi(V_1') = \emptyset,$$

and furthermore, by (4), $|V_2'| + 1 < \frac{k}{2} + 1$. Together with (2), we thus obtain that

$$|V_2' \cup \{r_1\}| = |V_2'| + 1 \leq \deg_L(u).$$

This yields the desired contradiction to (1), and thus proves (6).

Now, set

$$X := \{v \in L : \deg_{C \cup L}(v) \geq \frac{k}{2}\} \supseteq B.$$

We claim that

$$e(X, C) = 0. \tag{8}$$

Observe that then

$$X = B, \tag{9}$$

and,

$$e(B, C) = 0. \tag{10}$$

In order to see (8), suppose for contradiction that there exists an edge uv of G with $u \in X$ and $v \in C$. We define an embedding φ of $V_1' \cup V_2 \cup W_1^C \cup \{r_1, r_2\}$ in $V(G)$, where W_1^C is a certain subset of W_1, as follows.

Set $\varphi(r_1) := u$, and set $\varphi(r_2) := v$. Embed a subset V_1^C of V_1' in $N(u) \cap C$, and a subset $V_1^L = V_1' \setminus V_1^C$ in $N(u) \cap L$. We can do so by (2), and since by (4), $|V_1'| < \frac{k}{2}$.

Next, map $W_1^C := N(V_1^C) \cap W_1$ and V_2 to L, preserving all adjacencies. This is possible by (2). Indeed, observe that by the independence of S, each vertex in C has at least $\frac{k}{2}$ neighbours in L, while by (3), we have that

$$|V_1^L \cup W_1^C \cup V_2 \cup \{u\}| \leq |W_1 \cup V_2| + 1 < \frac{k}{2} + 1.$$

We have hence mapped V_1', V_2, W_1^C and the vertices r_1 and r_2 in a way so that the neighbours of $(V_1 \setminus V_1') \cup (W_1 \setminus W_1^C) \cup W_2$ are mapped to L. This yields the desired contradiction to (1). We have thus shown (8), and consequently, also (9) and (10).

Observe that

$$D \neq \emptyset. \tag{11}$$

Indeed, otherwise $C \neq \emptyset$ and thus by (8), we have that $A \neq \emptyset$. By (9), this implies that $D \neq \emptyset$, contradicting our assumption.
Next, we claim that there is a vertex $w \in N$ with

$$\deg_{C \cup L}(w) \geq \frac{k}{4}. \tag{12}$$

Indeed, suppose otherwise. Using (9) and (11), we obtain that

$$|A \setminus N|\frac{k}{2} + |N|\frac{3k}{4} \leq e(A, D) < |D|\frac{k}{2}.$$

Dividing by $\frac{k}{4}$, it follows that

$$2|A| + |N| < 2|D|.$$

Together with (7), this yields

$$|D| > |A| + |B| \geq \frac{n}{2},$$

a contradiction, since by assumption $|D| \leq |S| \leq \frac{n}{2}$. This proves (12).

Using a similar argument as for (8), we can now show that

$$|V_1'| \geq \frac{k}{4}. \tag{13}$$

Indeed, otherwise use (12) in order to map r_1 to w, r_2 to any $u \in N(w) \cap B$, and embed V_1' in $C \cup L$, and V_2 and W_1^C (defined as above) in L, preserving adjacencies. This yields the desired contradiction to (1).

Observe that (13) implies that $\frac{k}{4} \leq |V_1'| \leq |W_1|$, and hence, by (3),

$$|V_2| < \frac{k}{4}. \tag{14}$$

We claim that moreover

$$|V_1' \cup W_2| \geq \frac{k}{2}. \tag{15}$$

Suppose for contradiction that this is not the case. We shall then define an embedding φ of $V_1' \cup V_2 \cup \{r_1, r_2\} \cup W_2^C$ in $V(G)$, for a certain $W_2^C \subseteq W_2$, as follows.
Set $\varphi(r_2) := w$, and choose for $\varphi(r_1)$ any vertex $u \in N(w) \cap B$. Map a subset V_2^C of V_2 to $N(w) \cap C$, and map $V_2^L := V_2 \setminus V_2^C$ to $N(w) \cap L$. This is possible, as by (2), by (12), and by (14), we have that $\deg_{C \cup L}(w) \geq |V_2| + 1$. We may assume that we chose V_2^C so that it contains as many vertices from $V_2 \setminus V_2'$ as possible. We now plan to map $W_2^C := N(V_2^C) \cap W_2$ to $V(G)$. Observe that by our choice of V_2^C, either $V_2^C \subseteq V_2 \setminus V_2'$, and hence W_2^C is empty, or $V_2^L \subseteq V_2'$. In the latter case, it follows that

$$|V_2^L| \leq |W_2 \setminus W_2^C|,$$

and by our assumption that $|V_1' \cup W_2| < \frac{k}{2}$, we obtain that

$$|V_2^L \cup W_2^C \cup \{r_2\}| \leq |W_2| + 1 < \frac{k}{2} + 1.$$

Thus, by (2), for each $v \in C$, we have that $\deg(v) \geq |V_2^L \cup W_2^C| + 1$. Observe that (10) implies that $u \notin N(C)$. So, we can map W_2^C to L, preserving all adjacencies.
Next, we shall map V_1' to $N(u) \cap L$. We can do so, since

$$|V_1' \cup V_2^L \cup \{r_2\}| < \frac{k}{2} + 1$$

by (4), and since, in the case that $W_2^C = \emptyset$, we use by our assumption that (15) does not hold to see that

$$|V_1' \cup V_2^L \cup W_2^C \cup \{r_2\}| \leq |V_1' \cup W_2| + 1 < \frac{k}{2} + 1.$$

Hence, by (2), in either case it follows that

$$\deg_L(u) \geq |V_1' \cup V_2^L \cup W_2^C \cup \{r_2\}|.$$

We have thus embedded all of $V(T)$ except $(V_1 \setminus V_1') \cup (W_2 \setminus W_2^C) \cup W_1$ whose neighbours have their image in L. This yields a contradiction to (1), and hence proves (15).

Now, let $x \in \mathbb{Q}$ be such that $|V_1'| = x \cdot \frac{k}{2}$. Then, by (15),

$$|W_2| \geq (1 - x)\frac{k}{2}.$$

On the other hand, since $|W_1| \geq |V_1'| = x\frac{k}{2}$, and $|V(T)| = k+1$, we have that

$$\begin{aligned}|(V_1 \setminus V_1') \cup V_2 \cup W_2| &= |V(T) \setminus (V_1' \cup W_1 \cup \{r_1, r_2\})| \\ &< (1-x)k.\end{aligned}$$

Combining these inequalities, we obtain that

$$\begin{aligned}|V_1 \cup V_2| &= |V_1' \cup (V_1 \setminus V_1') \cup V_2| \\ &< x\frac{k}{2} + (1-x)k - (1-x)\frac{k}{2} \\ &= \frac{k}{2}. \end{aligned} \tag{16}$$

The now gained information on the structure of T enables us to show next that for each vertex v in $\tilde{N} := N(B \cup C) \cap L$ it holds that

$$\deg_L(v) < \frac{k}{4}. \tag{17}$$

Suppose for contradiction that this is not the case, i.e., that there exists a $v \in \tilde{N}$ with $\deg_L(v) \geq \frac{k}{4}$. We define an embedding φ of $V(T) \setminus (W_1 \cup W_2)$ in $V(G)$ so that $N(W_1 \cup W_2)$ is mapped to L.
Set $\varphi(r_2) := v$ and choose for $\varphi(r_1)$ any vertex $u \in N(v) \cap (B \cup C)$. By (14), and since we assume that (17) does not hold, we can map V_2 to $N(v) \cap L$. Moreover, since by (2) and (16) we have that

$$\deg_L(u) \geq |V_1 \cup V_2 \cup \{r_2\}|,$$

we can map V_1 to $N(u) \cap L$. We have hence mapped all of $V(T)$ but $W_1 \cup W_2$ to L, which yields the desired contradiction to (1) and thus establishes (17).

We shall finally bring (17) to a contradiction. We use (5), (9), (10) and (17) to obtain that

$$\begin{aligned}|D|\frac{k}{2} &\geq e(D, L) \\ &\geq |A \setminus \tilde{N}|\frac{k}{2} + |\tilde{N}|\frac{3k}{4} - e(C, \tilde{N}) + |B|k - e(B, \tilde{N}) \\ &\geq |A|\frac{k}{2} + |\tilde{N}|\frac{k}{4} + |B|k - e(B \cup C, \tilde{N}).\end{aligned}$$

Since $|S| \leq |L|$ by assumption, this inequality implies that

$$\begin{aligned}|B|\frac{k}{2} + |C|\frac{k}{2} + |\tilde{N}|\frac{k}{4} &\leq |B|\frac{k}{2} + (|A| + |B| - |D|)\frac{k}{2} + |\tilde{N}|\frac{k}{4} \\ &\leq e(B \cup C, \tilde{N}) \\ &\leq |\tilde{N}|\frac{k}{2},\end{aligned}$$

where the last inequality follows from the fact that $\tilde{N} \subseteq A = L \setminus X$, by (9). Using (17), a final double edge-counting now gives

$$\begin{aligned}(|A| + |B| + |C|)\frac{k}{2} &\leq |A|\frac{k}{2} + |\tilde{N}|\frac{k}{4} \\ &\leq e(A, S) \\ &< |D|\frac{k}{2} + |C|k \\ &= |S|\frac{k}{2} + |C|\frac{k}{2},\end{aligned}$$

implying that $|L| < |S|$, a contradiction. This completes the proof of Theorem 2.

5 Caterpillars

In this section, we shall prove Theorem 3. We shall actually prove something stronger, namely Lemmas 6 and 7.

A *caterpillar* is a tree T where each vertex has distance at most 1 to some central path $P \subseteq T$. In this paper, we shall consider a special subclass of caterpillars, namely those that have at most two vertices of degree greater than 2. Observe that any such caterpillar T can be obtained from a path P by identifying two of its vertices, v_1 and v_2, with the centres of stars. We shall write $T = C(a, b, c, d, e)$, if P has length $a + c + e$, and v_1 and v_2 are the $(a + 1)$th and $(a + c + 1)$th vertex on P, and have b, resp. d, neighbours outside P. Therefore, if $a, e > 0$, then $C(a, b, c, d, e)$ has $b + d + 2$ leaves.

We call P the *body*, and v_1 and v_2 the *joints* of the caterpillar. For illustration, see Figure 1.

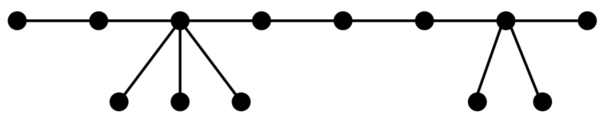

Figure 1: The caterpillar $C(2, 3, 4, 2, 1)$ or $C(2, 3, 4, 3, 0)$.

So $\mathcal{T}(k, \ell, c)$, as defined in the introduction, denotes the class of all caterpillars $C(a, b, c, d, e)$ with $b + d = \ell$, and $a + b + c + d + e = k$. We can thus state the result of Bazgan, Li, and Woźniak mentioned in the introduction as follows.

Theorem 5 (Bazgan, Li, Woźniak [3]). *Let $k, \ell, c \in \mathbb{N}$, and let $T = C(a, 0, c, d, e)$ be a tree from $\mathcal{T}(k, \ell, c)$. Let G be a graph so that at least half of the vertices of G have degree at least k. Then T is a subgraph of G.*

We shall now prove (separately) the two subcases of Theorem 3. Our first lemma deals with the case when c is even.

Lemma 6. *Let $k, \ell, c \in \mathbb{N}$ so that c is even and $\ell \geq c$. Let $T \in \mathcal{T}(k, \ell, c)$, and let G be a graph such that at least half of the vertices of G have degree at least k. Then T is a subgraph of G.*

Proof. Observe that we may assume that $\ell \geq 2$. Let v_1 and v_2 be the joints of T, and let P be its body. As above, denote by L the set of those vertices of G that have degree at least k and set $S := V(G) \setminus L$. We may assume that S is independent.
By Theorem 5, there is a path P_k of length k in G. Let φ be an embedding of $V(P)$ in $V(P_k)$ which maps the starting vertex of P to the starting vertex u of P_k. Now, if both $u_1 := \varphi(v_1)$ and $u_2 := \varphi(v_2)$ are in L, then we can easily extend φ to $V(T)$.
On the other hand, if both u_1 and u_2 lie in S, then we can 'shift' the image of $V(P)$ along P_k by 1 away from u, i. e., we map each vertex $v \in \varphi(V(P))$ to its neighbour on vP_k. Then, the composition of φ and the shift maps both v_1 and v_2 to L, and can thus be extended to an embedding of $V(T)$.
To conclude, assume that one of the two vertices u_1 and u_2 lies in L and the other lies in S. As c is even and S is independent, it follows that there are two consecutive vertices w_1 and w_2 (in this order) on $u_1 P_k u_2$ which lie in L. Similarly as above, shift the image of $V(P)$ away from u. In fact, we repeat this shift until u_1 is shifted to w_1. If the composition of φ with the iterated shift maps v_2 to L, we are done. Otherwise, we shift the image of $V(P)$ once more. Then both v_1 and v_2 are mapped to L, and we are done.
Observe that in total, we have shifted the image of $V(P)$ at most c times. We could do so, since $|P_k| - |P| = \ell \geq c$ by assumption. ☐

We now allow c to be odd, restricting the choice of k.

Lemma 7. *Let $k, \ell, c, n \in \mathbb{N}$ be such that $\ell \geq c$. Let $T = C(a, b, c, d, e)$ be a tree in $\mathcal{T}(k, \ell, c)$, and let G be a graph of order n such that at least $n/2$ vertices of G have degree at least k. Suppose that*

(i) $k \geq \lfloor n/2 \rfloor + 2 \min\{a, e\}$, *if* $\max\{a, e\} \leq k/2$, *and*

(ii) $k \geq \lfloor n/4 \rfloor + a + e + 1$, *if* $\max\{a, e\} > k/2$.

Then T is a subgraph of G.

Observe that in case (ii) of Lemma 7 it follows that

$$k > \lfloor n/4 \rfloor + \min\{a, e\} + \max\{a, e\} > \lfloor n/4 \rfloor + \min\{a, e\} + k/2,$$

and hence, as in (i),

$$k \geq \lfloor n/2 \rfloor + 2 \min\{a, e\}.$$

Proof of Lemma 7. As before, set $L := \{v \in V(G) : \deg(v) \geq k\}$ and set $S := V(G) \setminus L$. We may assume that S is independent, and that that $a, e \neq 0$. Because of Theorem 5, we may moreover assume that $b, d > 0$ (and thus $\ell \geq 2$), and by Lemma 6, that c is odd. Set æ $:= \min\{a, e\}$ and set Æ $:= \max\{a, e\}$.
Suppose that $T \not\subseteq G$. Using the same shifting arguments as in the proof of Lemma 6, we may assume that every path in G of length at least k zigzags between L and S, except possibly on its first a and its last e edges. In fact, as c is odd, we can even assume that every path in G of length at least $k-1$ zigzags between L and S, except possibly on its first a and its last e edges.
As paths are symmetric, we may actually assume that every path $Q = x_0 \dots x_m$ in G of any length $m \geq k-1$ zigzags on its subpaths $x_{\text{æ}} Q x_{m-\text{Æ}}$ and $x_{\text{Æ}} Q x_{m-\text{æ}}$. Observe that these subpaths overlap exactly if Æ $\leq m/2$. Our aim is now to find a path that does not zigzag on the specified subpaths, which will yield a contradiction.
So, let $\mathcal{Q}$ be the set of those subpaths of G that have length at least $k-1$ and end in L. Observe that by Theorem 5, and since S is independent, $\mathcal{Q} \neq \emptyset$. Among all paths in $\mathcal{Q}$, choose $Q = x_0 \dots x_m$ so that it has a maximal number of vertices in L.
This choice of Q guarantees that $N(x_m) \subseteq S \cup V(Q)$. Observe that the remark after the statement of Lemma 7 implies that in both cases (i) and (ii) ,

$$\begin{aligned} \deg(x_m) \geq k &> \lfloor n/2 \rfloor + 2\text{æ} \\ &\geq |S| + 2\text{æ}. \end{aligned}$$

Since æ > 0, we thus obtain that x_m has a neighbour $x_s \in L \cap V(Q)$ with

$$s \in [\text{æ}, m - \text{æ} - 1].$$

Moreover, in the case that Æ $> m/2$, condition (ii) of Lemma 7 implies that

$$\begin{aligned} \deg(x_m) \geq k &\geq 2(\lfloor n/4 \rfloor + a + e + 1) - k \\ &\geq \lfloor n/2 \rfloor - 1 + 2\text{æ} + 2\text{Æ} + 2 - (m+1) \\ &= |S| + 2\text{æ} + 2\text{Æ} - m. \end{aligned}$$

Hence, in this case we can guarantee that

$$s \in [\text{æ}, m - \text{Æ} - 1] \cup [\text{Æ}, m - \text{æ} - 1].$$

Now, consider the path Q^* we obtain from Q by joining the subpaths x_1Qx_s and $x_{s+1}Qx_m$ with the edge x_sx_m. Then Q^* is a path of length $m \geq k-1$ which contains the $L-L$ edge x_sx_m. Note that x_sx_m is neither one of the first æ nor of the last æ edges on Q^*. Furthermore, in the case that $\text{Æ} > m/2$, we know that x_sx_m is none of the middle $2\text{Æ} - m$ edges on Q^*. This contradicts our assumption that every path of length at least $k-1$ zigzags between L and S except possibly on these subpaths. □

It remains to prove Theorem 3.

Proof of Theorem 3. Assume we are given graphs G and $T \in \mathcal{T}(k, \ell, c)$ as in Theorem 3. If c is even, it follows from Lemma 6 that $T \subseteq G$. So assume that $\ell + c \geq \lfloor n/2 \rfloor$. We shall now use Lemma 7 to see that $T \subseteq G$. Suppose that $T = C(a, b, c, d, e)$. For $\max\{a, e\} \leq k/2$, it suffices to observe that

$$k - 2\min\{a, e\} \geq k - a - e = \ell + c \geq \lfloor n/2 \rfloor.$$

On the other hand, for $\max\{a, e\} > k/2$, observe that

$$k - a - e = \ell + c \geq \lfloor n/2 \rfloor \geq \lfloor n/4 \rfloor + 1,$$

since we may assume that $n \geq 4$, as otherwise Theorem 3 holds trivially. □

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