Short Cycle Covers of Cubic Graphs

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Abstract

The Shortest Cycle Cover Conjecture asserts that the edges of every bridgeless graph with m edges can be covered by cycles of total length at most $7m/5 = 1.4m$. We show that every cubic bridgeless graph has a cycle cover of total length at most $34m/21 \approx 1.619m$.

1 Introduction

Cycle covers of graphs form a prominent topic in graph theory closely related to several deep and open problems. A *cycle* in a graph is a subgraph with all degrees even. A *cycle cover* is a collection of cycles such that each edge is contained in at least one of the cycles (each edge is *covered*). The Cycle Double Cover Conjecture of Seymour [25] and Szekeres [26] asserts that every bridgeless graph G has a collection of cycles containing each edge of G exactly twice (*cycle double cover*). In fact, it was conjectured by Celmins [3] and Preissmann [23] that every graph has such a cycle cover consisting of five cycles.

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Cycle Double Cover Conjecture is known to be implied by several other conjectures, e.g., Berge-Fulkerson Conjecture [9] asserting that every cubic bridgeless graph G has 6 perfect matchings covering each edge of G twice. Another conjecture that implies the Cycle Double Cover Conjecture is the Shortest Cycle Cover Conjecture of Alon and Tarsi [1] which we study in this paper. The Shortest Cycle Cover Conjecture asserts that every bridgeless graph with m edges has a cycle cover of total length at most $7m/5$ (the *length of a cycle* is the number of edges contained in it and the *length of a cycle cover* is the sum of the lengths of its cycles). The reduction of the Cycle Double Cover Conjecture to the Shortest Cycle Cover Conjecture can be found in the paper Jamshy and Tarsi [14].

The best known general result on short cycle covers is due to Alon and Tarsi [1] and Bermond, Jackson and Jaeger [2]: every bridgeless graph with m edges has a cycle cover of total length at most $5m/3 \approx 1.667m$. As it is the case with most conjectures in this area, it seems that cubic bridgeless graphs form a class of graphs essential in any approach to prove the Shortest Cycle Cover Conjecture. There are several improvements of the general bound for cubic bridgeless graphs: Jackson [12] showed that every cubic bridgeless graph with m edges has a cycle cover of total length at most $64m/39 \approx 1.641m$ and Fan [5] later showed that every such graph has a cycle cover of total length at most $44m/27 \approx 1.630m$. There are several improvements of these bounds under additional assumptions on the girth, connectivity, etc. of the considered graph or assumptions on the existence of a nowhere-zero 4-/5-flow, see e.g. [6, 11, 15, 24]. The reader is referred to the monograph of Zhang [27] for further exposition of such results.

In this paper, we improve the known bound for cubic bridgeless graphs. In particular, we show that every cubic bridgeless graph with m edges has a shortest cycle cover of total length at most $34m/21 \approx 1.619m$. As it is the case in the previous results in [5, 12], the constructed cycle cover also consists of three cycles (note that there are bridgeless graphs with m edges with no cycle covers comprised of three cycles of length less than $22m/15 \approx 1.467m$ [7]). Let us mention that the result of Fan [5] is extended in another direction in [16] where it is shown that every bridgeless graph with minimum degree three and m edges has a cycle cover of total length at most $44m/27 \approx 1.630m$ (the bound matching Fan's bound, i.e., a bound worse than the one presented here, but it is established for a larger class of graphs).

The known improvements of the original bound of $5m/3 \approx 1.667m$ on the total length of a shortest cycle cover of a bridgeless graph with m edges may seem to be rather minor, however, obtaining a bound below $8m/5 = 1.600m$ could be quite tricky since this bound is implied by Tutte's 5-Flow Conjecture [15].

2 Notation

Let us briefly introduce the notation used throughout this paper. We only focus on those terms where the confusion could arise and refer the reader to standard graph theory textbooks, e.g. [4], for exposition of other graph theory notions.

Graphs considered in this paper can have loops and multiple edges. If E is a set of edges of a graph G, then $G \setminus E$ denotes the graph with the same vertex set and with the edges of E removed. For an edge e of G, G/e is the graph obtained by contracting the edge e, i.e., G/e is the graph with the end-vertices of e identified, the edge e removed and all the other edges preserved. In particular, the edges parallel to e become loops in G/e. Also note that if e is a loop, then $G/e = G \setminus e$. Finally, for a set E of edges of a graph G, G/E denotes the graph obtained by contracting all edges contained in E. If G is a graph and v a vertex of G of degree two, then the graph obtained from G by *suppressing* the vertex v is the graph obtained from G by contracting one of the edges incident with v, i.e., the graph obtained by replacing the two-edge path with the inner vertex v by a single edge.

An *edge-cut* in a graph G is a set E of edges such that the vertices of G can be partitioned into two sets A and B such that E contains precisely the edges with one end-vertex in A and the other in B. Such an edge-cut is also denoted by $E(A, B)$ and its size by $e(A, B)$. We abuse this notation a little bit and also use $e(A, B)$ for the number of edges between any two disjoint sets A and B which do not necessarily form a partition of the vertex set of G. An edge forming an edge-cut of size one is called a *bridge* and graphs with no edge-cuts of size one are said to be *bridgeless*. Note that we do not require edge-cuts to be minimal sets E such that $G \setminus E$ has more components than G. A graph G with no edge-cuts of odd size less than k is said to be *k-odd-connected*. For every set F of edges of G, cuts in G/F correspond to cuts (of the same size) in G. Therefore, if G has no edge-cuts of size k, then also G/F has no edge-cuts of size k.

As said before, a *cycle* of a graph G is a subgraph of G with all vertices of even degree. A *circuit* is a connected subgraph with all vertices of degree

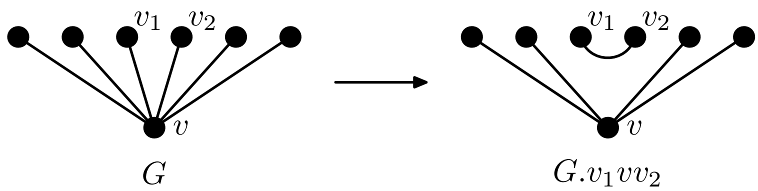

Figure 1: Splitting the vertices v_1 and v_2 from the vertex v.

two and a 2-*factor* is a spanning subgraph with all vertices of degree two.

3 Vertex splitting

In the proof of our main result, we will need to construct a nowhere-zero 4-flow of a special type. In order to exclude some "bad" nowhere-zero flows, we will first modify a considered graph in such a way that some of its edges must get the same flow value. This goal will be achieved by splitting some of the vertices of the considered graph. Let G be a graph, v a vertex of G and v_1 and v_2 some of the neighbors of v in G. Let $G.v_1vv_2$ be the graph obtained by removing the edges vv_1 and vv_2 from G and adding the edge v_1v_2 (see Figure 1). The graph $G.v_1vv_2$ is said to be obtained by *splitting the vertices v_1 and v_2 from the vertex v*.

Classical (and deep) results of Fleischner [8], Mader [21] and Lovász [19] assert that it is possible to split vertices without creating new small edge-cuts. Let us now formulate one of the corollaries of their results.

Lemma 1. *Let G be a 5-odd-connected graph. For every vertex v of G of degree four, six or more, there exist two neighbors v_1 and v_2 of the vertex v such that the graph $G.v_1vv_2$ is also 5-odd-connected.*

Zhang [28] proved a version of Lemma 1 where only some pairs of vertices are allowed to be split off.

Lemma 2. *Let G be an ℓ-odd-connected graph for an odd integer ℓ. For every vertex v of G with neighbors $v_1, \ldots, v_k$, $k \neq 2, \ell$, there exist two neighbors v_i and v_{i+1} such that the graph $G.v_ivv_{i+1}$ is also ℓ-odd-connected (indices are modulo k).*

However, any of these results is not sufficient for our purposes since we need to specify more precisely which pair of the neighbors of v should be

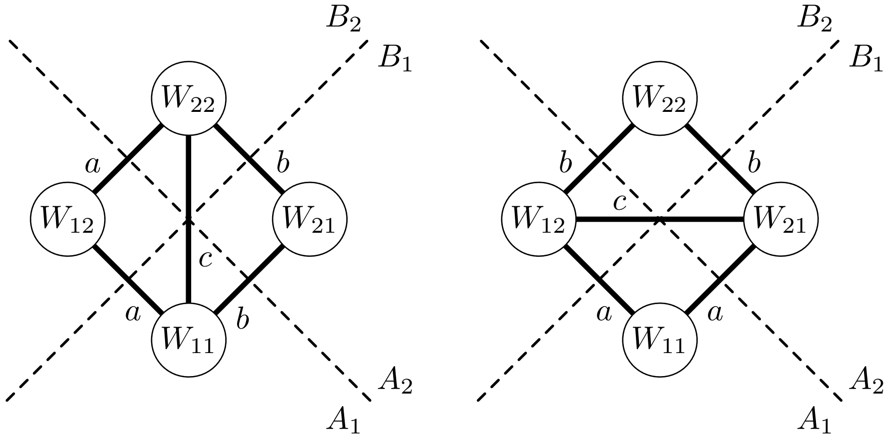

Figure 2: The two configurations described in the statement of Lemma 3.

split from v. This is guaranteed by the lemmas we establish in the rest of this section. Let us remark that Lemma 2 can be obtained as a consequence of our results. We start with a modification of the well-known fact that "minimal odd cuts do not cross" for the situation where small even cuts can exist.

Lemma 3. *Let G be an ℓ-odd-connected graph (for some odd integer ℓ) and let $E(A_1, A_2)$ and $E(B_1, B_2)$ be two cuts of G of size ℓ. Further, let $W_{ij} = A_i \cap B_j$ for $i, j \in \{1, 2\}$. If the sets W_{ij} are non-empty for all $i, j \in \{1, 2\}$, then there exist integers a, b and c such that $a + b + c = \ell$ and one of the following holds:*

- $e(W_{11}, W_{12}) = e(W_{12}, W_{22}) = a$, $e(W_{11}, W_{21}) = e(W_{21}, W_{22}) = b$, $e(W_{11}, W_{22}) = c$ *and* $e(W_{12}, W_{21}) = 0$, *or*
- $e(W_{12}, W_{11}) = e(W_{11}, W_{21}) = a$, $e(W_{12}, W_{22}) = e(W_{22}, W_{21}) = b$, $e(W_{12}, W_{21}) = c$ *and* $e(W_{11}, W_{22}) = 0$.

See Figure 2 for an illustration of the two possibilities.

Proof. Let w_{ij} be the number of edges with exactly one end-vertex in W_{ij}. Observe that

$$w_{11} + w_{12} = e(A_1, A_2) + 2e(W_{11}, W_{12}) = \ell + 2e(W_{11}, W_{12}) \quad \text{and} \qquad (1)$$

$$w_{12} + w_{22} = e(B_1, B_2) + 2e(W_{12}, W_{22}) = \ell + 2e(W_{12}, W_{22}) \, .$$

In particular, one of the numbers w_{11} and w_{12} is even and the other is odd. Assume that w_{11} is odd. Hence, w_{22} is also odd. Since G is ℓ-odd-connected, both w_{11} and w_{22} are at least ℓ.

If $e(W_{11}, W_{12}) \le e(W_{12}, W_{22})$, then

$$\begin{aligned} w_{12} &= e(W_{11}, W_{12}) + e(W_{12}, W_{21}) + e(W_{12}, W_{22}) \\ &\ge e(W_{11}, W_{12}) + e(W_{12}, W_{22}) \ge 2e(W_{11}, W_{12}) \; . \end{aligned} \tag{2}$$

The equation (1), the inequality (2) and the inequality $w_{11} \ge \ell$ imply that $w_{11} = \ell$. Hence, the inequality (2) is an equality; in particular, $e(W_{11}, W_{12}) = e(W_{12}, W_{22})$ and $e(W_{12}, W_{21}) = 0$. If $e(W_{11}, W_{12}) \ge e(W_{12}, W_{22})$, we obtain the same conclusion. Since the sizes of the cuts $E(A_1, A_2)$ and $E(B_1, B_2)$ are the same, it follows that $e(W_{11}, W_{21}) = e(W_{21}, W_{22})$. We conclude that the graph G and the cuts have the structure as described in the first part of the lemma.

The case that w_{11} is even (and thus w_{12} is odd) leads to the other configuration described in the statement of the lemma. □

Next, we use Lemma 3 to characterize graphs where some splittings of neighbors of a given vertex decrease the odd-connectivity. In the statement of Lemma 4, the graph G is assumed to be simple just to avoid unnecessary technical complications in its proof; the lemma also holds for graphs with loops and parallel edges (with a suitable definition of vertex splitting).

Lemma 4. *Let G be a simple ℓ-odd-connected graph for an odd integer $\ell \ge 3$, v a vertex of G and $v_1, \ldots, v_k$ some neighbors of v. If every graph $G.v_ivv_{i+1}$, $i = 1, \ldots, k-1$, contains an edge-cut of odd size smaller than ℓ, the vertex set $V(G)$ can be partitioned into two sets V_1 and V_2 such that $v \in V_1$, $v_i \in V_2$ for $i = 1, \ldots, k$ and the size of the edge-cut $E(V_1, V_2)$ is ℓ.*

Proof. The proof proceeds by induction on k. The base case of the induction is that $k = 2$. Let $E(V_1, V_2)$ be an edge-cut of $G.v_1vv_2$ of odd size less than ℓ. By symmetry, we can assume that $v \in V_1$. If both $v_1 \in V_1$ and $v_2 \in V_1$, then $E(V_1, V_2)$ as an edge-cut of G has the same size as in $G.v_1vv_2$ which contradicts the assumption that G is ℓ-odd-connected. If $v_1 \in V_1$ and $v_2 \in V_2$, then $E(V_1, V_2)$ is also an edge-cut of G of the same size as in $G.v_1vv_2$ which is again impossible.

Hence, both v_1 and v_2 must be contained in V_2, and the size of the edge-cut $E(V_1, V_2)$ in G is larger by two compared to its size in $G.v_1vv_2$. Since G

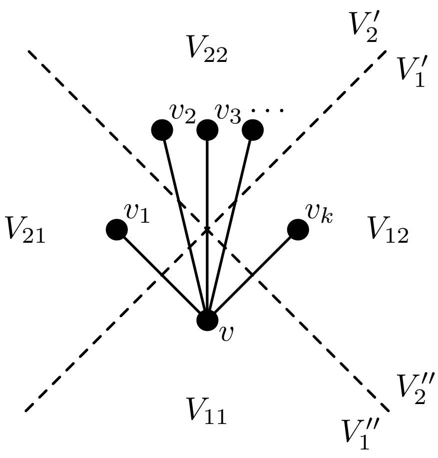

Figure 3: Notation used in the proof of Lemma 4.

has no edge-cuts of size $\ell - 2$, the size of the edge-cut $E(V_1, V_2)$ in $G.v_1vv_2$ is $\ell - 2$ and its size in G is ℓ. Hence, V_1 and V_2 form the partition of the vertices as in the statement of the lemma.

We now consider the case that $k > 2$. By the induction assumption, G contains a cut $E(V_1', V_2')$ of size ℓ such that $v \in V_1'$ and $v_1, \ldots, v_{k-1} \in V_2'$. Similarly, there is a cut $E(V_1'', V_2'')$ of size ℓ such that $v \in V_1''$ and $v_2, \ldots, v_k \in V_2''$. Let $V_{ij} = V_i' \cap V_j''$ for $i, j \in \{1, 2\}$ (see Figure 3). Apply Lemma 3 for the graph G with $A_i = V_i'$ and $B_i = V_i''$. Since $e(V_{11}, V_{22}) = e(V_1' \cap V_1'', V_2' \cap V_2'') \geq k - 2 > 0$, the first case described in Lemma 3 applies and the size of the cut $E(V_1' \cap V_1'', V_2' \cup V_2'')$ is ℓ. Hence, the cut $E(V_1' \cap V_1'', V_2' \cup V_2'')$ is a cut of size ℓ $\square$

We finish this section with a series of lemmas that we need in Section 4. All these lemmas are simple corollaries of Lemma 4.

Lemma 5. *Let G be a simple 5-odd-connected graph, and let v be a vertex of degree four and v_1, v_2, v_3 and v_4 its four neighbors. Then, the graph $G.v_1vv_2$ or the graph $G.v_2vv_3$ is also 5-odd-connected graph.*

Proof. Observe that the graph $G.v_1vv_2$ is 5-odd-connected if and only if the graph $G.v_3vv_4$ is 5-odd-connected. Lemma 4 applied for $\ell = 5$, the vertex v, $k = 4$ and the vertices v_1, v_2, v_3 and v_4 yields that the vertices of G can be partitioned into two sets V_1 and V_2 such that $v \in V_1$, $\{v_1, v_2, v_3, v_4\} \subseteq V_2$ and $e(V_1, V_2) = 5$. Hence,

$$e(V_1 \setminus \{v\}, V_2 \cup \{v\}) = e(V_1, V_2) - 4 = 5 - 4 = 1 .$$

This contradicts our assumption that G has no edge-cuts of size one. ☐

Lemma 6. *Let G be a simple 5-odd-connected graph, and let v be a vertex of degree six and $v_1, \dots, v_6$ its neighbors. At least one of the graphs $G.v_1vv_2$, $G.v_2vv_3$ and $G.v_3vv_4$ is also 5-odd-connected.*

Proof. Lemma 4 applied for $\ell = 5$, the vertex v, $k = 4$ and v_1, v_2, v_3 and v_4 yields that the vertices of G can be partitioned into two sets V_1 and V_2 such that $v \in V_1$, $\{v_1, v_2, v_3, v_4\} \subseteq V_2$ and $e(V_1, V_2) = 5$. If V_2 contains σ neighbors of v (note that $\sigma \geq 4$), then

$$e(V_1 \setminus \{v\}, V_2 \cup \{v\}) = e(V_1, V_2) - \sigma + (6 - \sigma) = 11 - 2\sigma$$

which is equal to 1 or 3 contradicting the fact that G is 5-odd-connected. ☐

Lemma 7. *Let G be a simple 5-odd-connected graph, and let v be a vertex of degree eight and $v_1, \dots, v_8$ its neighbors. At least one of the graphs $G.v_ivv_{i+1}$, $i = 1, \dots, 7$, is also 5-odd-connected.*

Proof. Since there is no partition of the vertices of G into two parts V_1 and V_2 such that $v \in V_1$, $v_i \in V_2$ for $i = 1, \dots, 8$, and $e(V_1, V_2) = 5$, Lemma 4 applied for $\ell = 5$, the vertex v, $k = 8$ and the vertices v_i, $i = 1, \dots, 8$ yields the statement of the lemma. ☐

Finally, we will also need in Section 4 the following corollary of Lemma 4.

Lemma 8. *Let G be a simple 5-odd-connected graph, and let v be a vertex of degree eight and $v_1, \dots, v_8$ its neighbors. Suppose that $G.v_1vv_2$ is 5-odd-connected. At least one of the following graphs is also 5-odd-connected: $G.v_1vv_2.v_3vv_4$, $G.v_1vv_2.v_7vv_8$, $G.v_1vv_2.v_8vv_3.v_4vv_5$ and $G.v_1vv_2.v_8vv_3.v_4vv_6$.*

Proof. The degree of the vertex v in $G.v_1vv_2$ is six. By Lemma 6, at least one of the graphs $G.v_1vv_2.v_3vv_4$, $G.v_1vv_2.v_3vv_8$ and $G.v_1vv_2.v_7vv_8$ is 5-odd-connected.

If $G.v_1vv_2.v_3vv_8$ is 5-odd-connected, we apply Lemma 5 for the vertex v and its neighbors v_5, v_4, v_6 and v_7 (in this order). Hence, the graph $G.v_1vv_2.v_8vv_3.v_4vv_5$ (which is homeomorphic to $G.v_1vv_2.v_8vv_3.v_6vv_7$) or the graph $G.v_1vv_2.v_8vv_3.v_4vv_6$ is 5-odd-connected. ☐

4 Rainbow lemma

In this section, we present a generalization of an auxiliary lemma referred to as the Rainbow Lemma.

Lemma 9 (Rainbow Lemma). *Every cubic bridgeless graph G contains a 2-factor F such that the edges of G not contained in F can be colored with three colors, red, green and blue in the following way:*

- *every even circuit of F contains an even number of vertices incident with red edges, an even number of vertices incident with green edges and an even number number of vertices incident with blue edges, and*
- *every odd circuit of F contains an odd number of vertices incident with red edges, an odd number of vertices incident with green edges and an odd number number of vertices incident with blue edges.*

In the rest, a 2-factor F with an edge-coloring satisfying the constraints given in Lemma 9 will be called a *rainbow 2-factor*. Rainbow 2-factors implicitly appear in, e.g., [5, 18, 20], and are related to the notion of parity 3-edge-colorings from the Ph.D. thesis of Goddyn [10].

A key ingredient in the proof of Lemma 9 is the following classical result of Jaeger:

Theorem 10 (Jaeger [13]). *If G is a 5-odd-connected graph, then G has a nowhere-zero 4-flow.*

A classical result of Petersen [22] asserts that every cubic bridgeless graph has a perfect matching. The following strengthening of this result, which appears, e.g., in [17, 27], is another ingredient for the proof of Rainbow Lemma.

Theorem 11. *Every cubic bridgeless graph G has a 2-factor such that the graph G/F is 5-odd-connected.*

We are now ready to provide a modification of the Rainbow Lemma that is used in the proof of our main theorem. In addition to the constraints on the edge-coloring of a perfect matching given in Lemma 9, we exclude certain color patterns from appearing on the edges incident with cycles of specific lengths. Let us be more precise. The *pattern* of a circuit $C = v_1 \dots v_k$ of a rainbow 2-factor F is $X_1 \dots X_k$ where X_i is the color of the edge incident with the vertex v_i; we use R to represent the red color, G the green color and B the blue color. Two patterns are said to be *symmetric* if one of them can be obtained from the other by a rotation, a reflection and/or a permutation of the red, green and blue colors. For example, the patterns RRGBGB and RBRBGG are symmetric but the patterns RRGBBG and RRGBGB are not.

Let us now state and prove a generalization of the Rainbow Lemma.

Lemma 12. *Every cubic bridgeless graph G contains a rainbow 2-factor F such that*

- *no circuit of the 2-factor F has length three,*
- *every circuit of length four has a pattern symmetric to RRRR or RRGG, and*
- *every circuit of length eight has a pattern symmetric to one of the following 16 patterns:*
  RRRRRRRR, RRRRRRGG, RRRRGGGG, RRRRGGBB, RRGGRRGG, RRGGRRBB, RRRRGRRG, RRRRGBBG, RRGGRGGR, RRGGRBBR, RRGGBRRB, RRRRGRGR, RRRGBGBR, RRGRGRGG, RRGRBRBG and RRGGBGBG.

Proof. By Theorem 11, there exists a 2-factor F such that the graph G/F is 5-odd-connected. Since G is cubic and G/F 5-odd-connected, the 2-factor F contains no circuits of length three. Let H_0 be the graph obtained from the graph G/F by subdividing each edge three times. Clearly, H_0 is also 5-odd-connected. This modification of G/F to H_0 is needed only to simplify our arguments later in the proof since it guarantees that the graph is simple and thus we can easily apply Lemmas 5–7 and 8.

In a series of steps, we iteratively modify the graph H_0 to graphs H_1, H_2, etc. During this process, the degree of each vertex of H_i is the same as in H_0 though some vertices may be removed (and thus not present in H_i). All the graphs H_1, H_2, ... will be simple and 5-odd-connected.

If the graph H_i contains a vertex v of degree four, then the vertex v corresponds to a circuit C of length four in F. Let v_1, v_2, v_3 and v_4 be the neighbors of v in the order in which the edges vv_1, vv_2, vv_3 and vv_4 correspond to edges incident with the circuit C. To obtain H_{i+1}, we consider among the graphs $H_i.v_1vv_2$ and $H_i.v_2vv_3$ one that is 5-odd-connected (by Lemma 5 at least one of them is). The graph H_{i+1} is then obtained by suppressing the

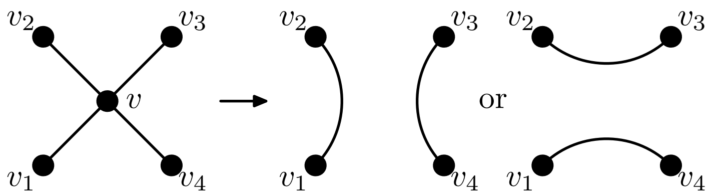

Figure 4: Reduction of a vertex of degree four in a graph H_i in the proof of Lemma 12.

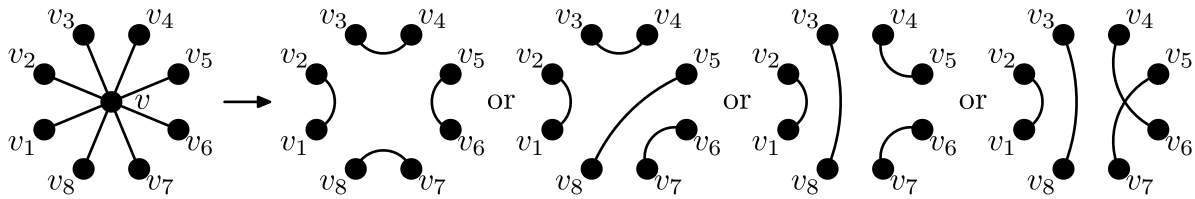

Figure 5: Reduction of a vertex of degree eight in a graph H_i in the proof of Lemma 12.

vertex v in this graph; see Figure 4. Clearly, H_{i+1} is 5-odd-connected and all the vertices of H_{i+1} have the same degree as in H_i.

If the graph H_i contains a vertex v of degree eight, we proceed as follows. The vertex v corresponds to a circuit C of F of length eight; let $v_1, \ldots, v_8$ be the neighbors of v in H_i in the order in that they correspond to the edges incident with the circuit C. By Lemma 7, we can assume that the graph $H_i.v_1vv_2$ is 5-odd-connected (for a suitable choice of the cyclic rotation of the neighbors of v).

By Lemma 8, at least one of the following graphs is 5-odd-connected: $H_i.v_1vv_2.v_3vv_4$, $H_i.v_1vv_2.v_7vv_8$, $H_i.v_1vv_2.v_8vv_3.v_4vv_5$ and $H_i.v_1vv_2.v_8vv_3.v_4vv_6$. If the graph $H_i.v_1vv_2.v_3vv_4$ is 5-odd-connected, we then apply Lemma 5 to the graph $H_i.v_1vv_2.v_3vv_4$ and conclude that the graph $H_i.v_1vv_2.v_3vv_4.v_5vv_6$ or the graph $H_i.v_1vv_2.v_3vv_4.v_6vv_7$ is 5-odd-connected. Since the case that the graph $H_i.v_1vv_2.v_7vv_8$ is 5-odd-connected is symmetric to the case of the graph $H_i.v_1vv_2.v_3vv_4$, it can be assumed that (at least) one of the following four graphs is 5-odd-connected: $H_i.v_1vv_2.v_3vv_4.v_5vv_6$, $H_i.v_1vv_2.v_3vv_4.v_6vv_7$, $H_i.v_1vv_2.v_8vv_3.v_4vv_5$ and $H_i.v_1vv_2.v_8vv_3.v_4vv_6$. Let H_{i+1} be the graph obtained by suppressing the vertex v in a 5-odd-connected graph in this list (see Figure 5).

We eventually reach a 5-odd-connected graph H_k with no vertices of degree four or eight. By Theorem 10, the graph H_k has a nowhere-zero 4-flow and, equivalenty, it has a nowhere-zero $\mathbb{Z}_2^2$-flow. This flow yields a nowhere-zero $\mathbb{Z}_2^2$-flow in $H_{k-1}, \ldots, H_0$ and eventually in G/F: in each step, we subdivide some edges (and give them the flow-value of the original edge), and identify some vertices. In particular, the pairs of edges split away from a vertex are assigned the same flow value.

The nowhere-zero $\mathbb{Z}_2^2$-flow of G/F gives the coloring of the edges: the edges with the flow value $(0, 1)$ are colored red, those with the flow value $(1, 0)$ green and those with the value $(1, 1)$ blue. Since the colors of the edges of G/F correspond to a nowhere-zero 4-flow of G/F, F is a rainbow 2-factor with respect to this edge-coloring.

We now verify that the edge-coloring of G/F also satisfies the additional two constraints given in the statement. Let us start with the first constraint, and let C be a circuit of F of length four, v the vertex of G/F corresponding to C and e_1, e_2, e_3 and e_4 the four (not necesserily distinct) edges leaving C in G. In H_0, the edge e_i corresponds to an edge vv_i for a neighbor v_i of v. During the construction of H_k, either the vertices v_1 and v_2 or the vertices v_2 and v_3 are split away from v. In the former case, the colors of the edges e_1 and e_2 are the same and the colors of the edges e_3 and e_4 are the same; in the latter case, the colors of the edges e_1 and e_4 and the colors of the edges e_2 and e_3 are the same. In both cases, the pattern of C is symmetric to RRRR or RRGG.

Let C be a circuit of F of length eight, v the vertex of G/F corresponding to C and e_1, $\ldots$, e_8 the eight edges leaving C in G (note that some of the edges e_1, $\ldots$, e_8 can be the same). Let c_i be the color of the edge e_i. Based on the splitting, one of the following four cases (up to symmetry) applies:

1. $c_1 = c_2$, $c_3 = c_4$, $c_5 = c_6$ and $c_7 = c_8$,
2. $c_1 = c_2$, $c_3 = c_4$, $c_5 = c_8$ and $c_6 = c_7$,
3. $c_1 = c_2$, $c_3 = c_8$, $c_4 = c_5$ and $c_6 = c_7$, and
4. $c_1 = c_2$, $c_3 = c_8$, $c_4 = c_6$ and $c_5 = c_7$.

In the first case, the pattern of the circuit C is symmetric to RRRRRRRR, RRRRRRGG, RRRRGGGG, RRRRGGBB, RRGGRRGG or RRGGRRBB. In the second case, the pattern of the circuit C is symmetric to RRRRRRRR, RRRRRRGG, RRRRGGGG, RRRRGGBB, RRRRGRRG, RRRRGBBG, RRGGRGGR, RRGGRBBR or RRGGBRRB. The third case is symmetric to the second one (see Figure 5). In the last case, the pattern of C

is symmetric to RRRRRRRR, RRRRRRGG, RRRRGRGR, RRRRGRRG, RRRRGGGG, RRGRGRGG, RRRGBGBR, RRGRBRBG, RRGGBGBG or RRRRGBBG. In all the four cases, the pattern of C is one of the patterns listed in statement of the lemma. □

5 Intermezzo

In order to help the reader to follow the arguments presented in the next section, we reprove a restricted version of the classical result of Alon and Tarsi [1] and Bermond, Jackson and Jaeger [2]: every cubic bridgeless graph with m edges has a cycle cover of length at most $5m/3$. We restrict our attention to cubic graphs only and use a technique similar to that used by Fan in [5]. In the next section, we refine the presented proof to improve the bound.

Let us now introduce additional notation used in the proof of Theorem 13. Let G be a cubic grapha and F a 2-factor of G. For a circuit C contained in F and for a set of edges of E such that $C \cap E = \emptyset$, we define $C(E)$ to be the set of vertices of C incident with the edges of E. Our goal in the proof will be to extend a certain set E of edges of G to a cycle by adding edges of F. This is impossible if $|C(E)|$ is odd for any circuit C of F. If $|C(E)|$ is even, we partition the edges of C into two sets $C(E)^A$ and $C(E)^B$ such that each of them induces paths with end-vertices being the vertices of $C(E)$. If $C(E) = \emptyset$, then $C(E)^A$ contains no edges of C and $C(E)^B$ contains all the edges of C (or vice versa). Observe that for every set E of edges not contained in F, adding one of the sets $C(E)^A$ and $C(E)^B$ for each circuit C of F yields a cycle of G. In the rest, we will always assume that the number of edges of $C(E)^A$ does not exceed the number of edges of $C(E)^B$.

Theorem 13. *Every cubic bridgeless graph G with m edges has a cycle cover of length at most $5m/3$.*

Proof. We first apply Lemma 9 to G and obtain a rainbow 2-factor F. Let $\mathcal{R}$, $\mathcal{G}$ and $\mathcal{B}$ be the sets of red, green and blue edges and r, g and b their numbers. By symmetry, we can assume that $r \leq g \leq b$. Also observe that $r + g + b = m/3$.

The desired cycle cover of G, which is comprised of three cycles, is defined as follows. The first cycle $\mathcal{C}_1$ contains all the red and green edges and the edges of $C(\mathcal{R} \cup \mathcal{G})^A$ for all circuits C of the 2-factor F. The second cycle

$\mathcal{C}_2$ contains all the red and green edges and the edges of $C(\mathcal{R} \cup \mathcal{G})^B$ for all circuits C of F. Finally, the third cycle $\mathcal{C}_3$ contains all the red and blue edges and the edges of $C(\mathcal{R} \cup \mathcal{B})^A$ for all circuits C of F.

It remains to verify that the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ cover all edges and to estimate their total length. Each edge of F is covered once by either the cycle $\mathcal{C}_1$ or $\mathcal{C}_2$; since $|C(E)^A| \leq |C(E)^B|$ for every circuit C of F, at most half of the edges of F is also covered by the cycle $\mathcal{C}_3$. Since each red edge is covered three times, each green edge twice and each blue edge once, the total length of the constructed cycle cover is at most:

$$3r + 2g + b + |F| + |F|/2 \leq 2(r + g + b) + 3|F|/2 = 2m/3 + m = 5m/3 .$$

The proof of the theorem is now finished. □

6 Main result

We are now ready to prove the main result of this paper.

Theorem 14. *Every cubic bridgeless graph G with m edges has a cycle cover comprised of three cycles of total length at most $34m/21$.*

Proof. We present two bounds on the length of a cycle cover of G and the bound claimed in the statement of the theorem is eventually obtained by combining the two presented bounds. In both bounds, the constructed cycle cover will consist of three cycles. Fix a rainbow 2-factor F and an edge-coloring of the edges not contained in F with the red, green and blue colors as described in Lemma 12. Let $\mathcal{R}$, $\mathcal{G}$ and $\mathcal{B}$ be the sets of the red, green and blue edges, respectively, and let r, g and b be their numbers. Finally, let d_ℓ be the number of circuits of lengths ℓ contained in F. By Lemma 12, $d_3 = 0$.

The first cycle cover. Before we proceed with constructing the first cycle cover, recall the notation of $C(E)^A$ and $C(E)^B$ introduced before Theorem 13. The cycle cover is comprised of three cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$. The cycle $\mathcal{C}_1$ contains all red and green edges, the cycle $\mathcal{C}_2$ contains all red and blue edges and the cycles $\mathcal{C}_3$ contains all green and blue edges. In addition, the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ contain some edges of the 2-factor F as described further.

Let C be a circuit of the 2-factor F. Lemma 12 allows us to assume that if the length of C is four, then the pattern of C is either BBBB or GGBB (otherwise, we can—for the purpose of extending the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ to C—switch the roles of the red, green and blue colors and the roles of the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ in the remaining analysis; note that we are not recoloring the edges, just apply the arguments presented in the next paragraphs with respect to a different permutation of colors). Similarly, we can assume that the pattern of the circuit C of length eight is one of the following 16 patterns:

BBBBBBBB, BBBBBBRR, BBBBRRRR, BBBBRRGG,
BBRRBBRR, BBRRBBGG, BBBBRBBR, BBBBRGGR,
BBRRBRRB, BBRRBGGB, RRGGBRRB, BBBBRBRB,
BBBRGRGB, RRBRBRBB, BBRBGBGR and BBRRGRGR.

Let us now choose edges of the circuit C that are included in the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$. The cycle $\mathcal{C}_1$ contains the edges of $C^1 = C(\mathcal{R}\cup\mathcal{G})^A$. The cycle $\mathcal{C}_2$ contains the edges C^2 of either $C(\mathcal{R}\cup\mathcal{B})^A$ or $C(\mathcal{R}\cup\mathcal{B})^B$—we choose the set with smaller intersection with $C(\mathcal{R}\cup\mathcal{G})^A$. Finally, the edges included to $\mathcal{C}_3$ are chosen so that every edge of C is covered odd number of times; explicitly, the edges $C^3 = C^1 \triangle C^2 \triangle C$ are included to $\mathcal{C}_3$. Note that C^3 is either $C(\mathcal{G}\cup\mathcal{B})^A$ or $C(\mathcal{G}\cup\mathcal{B})^B$. In particular, the sets $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ form cycles.

We now estimate the number of the edges of C contained in $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$. The sum of the numbers edges contained in each of the cycles is:

$$\begin{aligned} |C^1| &+ |C^2| + |C^1 \triangle C^2 \triangle C| \\ &= |C^1 \cup C^2| + |C^1 \cap C^2| + |C \setminus (C^1 \cup C^2)| + |C^1 \cap C^2| \\ &= |C| + 2|C^1 \cap C^2| \end{aligned}$$

Since $|C^1| = |C(\mathcal{R}\cup\mathcal{G})^A| \leq |C(\mathcal{R}\cup\mathcal{G})^B|$, the number of edges of C^1 is at most $\ell/2$. By the choice of C^2, $|C^1 \cap C^2| \leq |C^1|/2 \leq \ell/4$. Hence, the sets $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ contain at most $\ell + 2\lfloor \ell/4 \rfloor$ edges of the circuit C.

The estimate on the number of edges of C contained in the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ can further be improved if the length of the circuit C is four: if the pattern of C is BBBB, then $C^1 = C(\mathcal{R}\cup\mathcal{G})^A = \emptyset$ and thus $C^1 \cap C^2 = \emptyset$. If the pattern is GGBB, then $C^1 \cap C^2 = C(\mathcal{R}\cup\mathcal{G})^A \cap C(\mathcal{R}\cup\mathcal{B})^A = \emptyset$. In both the cases, the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ contain (at most) $|C| + 2|C^1 \cap C^2| = \ell = 4$ edges of C.

Similarly, the estimate on the number of edges of C contained in the cycles can be improved if the length of C is eight. As indicated in Figure 6, it holds that $|C^1| = |C(\mathcal{R}\cup\mathcal{G})^A| \leq 3$. Hence, $|C^1 \cap C^2| \leq |C^1|/2 \leq 3/2$.

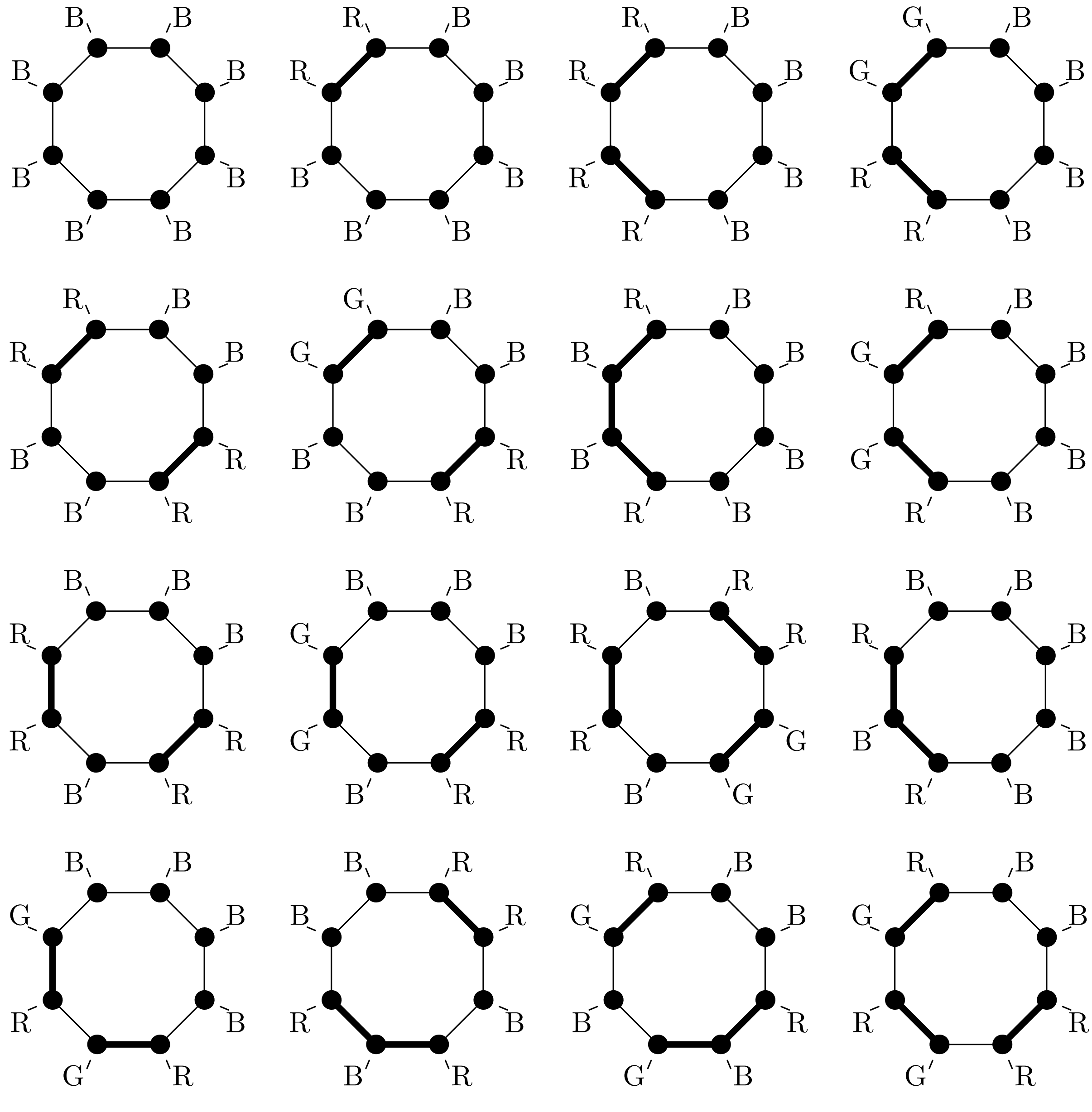

Figure 6: The sets $C^1 = C(\mathcal{R} \cup \mathcal{G})^A$ for circuits C with length eight; the edges contained in the set are drawn bold.

Consequently, $|C^1 \cap C^2| \leq 1$ and the number of edges of C included in the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ is at most $|C| + 2|C^1 \cap C^2| \leq 8 + 2 = 10$.

Based on the analysis above, we can conclude that the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ contain at most the following number of edges of the 2-factor F in total:

$$\begin{aligned} & 2d_2 + 4d_4 + 7d_5 + 8d_6 + 9d_7 + 10d_8 + 13d_9 + 14d_{10} + 15d_{11} + \sum_{\ell=12}^{\infty} \frac{3\ell}{2} d_\ell \\ = & \frac{3}{2} \sum_{\ell=2}^{\infty} \ell d_\ell - d_2 - 2d_4 - \frac{1}{2}d_5 - d_6 - \frac{3}{2}d_7 - 2d_8 - \frac{1}{2}d_9 - d_{10} - \frac{3}{2}d_{11} \ . \end{aligned} \quad (3)$$

Since the 2-factor F contains $2m/3$ edges, the estimate (3) translates to:

$$m - d_2 - 2d_4 - \frac{1}{2}d_5 - d_6 - \frac{3}{2}d_7 - 2d_8 - \frac{1}{2}d_9 - d_{10} - \frac{3}{2}d_{11} \ . \quad (4)$$

Since each red, green or blue edge is contained in exactly two of the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ and there are $m/3$ such edges, the total length of the cycle cover of G formed by $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ does not exceed:

$$\frac{5m}{3} - d_2 - 2d_4 - \frac{1}{2}d_5 - d_6 - \frac{3}{2}d_7 - 2d_8 - \frac{1}{2}d_9 - d_{10} - \frac{3}{2}d_{11} \ . \quad (5)$$

This finishes the construction and the analysis of the first cycle cover of G.

The second cycle cover. We keep the 2-factor F and the coloring of the edges of G by red, green and blue colors fixed. As long as the graph $H = G/F$ contains a red circuit, choose a red circuit of $H = G/F$ and recolor its edges with blue. Similarly, recolor edges of green circuits with blue. The modified edge-coloring still gives a rainbow 2-factor but the two additional constraints given in Lemma 12 need not be met anymore. Let $\mathcal{R}'$, $\mathcal{G}'$ and $\mathcal{B}'$ be the sets of red, green and blue in the modified edge-coloring and r', g' and b' their cardinalities.

The construction of the cycle cover now follows the lines of the proof of Theorem 13. The first cycle $\mathcal{C}_1$ is formed by the red and green edges and the edges of $C(\mathcal{R}' \cup \mathcal{G}')^A$ for every circuit C of the 2-factor F. The cycle $\mathcal{C}_2$ is also formed by the red and green edges and it contains the edges of $C(\mathcal{R}' \cup \mathcal{G}')^B$ for every circuit C of F. Finally, the cycle $\mathcal{C}_3$ is formed by the red and blue edges and the edges of $C(\mathcal{R}' \cup \mathcal{B}')^A$ for every circuit C of F. Clearly, the sets $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ are cycles of G and they cover all the edges of G.

Let us now estimate the lengths of the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$. Each red edge is contained in all the three cycles, each green edge in two cycles and each blue edge in one cycle. Each edge of a circuit C of length ℓ of the 2-factor F is contained either in $\mathcal{C}_1$ or in $\mathcal{C}_2$ and at most half of the edges of C is also contained in the cycle $\mathcal{C}_3$. Hence, the total length of the cycles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ is at most:

$$3r' + 2g' + b' + \sum_{\ell=2}^{\infty} \left\lfloor \frac{3\ell}{2} \right\rfloor d_\ell \,. \tag{6}$$

Since the red edges form an acyclic subgraph of G/F, the number of red edges is at most the number of the cycles of F, i.e., $d_2 + d_3 + d_4 + d_5 + \ldots$ Similarly, the number of green edges does not exceed the number of the cycles of F. Since $r' + g' + b' = m/3$, the expression (6) can be estimated from above by

$$\frac{m}{3}+2r'+g'+\sum_{\ell=2}^{\infty} \left\lfloor \frac{3\ell}{2} \right\rfloor d_\ell \le \frac{m}{3}+\sum_{\ell=2}^{\infty} 3d_\ell+\sum_{\ell=2}^{\infty} \left\lfloor \frac{3\ell}{2} \right\rfloor d_\ell = \frac{m}{3}+\sum_{\ell=2}^{\infty} \left\lfloor \frac{3\ell}{2}+3 \right\rfloor d_\ell \,. \tag{7}$$

Since the 2-factor F contains $2m/3 = 2d_2 + 3d_3 + 4d_4 + \ldots$ edges, the bound (7) on the number of edges contained in the constructed cycle cover can be rewritten to

$$\frac{m}{3} + \frac{7 \cdot 2 \cdot m}{4 \cdot 3} + \sum_{\ell=2}^{\infty} \left(\left\lfloor \frac{3\ell}{2} + 3 \right\rfloor - \frac{7\ell}{4} \right) d_\ell \le \frac{3m}{2} + \sum_{\ell=2}^{10} \left(\left\lfloor \frac{3\ell}{2} + 3 \right\rfloor - \frac{7\ell}{4} \right) d_\ell \,. \tag{8}$$

Note that the last inequality follows from the fact that $\left\lfloor \frac{3\ell}{2} + 3 \right\rfloor - \frac{7\ell}{4} \le \frac{3\ell}{2} - \frac{7\ell}{4} + 3 = 3 - \frac{\ell}{4} \le 0$ for $\ell \ge 12$ and the expression $\left\lfloor \frac{3\ell}{2} + 3 \right\rfloor - \frac{7\ell}{4} = -1/4$ is also non-positive for $\ell = 11$. The estimate (8) can be expanded to the following form (recall that $d_3 = 0$):

$$\frac{3m}{2} + \frac{5}{2}d_2 + 2d_4 + \frac{5}{4}d_5 + \frac{3}{2}d_6 + \frac{3}{4}d_7 + d_8 + \frac{1}{4}d_9 + \frac{1}{2}d_{10} \,. \tag{9}$$

The length of the shortest cycle cover of G with three cycles exceeds neither the bound given in (5) nor the bound given in (9). Hence, the length of such cycle cover of G is bounded by any convex combination of the two bounds, in particular, by the following:

$$\frac{5}{7} \cdot \left(\frac{5m}{3} - d_2 - 2d_4 - \frac{1}{2}d_5 - d_6 - \frac{3}{2}d_7 - 2d_8 - \frac{1}{2}d_9 - d_{10} - \frac{3}{2}d_{11} \right) +$$

$$\frac{2}{7} \cdot \left(\frac{3m}{2} + \frac{5}{2}d_2 + 2d_4 + \frac{5}{4}d_5 + \frac{3}{2}d_6 + \frac{3}{4}d_7 + d_8 + \frac{1}{4}d_9 + \frac{1}{2}d_{10} \right) =$$

$$\frac{34m}{21} - \frac{6}{7}d_4 - \frac{2}{7}d_6 - \frac{6}{7}d_7 - \frac{8}{7}d_8 - \frac{2}{7}d_9 - \frac{4}{7}d_{10} - \frac{15}{14}d_{11} \leq \frac{34m}{21} \ .$$

The proof of Theorem 14 is now completed. □

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