Finite dualities and map-critical graphs on a fixed surface

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Abstract

Let $\mathcal{K}$ be a class of graphs. Then, $\mathcal{K}$ is said to have a finite duality if there exists a pair $(\mathcal{F}, U)$, where $U \in \mathcal{K}$ and $\mathcal{F}$ is a finite set of graphs, such that for any graph G in $\mathcal{K}$ we have $G \leq U$ if and only if $F \not\leq G$ for all $F \in \mathcal{F}$ (" $\leq$ " is the homomorphism order). We prove that the class of planar graphs has no finite duality except for two trivial cases. We also prove that a 5-colorable toroidal graph U obtains a finite duality on a given fixed surface if and only if the core of U is K_5. In a sharp contrast, for a higher genus orientble surface S we show that Thomassen's result [15] implies that the class, $\mathcal{G}(S)$, of all graphs embeddable in S has a number of finite dualities. Equivalently, our first result shows that for every planar core graph H (except K_1 and K_4) there are infinitely many minimal planar obstructions for H-coloring, whereas our later result gives a converse of Thomassen's theorem [15] for 5-colorable graphs on the torus.

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1 Introduction

Finite dualities relate descriptive complexity of constrained satisfaction problems (or H-coloring problems) to properties of the induced homomorphism quasi-order. This appears as an interesting question in the context of various areas of mathematics, such as graph theory, logic and algorithms. Note that the existence of a non-trivial finite duality in a class of graphs $\mathcal{K}$ implies the existence of a polynomial time algorithm for the H-coloring problem in $\mathcal{K}$. We present this paper in light of this growing interest, by showing that certain minor closed classes of graphs have dualities while we prove that some others, such as the set of planar graphs have none.

To be precise, let $\mathcal{K}$ be a class of graphs. Then, $\mathcal{K}$ has a *finite duality* if there exists a pair $(\mathcal{F}, U)$, where $U \in \mathcal{K}$ and $\mathcal{F}$ is a finite set of graphs, such that for any graph G in $\mathcal{K}$ one and only one of the following holds: either G is below U (by the homomorphism order) or F is below G for some $F \in \mathcal{F}$ (A schematic description of the concept is given in Figure 1). We also say $(\mathcal{F}, U)$ is a *dual pair* of $\mathcal{K}$. A finite duality is *trivial* if $\mathcal{F} = \emptyset$ or $U = K_1$.

A widely studied notion in graph theory is the notion of a k-color-critical graph: A graph G is *k-color-critical* if G is not $(k-1)$-colorable but its every proper subgraph is. This is generalized by what we call 'map-critical' or more specific 'H-critical' graph: For any graph H, if a graph G is not homomorphic to H but its every proper subgraph is, then G is called *H-critical.* Thus, k-color-critical graphs are exactly the K_{k-1}-critical graphs, where K_n denotes the complete graph on n vertices. The problem of deciding if $G \leq H$ is known as the *H-coloring* problem.

An easy observation (Proposition 6) shows that the existence a dual-pair $(\mathcal{F}, U)$ in $\mathcal{K}$ is equivalent to non-existence of infinite U-critical graphs in $\mathcal{K}$. The interesting case is, however, when the duality is not trivial. For example, by the Four-color-theorem, we know $(\emptyset, K_4)$ is a dual-pair for planar graphs. Of course, we do not have a finite list of planar graphs in $\mathcal{F}$ listed as obstructions for K_4-coloring. In fact, in the class of toroidal graphs we shall see that K_4 attains no finite duality.

We start by reviewing the basic definitions. We assume graphs are finite, undirected and simple (with no loops and parallel edges). Let G, G' be graphs. A *homomorphism* from G to G' is a mapping $f{:}V(G) \to V(G')$ which preserves adjacency. That is, if $uv \in E(G)$ then $f(u)f(v) \in E(G')$.

We write $G \leq G'$ if there is a homomorphism from G to G'. The notation $G < G'$ means $G \leq G' \not\leq G$, whereas $G \sim G'$ means $G \leq G' \leq G$. The notation $G \parallel G'$ means $G \not\leq G' \not\leq G$ and we say G and G' are *incomparable.* If $S = G_1, G_2, \ldots$ is a sequence of pairwise incomparable graphs, we say S is an *anti-chain.* If $G \sim G'$, we say G and G' are *hom-equivalent.* The smallest graph H for which $G \sim H$ is called the *core* of G. For finite graphs, the core is uniquely determined up to an isomorphism. It can also be seen that H is an induced subgraph of G. See [4] for introduction to graphs and their homomorphisms.

We say a graph G is a *minor* of G', written $G \preceq G'$, if G can be obtained from G' by deleting and contracting edges of G'. The class of graphs that is closed under the minor relation is called *a minor closed class.* A celebrated theorem of Robertson and Seymour [14] states that graphs are well-quasi-ordered (wqo) under the minor relation $\preceq$. However, this is not true for the homomorphism relation $\leq$: it has been shown [4] that even simple classes such as the class of all directed paths, or for the undirected case, a proper subclass of K_4-minor-free graphs [5],[13] (series-parallel graphs) can be used to represent any countable partial order. It is interesting to consider problems relating the minor and homomorphism relations on graphs. One such question is finite duality which we defined above.

If $(\mathcal{F}, U)$ is a dual-pair in $\mathcal{K}$, we observe that we do not require the elements of $\mathcal{F}$ to be in $\mathcal{K}$, in our definition of duality. For each $F \in \mathcal{F}$ the number of distinct homomorphic images of F in $\mathcal{K}$ is finite, since F is a finite graph. Hence, we could replace F by a finite number of its images from $\mathcal{K}$. On the other hand, if F maps to no graph in $\mathcal{K}$ then we might as well delete F from $\mathcal{F}$. Therefore, without loss of generality, we may assume that $\mathcal{F} \subseteq \mathcal{K}$.

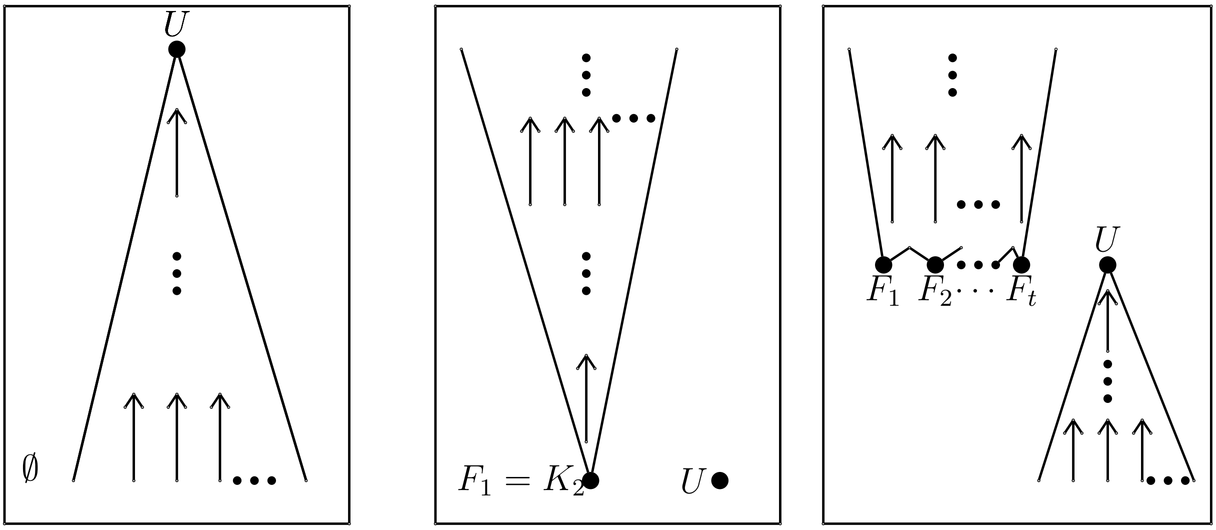
It should be explicitly mentioned that our concept of dualities for a class $\mathcal{K}$ is different from the concept of restricted dualities as defined in [10], [11]: the dual graph U is supposed to be in the class $\mathcal{K}$ while for the restricted dualities we accept arbitrary U. It has been proved in [10] that the class of all planar graphs has all restricted dualities (for any finite set $\mathcal{F}$ of connected graphs).

A natural question is which classes of undirected graphs have duality. Of course some dualities exist for every class of graphs. For example, one can clearly see that $(\{K_2\}, K_1)$ is a dual-pair to any class of graphs. Another similar example is if the class $\mathcal{K}$ contains a maximum U, then $(\emptyset, U)$ is a dual pair. Recall that these two types of dualities are called trivial (See Figure 1). Henceforth, we say $\mathcal{K}$ has *a finite duality property* (or $\mathcal{K}$ *has a duality*, for short) only if $\mathcal{K}$ has a non-trivial finite duality. In this paper we address the question whether a minor closed class of graphs has a duality.

Remark. It is interesting to note that it may well happen that every minor closed class contains a maximum graph (in the homomorphism order). In fact this question is equivalent to the Hadwiger Conjecture as shown independently in [6] and [9], see also [3].

If a class $\mathcal{K}$ is linearly ordered by homomorphism then clearly it has

Figure 1: Schematic diagram of trivial (left and middle) versus non-trivial (right) finite dualities for a class $\mathcal{K}$. The arrows show the homomorphism hierarchy. Trivial dualities of a class $\mathcal{K}$ partition $\mathcal{K}$ trivially.



dualities. Of course then if the elements of $\mathcal{K}$ can be listed as $\{G_1 < G_2 < \dots\}$ or as $\{G_1 > G_2 > \dots\}$, then trivially $(\{G_{i+1}\}, G_i)$ or $(\{G_i\}, G_{i+1})$ respectively, obtains a dual pair. Note, however, that in this case one of the two partitioning of the class is finite. The set of cycles is an obvious example. A bit more non-trivial example, but which turns out to be totally ordered, can be found in [12]. In [7], Nešetřil asked if minor closed classes with non-trivial dualities can be characterized. Our paper is motivated by this problem. The following are the main results of this paper:

Theorem 1. *The class of planar graphs has no non-trivial dualities on any surface. In other words, for every planar graph H (except K_1 and K_4) there exist infinitely many minimal planar obstructions for H-coloring.*

Theorem 2. *Let $\mathcal{G}(S)$ be the class of all graphs embeddable in an orientable surface S of positive genus such that the clique K_N embeds in S, for some positive integer $N \geq 5$. Then, there exists is a dual pair $(\mathcal{F}, K_k)$ in $\mathcal{G}(S)$ for each k, $5 \leq k \leq N$.*

Theorem 3. *Let U be a 5-colorable toroidal graph. Then $(\mathcal{F}, U)$ is a dual-pair in $\mathcal{G}(S)$ for a fixed orientable surface S if and only if the core of U is K_5. In other words, for every 5-colorable toroidal graph U there exist infinitely many minimal obstructions for U-coloring on a fixed surface if and only if U is K_5-free.*

The next proposition shows that Theorem 3 can not be extended for k-colorable graphs when $k \geq 7$, leaving the case $k = 6$ as the only open case.

Proposition 4. *For each integer $k \geq 7$, there exists a k-colorable core graph U which is not a clique and $(\mathcal{F}, U)$ is a dual-pair in $\mathcal{G}(S)$ for some fixed surface S.*

The dualities may be abundant for other types of minor closed classes. For example we have:

Theorem 5. *For every integer $k, k \geq 4$, the class $\mathcal{G}/K_k$ of all graphs with no K_k minor, contains a proper minor closed subclass $\mathcal{K}_{k-1}$ containing $(k-1)$-chromatic graphs and that $\mathcal{K}_{k-1}$ has infinitely many finite dualities.*

In the next three sections we prove Theorem 1 through Theorem 5. In proving Theorem 2, 3 and 5, we use a result of Thomassen [15], and also of our earlier papers [12], [13]. In the last section we offer some open problems.

2 No dualities for planar graphs

In this section we prove that planar graphs have no dualities. The reader will notice below that we have propositions that apply not only to planar but to all graphs. These will be very useful in the next section where non-planar graphs are studied. Recall that the concept of k-color-critical graphs is generalized by H-critical graphs for any graph H. We use the following observation throughout this paper:

Proposition 6. *Let $\mathcal{K}$ be a class of graphs and let $H \in \mathcal{K}$. Then $\mathcal{K}$ has a dual-pair $(\mathcal{F}, H)$ for some finite set $\mathcal{F}$ if and only if the number of H-critical graphs in $\mathcal{K}$ is finite.*

Proof. Suppose H has only finitely many critical graphs $\{F_1, F_2, \ldots, F_t\} = \mathcal{F}$. Note that F_i is a subgraph of G implies $F_i \leq G$ for any i. Then obviously $(\mathcal{F}, H)$ is a dual-pair in $\mathcal{K}$.

Conversely, assume that $\mathcal{K}$ has a dual-pair $(\mathcal{F}, H)$, for some finite set $\mathcal{F}$. By definition, for every $F \in \mathcal{F}$ we have $F \not\leq H$. Suppose (for a contradiction) that $X = \{H_i\}_{i=1}^{\infty}$ is an infinite sequence of distinct H-critical graphs in $\mathcal{K}$. As $H_i \not\leq H$, for all i, we deduce that there exist $f_i : F \to H_i$ for some $F \in \mathcal{F}$ and for infinitely many H_i. For some large i, we know that $f_i(F)$ is a proper subgraph of H_i, as $|f_i(F)|$ is bounded by a constant $|F|$ for all i whereas $\{|H_i|\}_{i=1}^{\infty}$ is an unbounded sequence. But then H_i is H-critical, and so we have $F \leq f_i(F) \leq H$, a contradiction. $\square$

We use the notation W_n to denote the *wheel* on $n+1$ vertices, where a *hub* is a vertex v connected to all other n vertices that induce an n-cycle. We call the n vertices the *rims* of v. The edge between two consecutive rims is called a *rim-edge*. An edge connecting a hub to its rim is called *a spoke*. We also use a term *generalized-wheel* and denote it by $W_{n,h}$ by allowing the hub to be a clique $K_h, 1 \leq h$ such that each vertex of the hub is connected to all n rims.) Applying Proposition 6 obtains some partial results immediately:

Corollary 7. *Let $(\mathcal{F}, U)$ be a non-trivial dual-pair of planar graphs. Then U is not a clique.*

Proof. By non-triviality, either $k = 2$ or $k = 3$. The odd cycles $\{C_{2i+1}\}_{i=1}^{\infty}$ form a K_2-critical sequence and the odd wheels $\{W_{2i+1}\}_{i=1}^{\infty}$ form a K_3-critical sequence of planar graphs and hence refute these possibilities, respectively. □

We use a notation W_{2i+1}^{j} for a graph called a *subdivided-wheel*, where W_{2i+1}^{j} is obtained from W_{2i+1} by subdividing each spoke of W_{2i+1} into a path P_j of length $j, j \geq 1$.

Corollary 8. *Let $(\mathcal{F}, U)$ be a non-trivial dual-pair in $\mathcal{G}(S)$ for any orientable surface S. Then (i) If $U = K_k$ then $k \geq 5$, (ii) U is not a cycle, (iii) U is not a wheel.*

Proof. (i) Suppose U is a clique K_k and that $k < 5$. By Corollary 7 and non-triviality, we only need to consider the case $k = 4$ and a surface S with a positive genus. Then $\{W_{2i+1,2}\}_{i=1}^{\infty}$ is a K_4-critical sequence in $\mathcal{G}(S)$. (Note that $W_{2i+1,2}$ embeds for all $i \geq 1$ in any given positive genus surface S.) One may also use Fisk's construction [1] as another set of K_4-critical sequence. To prove (ii), suppose that U is a cycle. Since U is a core, by (i), we deduce that U is an odd cycle. The subdivided odd wheels $\{W_{2i+1}^{j}\}_{i=j}^{\infty}$, form a C_{2j+1}-critical sequence of planar graphs. To prove (iii), suppose that U is a wheel $W_{2k+1}, k \geq 1$. By (i), $U \neq K_4$ and so we have $k \geq 2$. But if $k \geq 2$, then for $i \geq k$, one can see that any homomorphism from W_{2i+1} to W_{2k+1} must map the hub of W_{2i+1} i to the hub of W_{2k+1}. With this in mind, we glue the hub of W_{2i+1} to a rim of W_{2k+1} to obtain H_i, and we deduce that $H_i \not\leq W_{2k+1}$ and that $\{H_i\}_{i=k}^{\infty}$ is a W_{2k+1}-critical sequence. □

The method of proof in Proposition 8 (iii) is a key idea that allows us to prove the main results of this paper, and so we have the following:

Proposition 9. *Let G be a graph and suppose $f : W_{2k+1,h} \to G$, where $h, k \geq 1$, for some homomorphism f. Then $f(W_{2k+1,h})$ contains an odd wheel subgraph $W_{2k'+1,h}$ of G such that the hub isomorphic to K_h of $W_{2k+1,h}$ is mapped to the hub isomorphic to K_h of $W_{2k'+1,h}, k' \leq k$.*

Proof. If $k = 1$, then we have $W_{3,h} = K_{3+h}$, a complete graph and we know $f(K_{3+h})$ is isomorphic to K_{3+h}. Since any subgraph K_h of K_{3+h} is a hub to the other three vertices of the clique, the proposition holds for $k = 1$. The case $k > 2$ easily follows by induction. One only needs to note that if $f(x) = f(y)$, then both x and y are rims. □

Remark. Proposition 9 holds for a plane graph G of any odd-girth $2j + 1$, $j \geq 1$, if we allow the odd wheel to be a subdivided wheel W^j_{2i+1} for all $j \geq 1$, for if we have $f(x) = f(y)$, and not both x and y are rims, then either a smaller odd cycle or a $K_{3,3}$-minor is obtained in $f(W^j_{2i+1})$, contrary to the hypothesis on G.

Proof of Theorem 1. Suppose we have a non-trivial dual-pair $(\mathcal{F}, U)$ of the class of all planar graphs. Then, by Corollary 7, U is not a clique. Let the odd girth of U be $2j + 1, j \geq 1$. We construct a U-critical sequence, contrary to Proposition 6.

Assume first that $j = 1$ and that U has no triangle-separation. Let v be a vertex on a triangle of U. We claim v is a hub of an odd wheel in U. If not, we glue the hub of W_{2i+1} to v on U and let the resulting graph be H_i. Since $U \subseteq H_i$ and that U is a core, the inequality $H_i \leq U$ implies the restriction of the map on the subgraph that is isomorphic to U in H_i is an automorphism of U. But then Proposition 9 implies that $H_i \not\leq U$, since v is not a hub of a wheel. Hence $H_i \not\leq U$ for all i. It follows that each H_i contains a U-critical subgraph H'_i containing W_{2i+1} for all i, a contradiction. As claimed v is a hub of an odd wheel.

Moreover, we claim that every vertex v of U is a hub of some odd wheel $W_{2k+1}(v), k \geq 2$ and the degree of v in U is exactly $2k + 1$. To see the first part of the claim, take any connected component K of U. (In fact it is easy to see that U is connected, for otherwise connecting the components of U by arbitrary long paths obtains a U-critical sequence.) Then, we claim the previous claim holds for each vertex in U. That is, starting from v contained in a triangle we observe that every neighbor of v is a rim of $W_{2k+1}(v)$ (because U has no separating triangle), and so contained in a triangle. Recursively this holds for each vertex of U. The second part of the claim is also clear form this.

Let a plane embedding of U be given and let v^* be a minimal degree vertex. Since U is a core we may assume that U is K_4-free, and so we note that $deg_U(v^*) = \delta = 5$.

We may assume that v^* is not on the outerface of U and that v_1, v_2, v_3, v_4, v_5 are the rims of v^* forming a 5-cycle C and separating v^* from the exterior, $ext(C)$. Let $C' = \{v^*, v_1, v_2\}$ be one of the five faces. Take new

Figure 2: Construction of H_i, by adding vertices u, u' and u_j, $j = 1, 2, \dots 2i{-}1$ in a face of U, for each $i \geq 1$

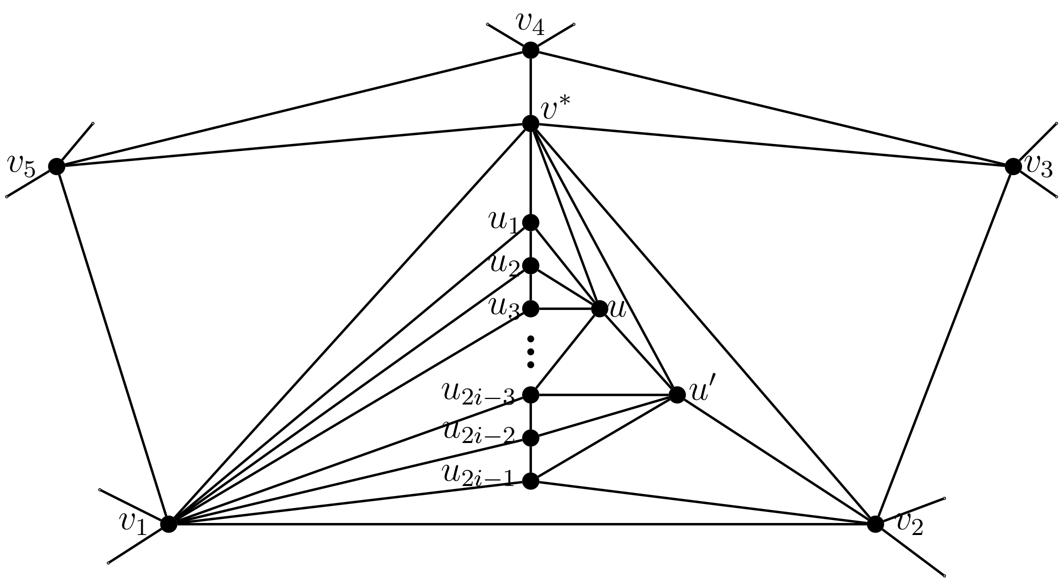
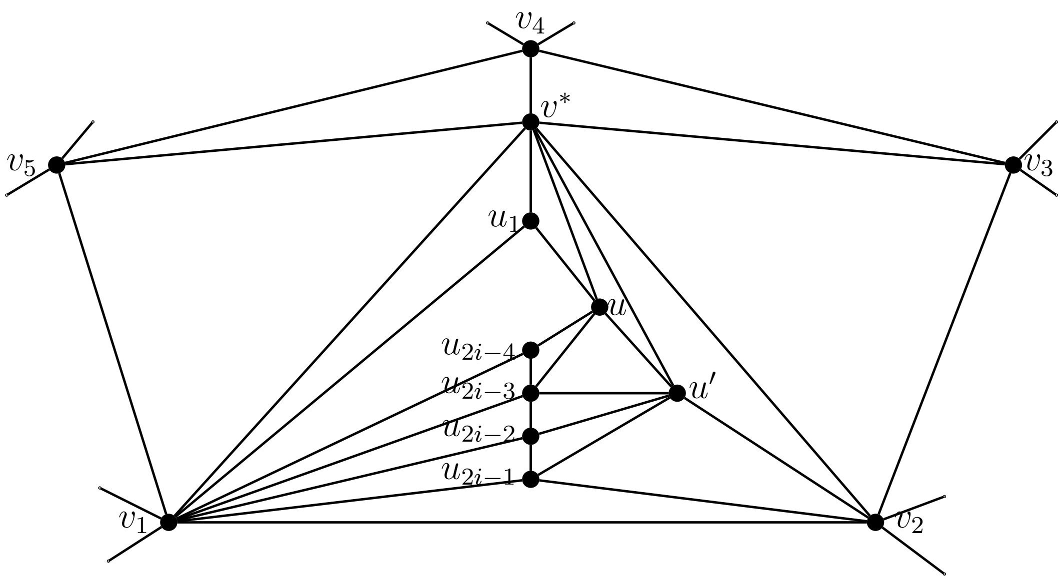


Figure 3: Any graph mapping to infinitely many H_i maps to U' depicted below and $U' \leq U$.



vertices $u, u', u_1, u_2, \ldots u_{2i-1}, i \geq 1$. Then add edges uv^* and $v_1u_r, 1 \leq r \leq 2i-1$, $uu_r, 1 \leq r \leq 2i-3$. Also add edges $u'u, u'v^*, u'v_2, u'u_{2i-3}, u'u_{2i-2}$ and $u'u_{2i-1}$. This procedure embeds in the interior, $Int(C')$, three new odd wheels $W(v_1) = W_{2i+1}$ with a hub at v_1, $W(u) = W_{2i-1}$ with a hub at u and $W(u_{2i-3}) = W_\delta$ with hub at u_{2i-3}. (See Figure 2).

First, we show that $H_i \not\leq U$ for all i. Since U is a core any homomorphism f from H_i to U is an automorphism when restricted to the copy of U in H_i. Hence the new edges of H_i are retracted to the edges of U. Since $v^*u \in E(H_i)$, it follows that $f(u) = v_i$, for some $i, 1 \leq i \leq 5$.

If $i = 2$ or 5, then $\{v_1, u_1, v^*, v_i = f(u)\}$ induces a K_4 subgraph in $f(H_i)$, a contradiction, as U is K_4-free. If $i = 3$ or 4, then v_2, v^*, and v_5 are common rims for $W(v_1)$ and $W(u)$ (with its hub now at v_i). This induces an edge from v_i to $v_j, |i-j| > 1$, which clearly induces a K_4 subgraph, a contradiction. Alternatively, we can also deduce that the edges v_iv_j, v^*v_i, and v^*v_j induce a separating triangle, also leading to a contradiction. (We prefer the later alternative as it will remain useful in proving the non-planar case, in the next section.) It follows that $f(u) = v_1$. But then, this reduces the $W(u_{2i-3})$ to a $W_{\delta-2} = K_4$, a contradiction. Hence $H_i \not\leq U$ for all i.

Next we show a U-critical subgraph H_i' of H_i must contain all of the new vertices we added. If not, then a subgraph of H_i missing any edge u_ju_{j+1} maps to the graph U' depicted in Figure 3. But then it can be seen easily that $U' \leq U$, if one identifies u_{2i-1}, u_{2i-3} and v^*. Also, deleting uu' allows us to map the resulting graph to U. We have the desired U-critical sequence.

Suppose now that U has a separating triangle C. We may assume that the interior of C has no more separating triangles. Since U is K_4-free, $int(C)$ contains more than one vertex. In fact, $int(C)$ contains an odd wheel of degree five that is disjoint from C, and the previous argument (for the case of no separating-triangle) holds for $Int(C)$. Note that our argument for $H_i \not\leq U$ is based on only local conditions: v^* and its neighbors and so, we find a U-critical sequence for any U of girth three.

The remaining case is the case $j > 1$. By the remark below Proposition 9, for $j > 1$, we deduce that each vertex of U is a hub of some subdivided odd wheel, and so $\delta(U) \geq 5$. But then, from Euler's formula we know that triangle-free planar graphs have at most $2n-4$ edges and this implies $\delta \leq 3$, a contradiction. We must have $j = 1$ and so this case does not occur. Hence, no non-trivial duality exist for planar graphs. $\square$

3 Dualities for other surfaces

In this section, we consider graphs on surfaces other than the sphere. For the torus, in particular, Thomassen's result [15] implies that for a 5-colorable graph U having a K_5 subgraph is a sufficient condition for finite duality. In this section we prove it is also necessary.

The theorem in [15] stated in terms of map-critical graphs terminology is as follows:

Theorem 10 (Thomassen [15]). *Let S be an orientable surface other than the sphere and let k be a natural number. Then there are only finitely many K_k-color critical graphs on S if and only if $k \geq 5$.*

Proof of Theorem 2. A direct consequence of Theorem 10 and Proposition 6. □

Proof of Theorem 3. Let U be a core graph on the torus and 5-colorable. If $U = K_5$, the result follows from Theorem 10.

Conversely, assume $U \neq K_5$. By Theorem 1, U is not planar unless $U = K_4$. But by Corollary 8, U can not be K_4. Since U is 5-colorable and a core, it is K_5-free. We construct a U-critical sequence on a the double torus.

We claim U is K_4-free. Suppose, for a contradiction, U has a subgraph K isomorphic to K_4. Then each edge of K is a hub of some double-wheel, for otherwise we glue the hub of $W_{2i+1,2}$ to the edge of K and by Proposition 9, we have $\{H_i\}_{i=1}^{\infty}$ a U-critical sequence on the double torus. Hence assume each edge of K is a hub to some double-wheel. Then, for every pair of edges e_i, e_j of K, their respective set of rim-edges R_i, R_j are disjoint, for otherwise their common rim-edge together with a triangle in K containing both e_i and e_j induces a K_5-subgraph, a contradiction. Recursively, we may apply the same argument to each edge we find in a K_4-subgraph and arrive at a subgraph U' of U such that every edge of U' is a hub of some double-wheel $W_{2k+1,2}$. Since U is K_5-free, we have $k \geq 2$. Take a vertex, $v \in V(U')$. Since it is an end vertex of a hub-edge e, it has degree at least six. It is also an end vertex of a spoke-edge e'. Since e' itself is a hub of another double-wheel whose rim-edges are disjoint form that of e, we deduce that v has at least two more neighbors. Hence, $\delta(U') \geq 8$. This contradicts, the fact that toroidal graphs have minimal degree at most six.

Assume now U is K_4-free. In addition, note that in this case, a vertex v^* of U can be chosen such that v^* is contained in neither a non-contactible nor a separating triangle of U. Then, the U-critical sequence construction mimics the construction in Theorem 1 as follows: the difference here is that $deg_U(v^*)$ can be greater than five for positive genus surfaces. However, since

v^* is contained in a triangle, by the same argument as in Theorem 1 we deduce that v^* is a hub of some odd-wheel $W(v^*) = W_{2d+1}$. Since v^* is not in a separating nor in a non-contractible triangle, it has degree $2d+1$ with all neighbors $v_1, v_2, \ldots, v_{2d+1}$ as rims of $W(v^*)$.

In Figure 2, we replace the edge uu' by a path P of length $2d-3$ and add edges from u_{2i-3} and from v^* to each new vertex of P to obtain three odd wheels: W_{2d+1}, with hub at u_{2i-3}, W_{2i+1} at v_1 and W_{2i-1} at u. The rest of the proof that obtains a U-critical sequence for a K_4-free graph is the same as the proof of Theorem1. The additional assumption that v^* is not contained in a non-contractible triangle may be necessary to deduce that $f(u) = v_1$ (as in the proof of Theorem 1), for otherwise there could be a vertex different from v_1 that is adjacent to the rims of $W(v_1)$. The desired U-critical sequence is obtained in each case and so the result follows for girth three U.

Finally assume U is triangle-free. Similar to the case of planar graphs we show that most of the vertices of U must have degree at least five and this will lead to a contradiction as follows: For an arbitrary vertex v of U, let C_{2j+1} be the smallest odd cycle of U containing v. Now we glue the hub of a subdivided wheel W^j_{2i+1} with v, for $i \geq j$ and obtain H_i. If $H_i \not\leq U$ for all i, we have found the desired U-critical sequence. If not, then either v itself is a hub of a subdivided wheel and hence $deg_U(v) \geq 5$ for U is triangle-free, or $3 \leq deg_U(v) \leq 4$ and there is an odd cycle C such that $|C| \geq 5$ and each vertex of C is at distance j from v, and C together with the j-paths to v contain a $K_{3,3}$ or a K_5 subdivision. This non-planarity condition does not occur at many vertices of U, since otherwise U will not be toroidal. By allowing p vertices, $p \geq 1$ to have degree at most 4, (so that each of the p vertices are contained in a $K_{3,3}$ or K_5 subdivision), from degree sum inequality we have $2e \geq 5(n-p) + 3p$. From Euler's formula we have $n - e + f = 2 - 2g$, and from face sum, $2e \geq 4f$, for triangle-free graphs. We arrive at the inequality $n \leq 2p$. One may generously allow p to be some small integer and we see that no such triangle-free toroidal counterexample exists. This completes the remaining case and the result follows. □

One can not generalize the above theorem for all k-colorable graphs, $k \geq 7$ due to the following lemma. It is, however, interesting to determine the missing case $k = 6$.

Lemma 11. *Let $W = W_{2k+1,h}$ and let S be any fixed surface where W is embeddable. If $h \geq 4$, then there are only finitely many W-critical graphs in $\mathcal{G}(S)$. If $h = 2$, then W has only finitely many critical graphs in a any higher genus surface if and only if $k = 1$.*

Proof. Let $W = W_{2k+1,h}$ be the generalized odd wheel with a clique-hub K_h.

Assume first that $h \geq 5$. Then, if G is W-critical, we deduce that $\delta(G) \geq 7$, because for any set of six vertices we can find a vertex adjacent to all six in W. From Eüler's equation for surfaces of genus g and from the degree sum equation, we have $(\delta - 6)|G| \leq 12(g-1)$.

If $h = 4$, we use Gallai's inequality for k-critical graphs [2]: $2e \geq \delta n + \frac{\delta-2}{(\delta+1)^2-3}n + 2\frac{\delta}{(\delta+1)^2-3}$ for $\delta = 6$ and obtain $|G| \leq 139(g-1)$ (a better upper bound of $96(g-1)$ is given in [16] using simple arguments). Nevertheless, g is fixed and so we have only finitely many W-critical graphs for $h \geq 4$.

To prove the second part of the statement, note that $W_{2k+1,2}$ for all $k \geq 1$ can be embedded in a surface of genus one. However, we observe that $W_{2k+1,2}$ is not edge-transitive unless $k = 1$. Take two double wheels $W_{2k+1,2}$ and $W_{2k'+1,2}$. Glue the hub edge of one with a rim-edge of the other to obtain a graph embeddable in the double-torus. Now by letting k' be arbitrary, we find a $W_{2k+1,2}$-critical sequence in $\mathcal{G}(S)$. Therefore, by this fact and by Thomassen's theorem [15], we know that $W_{2k+1,2}$ has only finitely many critical graphs if and only if $k = 1$. □

Proof of Proposition 4. A direct consequence of Lemma 11. □

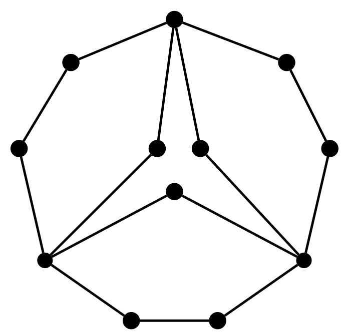
Remark. Note that double-wheel-critical graphs exist infinitely many on the double-torus. There remains a question if these double-wheels obtain finite duality on the torus. If they do, then perhaps these are the only examples that would be known as giving infinitely many distinct dual-pairs in a fixed surface.

4 Other minor closed classes with dualities

We show now existence of dual pairs in other types of minor closed classes. Although the main focus of this paper is graphs that are embedded in a fixed surface, we add this short section to show that without the topological constraint of embeddings, one can find infinitely many finite dualities in a fixed minor closed class. We shall define a specific minor closed class and construct the dual pairs explicitly.

Let $C_{2,k}$ denote the dual graph of the complete-bipartite graph $K_{2,k}$. We say that a graph G is a k-ear-face if G can be obtained from $C_{2,k}$ by subdividing some of the edges of $C_{2,k}$ a finite number of times. We say G is an *ear-face* when the integer k is irrelevant. An *ear* of an ear-face consists of one or two threads. Hence a cycle is not an ear-face. Recall that a *thread* has exactly two vertices of degree at least three and all other internal vertices are of degree two. Any connected proper subgraph of a thread is said to be *sub-thread.* An example of a 3-ear-face is the 3-*Pentagon* depicted in Fig-

Figure 4: The 3-Pentagon, $3\mathbb{P}$



ure 4. We can represent the graph by a vector $([2,3],[2,3],[2,3])$, indicating the even and odd lengths of the three ears of $3\mathbb{P}$. In general, for any $k \geq 2$ we may represent an ear-face G by $(E_1, E_2, \ldots, E_k)$, where each $E_i = [\epsilon_i, \delta_i]$ is an ear of G consisting two threads of length ϵ_i and δ_i, with ϵ_i even and δ_i odd. If both threads are of same parity, then we take the smaller length and disregard the longer thread, since a longer thread of same parity is redundant under homomorphism retraction. We let $\delta_i = \infty$ if E_i has only a thread of even length, and let $\epsilon_i = \infty$ if E_i has only a thread of odd length. If either δ_i or ϵ_i are set to be ∞ then E_i is called a *diminished-ear*. For example, if we delete a thread of length 3 from $3\mathbb{P}$ to obtain a 3-ear-face $G' = ([2,\infty],[2,3],[2,3])$, then G' has one diminished-ear.

Let $\mathcal{EF}$ denote the set of all ear-faces including all of their minors and any of their finite disjoint union. We can see by the next two lemmas that $\mathcal{EF}$ contains arbitrary long anti-chain and that it has finite duality.

Lemma 12. *[12] For any integer $N \geq 2$, and odd composite g, $g \neq 9$, there exists an anti-chain of length 2^N, consisting of odd-girth g ear-faces of size $O(N)$.*

To establish the finite-duality property the following lemma is useful.

Lemma 13 (Ear-folding lemma). *[12] Let G be an ear-face of odd-girth $g > 3$ with k ears, $k > 3$. Then there exists an ear-face G' of same odd-girth and $k-1$ ears such that $G \leq G'$.*

Existence of a finite duality in $\mathcal{EF}$ is now an easy consequence of the above lemma:

Corollary 14. $(\{K_3\}, 3\mathbb{P})$ *is a dual pair in $\mathcal{EF}$.*

Proof. Let $\mathcal{F} = \{K_3\}$. Then, by the Ear-folding lemma, every triangle-free k-ear-face of odd girth g is homomorphic to a 3-ear-face of same odd girth.

Any 3-ear-face G that has a thread length one maps to its smallest odd cycle, and so $G \leq C_5 \leq 3\mathbb{P}$. Otherwise each thread has either even length of at least 2 or odd length of at least 3 and so maps to the 3-pentagon, $3\mathbb{P}$. It follows, $G \leq 3\mathbb{P}$ if and only if $K_3 \not\leq 3\mathbb{P}$, for any $G \in \mathcal{EF}$. $\square$

In fact, it is easy to obtain the following:

Lemma 15. *$\mathcal{EF}$ contains infinitely many dual pairs.*

Proof. For any odd integer $g > 1$, let $\mathcal{F} = \{C_g\}$. Now the set X of core 3-ear-faces of odd girth $g+2$ that are not homomorphic to any other 3-ear-face of same odd girth is a finite set. This is true because if G is a 3-ear-face of odd girth $g+2$ and not homomorphic to another 3-ear-face of same odd girth, then each of its 3 ears must be of length exactly $g+2$. There are only finitely many positive pairs of integers ϵ, δ such that $\epsilon + \delta = g+2$ and so X is a finite set.

Next, let U be the graph obtained by taking a disjoint union of all 3-ear-faces in X. Take any $G \in \mathcal{EF}$. If G has odd girth $g' \leq g$, then $C_g \leq G$. Otherwise, $C_g \not\leq G$ and each component of G maps, by the Ear-folding lemma, to some 3-ear-face H of odd girth $g+2$. In turn, H maps to some H' in X. Hence, by transitivity of $\leq$, we have $G \leq U$. Since g is arbitrary we have infinitely many cases. $\square$

It is interesting to find other types of dual pairs in $\mathcal{EF}$. For instance, one can easily see that $(\{3\mathbb{P}\}, C_5)$ is a dual pair in $\mathcal{EF}$, because $3\mathbb{P}$ is the only ear-face of odd girth five that does not map to C_5, and we know any graph of odd girth at least seven in $\mathcal{G}/K_4$ maps to C_5. Are there any more dual-pairs? We present this problem in the last section.

We observe that $\mathcal{EF} \subset \mathcal{G}/K_4$. It is easy to deduce now that for all $k \geq 4$, there exists a minor closed class $\mathcal{K}_k \subset \mathcal{G}/K_k$ that has infinitely many dual pairs, by attaching a universal clique of size $k-4$ to every graph in $\mathcal{EF}$. A bit less trivial construction also can be given as follows: Suppose $G \in \mathcal{EF}$ is an-ear face. For $k \geq 1$, let $w_k(G)$ denote the graph we obtain from G by replacing each odd-ear of G by an odd-wheel of same odd length with a hub size k. Call $w_k(G)$, a *wheel-face* and denote the class of the graphs obtain by $\mathcal{WF}_k$. Note that $w_k(G) \in \mathcal{G}/K_{k+4}$. It is easy to generalize the above two theorems as follows:

Theorem 16. $(\{K_{k+3}\}, w_k(3\mathbb{P}))$ *is a dual pair in $\mathcal{WF}_k \subset \mathcal{G}/K_{k+4}, k \geq 0$.*

Theorem 17. *$\mathcal{WF}_k$ contains infinitely many dual pairs, for all $k \geq 0$.*

5 Conclusion

Several problems seem to arise that are related to our results in this paper. Of course the most interesting problem in this subject would be to characterize all of the dualities in any fixed orientable surface. We offer a specific problem that seems quite accessible. It also is a problem that covers the missing case in Proposition 4:

Problem 18. *Are there only finitely many $W_{2k+1,3}$-critical graphs on some fixed surface S, and some $k \geq 2$? Are there only finitely many $W_{2k+1,2}$-critical graphs on the torus for all $k \geq 2$?*

So far, for a fixed surface S the number of dualities obtained is finite. We have shown large wheels are good sources of finite dualities. A potential source of infinitely many distinct dualities on a fixed surface would be the double wheels. However, by Lemma 11 this case is excluded, except possibly for the torus.

Perhaps Thomassen's theorem [15] can be strengthened for core and homomorphism anti-chains. For example, we do not know the answer to the following:

Problem 19. *Is it true that for any surface S of genus $g \geq 2$, the number of distinct dualities in $\mathcal{G}(S)$ is finite?*

Recall that, there are only finitely many k-color-critical graphs on any fixed orientable surface S, if $k \geq 6$. However, we can find an infinite anti-chain (by homomorphism) of say, 7-chromatic graphs on that surface. To see this, take any 7-chromatic graph G of girth at least six and consider a surface S where G embeds. Now take any 3-chromatic core non-vertex-transitive graph H of girth at most five (for example the 3-pentagon in Figure 4). Note that the disjoint union $G \cup H$ is a core. In fact, if H' is constructed by concatenating several copies $H_1, H_2, \ldots, H_t$ of H (by attaching a vertex of H_i to a vertex of H_{i+1}) in a path-like structure such that H' is a core then $G \cup H'$ is a core as well (see [4],[5] for detail). Let a and b be two vertices of H for which no automorphism of H sends a to b. Using a and b as vertices of attachment, it is shown in [4],[5] that any partial order can be embedded in a class $\mathcal{K}$ of planar graphs constructed by concatenating finite number of copies of H that mimic arcs of directed paths. Now taking disjoint union of G with each element of an infinite anti-chain of $\mathcal{K}$ gives us an infinite anti-chain of 7-chromatic graphs in $\mathcal{G}(S)$. However, we note that the clique number ω of these graphs is quite small. We don't know if similar anti-chain can be formed using graph with large cliques.

This leads to the following question:

Problem 20. *Is it true that every anti-chain of the class $\mathcal{K} \subset \mathcal{G}(S)$ of all graphs containing K_7 subgraph is finite? Does $\mathcal{K}$ have only finitely many cores?*

Remark. Let us remark that one can construct infinitely many k-critical graphs in $\mathcal{G}/K_n, 3 \leq k \leq n-1$ and so, unlike the result we obtain from Thomassen's theorem, there are no dualities of the form $(\mathcal{F}, K_{k-1})$ in $\mathcal{G}/K_n$, for all $k \leq n-1$. This can be seen easily by considering odd wheels of appropriate hub-size. The remaining case $(\mathcal{F}, K_{n-1})$ is a dual-pair (trivially) if Hadwiger's conjecture holds for n, by setting $\mathcal{F} = \emptyset$.

The question of existence of a non-trivial dual-pair in $\mathcal{G}/K_n$ remains open for $n > 6$. We can prove the case $n = 4, 5$, and 6. The following is a question that considers the general case:

Problem 21. *Does a non-trivial dual-pair exist in $\mathcal{G}/K_k$ for $k > 6$?*

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