

Chromatic number for a generalization of Cartesian product graphs

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Abstract

Let \mathcal{G} be a class of graphs. The d -fold grid over \mathcal{G} , denoted \mathcal{G}^d , is the family of graphs obtained from d -dimensional rectangular grids of vertices by placing a graph from \mathcal{G} on each of the lines parallel to one of the axes. Thus each vertex belongs to d of these subgraphs. Let $f(\mathcal{G}; d) = \max_{G \in \mathcal{G}^d} \chi(G)$. If each graph in \mathcal{G} is k -colorable, then $f(\mathcal{G}; d) \leq k^d$. We show that this bound is best possible by proving that $f(\mathcal{G}; d) = k^d$ when \mathcal{G} is the class of all k -colorable graphs. We also show that $f(\mathcal{G}; d) \geq \left\lfloor \sqrt{\frac{d}{6 \log d}} \right\rfloor$ when \mathcal{G} is the class of graphs with at most one edge, and $f(\mathcal{G}; d) \geq \left\lfloor \frac{d}{6 \log d} \right\rfloor$ when \mathcal{G} is the class of graphs with maximum degree 1.

1 Introduction

The *Cartesian product* of graphs G_1, \dots, G_d is the graph with vertex set $V(G_1) \times \dots \times V(G_d)$ in which two vertices (v_1, \dots, v_d) and (v'_1, \dots, v'_d) are

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adjacent if they agree in all but one coordinate, and in the coordinate where they differ the values are adjacent vertices in the corresponding graph. The product can be viewed as a rectangular grid with copies of G_1, \dots, G_d placed on vertices forming lines parallel to the d axes. It is well-known (and easy to show) that the chromatic number of the Cartesian product of G_1, \dots, G_d is the maximum of the chromatic numbers of G_1, \dots, G_d .

In this paper, we consider bounds on the chromatic number of graphs in a family resulting from a more general graph operation. Instead of placing copies of the same graph G_i on all the lines parallel to the i -th axis, we may place different graphs from a fixed class. Let $[n]$ denote $\{1, \dots, n\}$. For a class \mathcal{G} of graphs, the d -fold grid over \mathcal{G} , denoted \mathcal{G}^d , is the family of all graphs formed by choosing a vertex set of the form $[n_1] \times \dots \times [n_d]$ and letting each set of vertices where all but one coordinate is fixed induce a graph from \mathcal{G} . For example, a Cartesian product of graphs in \mathcal{G} belongs to \mathcal{G}^d .

The study of the chromatic number and independence number of graphs in \mathcal{G}^d is motivated by an application in computational geometry. Frequency assignment problems for transmitters in the plane are modeled by coloring and independence problems on certain graphs (see [2, 6]). These graphs arise from sets of points using the Euclidean metric. When such problems are studied using the Manhattan metric (see [4]), results about grid structures give bounds for the number of frequencies needed. Motivated by such problems, Szegedy [13] posed the following open problem at the workshop ‘‘Combinatorial Challenges’’:

What is the maximum chromatic number of a graph $G \in \mathcal{G}^d$ when \mathcal{G} is the class \mathcal{B} of all bipartite graphs or when \mathcal{G} is the class \mathcal{S} of graphs containing at most one edge?

If each graph of \mathcal{G} is k -colorable, then every graph in \mathcal{G}^d has chromatic number at most k^d , since it is the union of d subgraphs, each of which is k -colorable. In particular, all graphs in \mathcal{B}^d are 2^d -colorable. We show that this bound is sharp, which is somewhat surprising since Cartesian products of bipartite graphs are bipartite. More generally, let $f(\mathcal{G}; d) = \max_{G \in \mathcal{G}^d} \chi(G)$. We show that if \mathcal{G} is the class of all k -colorable graphs, then $f(\mathcal{G}; d) = k^d$. We prove the existence of k^d -chromatic graphs in \mathcal{G}^d probabilistically, but an

explicit construction can then be obtained by building, for each n , a graph in \mathcal{G}^d that is “universal” in the sense that it contains all graphs in \mathcal{G}^d with vertex set $[n]^d$. This settles the first part of Szegedy’s question.

Determining $f(\mathcal{S}; d)$ is more challenging. Since the maximum degree of a graph in \mathcal{S}^d does not exceed d , and these graphs do not contain K_{d+1} , Brooks’ Theorem [3] implies that each graph in \mathcal{S}^d is d -colorable (when $d \geq 3$). Also graphs in \mathcal{S}^2 are bipartite, since cycles in such a graph alternate between horizontal and vertical edges.

When d is large, we can use a refinement of Brook’s Theorem obtained by Reed et al. [7, 10, 11, 12] to improve the upper bound.

Theorem 1 (Molloy and Reed [11]). *There exists a constant D_0 such that for all $D \geq D_0$ and k satisfying $k^2 + 2k < D$, every graph G with maximum degree D and $\chi(G) > D - k$ has a subgraph H with at most $D + 1$ vertices and $\chi(H) > D - k$.*

Theorem 1 implies that $f(\mathcal{S}; d) \leq d - \sqrt{d} + o(1)$. A still better upper bound follows from another result.

Theorem 2 (Johansson [9]). *The chromatic number of a triangle-free graph with maximum degree D is at most $O(D/\log D)$.*

This result, which was further strengthened by Alon et al. [1], implies that $f(\mathcal{S}; d) \in O(d/\log d)$.

We show that though the graphs contained in \mathcal{S}^d are very sparse, and it is natural to expect that the graphs contained in \mathcal{S}^d can be colored just with a few colors, $f(\mathcal{S}; d) \geq \left\lfloor \sqrt{\frac{d}{6 \log d}} \right\rfloor$. Our argument is again probabilistic. A similar argument yields $f(\mathcal{M}; d) \geq \left\lfloor \frac{d}{6 \log d} \right\rfloor$, where \mathcal{M} is the class of all matchings (i.e., graphs with maximum degree 1). This lower bound is asymptotically best possible, since the discussion above yields $f(\mathcal{M}; d) \in O(d/\log d)$.

2 Preliminaries

In this section, we make several observations used in the proofs of our subsequent lower bounds on $f(\mathcal{G}; d)$ for various \mathcal{G} . We start by recalling the

Chernoff Bound, an upper bound on the probability that a sum of independent random variables deviates greatly from its expected value (see [8] for more details).

Proposition 3. *If X is a random variable equal to the sum of N independent identically distributed 0, 1-random variables having probability p of taking the value 1, then the following holds for every $0 < \delta \leq 1$:*

$$\text{Prob}(X \geq (1 + \delta)pN) \leq e^{-\frac{\delta^2 pN}{3}} \quad \text{and} \quad \text{Prob}(X \leq (1 - \delta)pN) \leq e^{-\frac{\delta^2 pN}{2}}.$$

Next, we establish two technical claims. We begin with a standard bound on the number of subsets of a certain size.

Proposition 4. *For $\alpha > 2$, the number of N/α -element subset of an N -element set is at most $2^{\frac{N}{\alpha}(1+\log \alpha)}$.*

Proof. An N -element set has $\binom{N}{N/\alpha}$ subsets of size N/α . It is well known, see e.g. [5], that $\binom{N}{N/\alpha} \leq 2^{N \cdot H(1/\alpha)}$, where $H(p) = -p \log p - (1 - p) \log(1 - p)$ ($H(p)$ is called the entropy function). A simple calculation yields the upper bound:

$$\binom{N}{N/\alpha} \leq 2^{N \cdot (\frac{1}{\alpha} \log \alpha + \frac{\alpha-1}{\alpha} \log \frac{\alpha}{\alpha-1})} \leq 2^{N \cdot (\frac{1}{\alpha} \log \alpha + \frac{\alpha-1}{\alpha} \cdot \frac{1}{\alpha-1})} \leq 2^{\frac{N}{\alpha}(1+\log \alpha)}$$

□

The second claim is a straightforward upper bound on a certain type of product of expressions of the form $(1 - \varepsilon)$:

Proposition 5. *If a_1, \dots, a_m are nonnegative integers with sum n , then*

$$\prod_{i=1}^m \left(1 - \binom{a_i}{2} \frac{1}{x}\right) \leq \left(1 - \frac{1}{x}\right)^{n-m}$$

for any positive real x such that $x \leq \binom{\max_i a_i}{2}$.

Proof. Since $\sum a_i = n$, it suffices to show that

$$\left(1 - \binom{a}{2} \frac{1}{x}\right) \leq \left(1 - \frac{1}{x}\right)^{a-1} \tag{1}$$

for every nonnegative integer a . If $a \leq 1$, then the left side of (1) is 1 and the right side is at least 1. If $a \geq 2$, then (1) follows (by setting $k = a - 1$) from the well-known inequality

$$1 - \frac{k}{x} \leq \left(1 - \frac{1}{x}\right)^k,$$

which holds whenever $0 \leq k \leq x$. □

3 Products of k -colorable graphs

In this section, we prove that $f(\mathcal{B}; d) = 2^d$. Note again that after the probabilistic proof of existence, we can construct such graphs explicitly as explained in Introduction. Even so, the argument that they are not $(2^d - 1)$ -colorable remains probabilistic. We prove the result in the more general setting of k -colorable graphs.

Theorem 6. *For $d, k \in \mathbf{N}$, the d -fold grid over the class of k -colorable graphs contains a graph with chromatic number k^d .*

Proof. The claim holds trivially if $d = 1$, so we assume $d \geq 2$. The integers k and d are fixed, and N is a large integer to be chosen in terms of k and d . Consider the set $[N]^d$. For each $v \in [N]^d$, define a random d -tuple $X(v)$ such that $X(v)_i$ takes each value in $[k]$ with probability $1/k$, and the d coordinate variables are independent. Generate a graph G with vertex set $[N]^d$ by making two vertices u and v adjacent if they differ in exactly one coordinate and $X(u)_\ell \neq X(v)_\ell$, where ℓ is the coordinate in which u and v differ.

By construction, any set of vertices in G that all agree outside a fixed coordinate induce a complete multipartite graph with at most k parts. Hence G is in the d -fold grid over the k -colorable graphs. It will suffice to show that almost surely (as N tends to infinity) G does not have an independent set with more than $\frac{N^d}{k^{d-0.5}}$ vertices. This yields $\chi(G) \geq k^d$, since otherwise some color class would be an independent set of size at least $\frac{N^d}{k^{d-1}}$.

For an independent set A in G , let the *shade* of A be the function $\sigma: [d] \times [N]^{d-1} \rightarrow [k]$ defined as follows. For $z = (\ell; i_1, \dots, i_{\ell-1}, i_{\ell+1}, \dots, i_d) \in [d] \times$

$[N]^{d-1}$, consider the vertices in A of the form $(i_1, \dots, i_{\ell-1}, j, i_{\ell+1}, \dots, i_d)$. By the construction of G , the value of $X(v)_\ell$ is the same for each such vertex v , since vertices of A are nonadjacent. Let this value be $\sigma(z)$. If there is no vertex of A with this form, then let $\sigma(z) = 1$.

The union of independent sets with the same shade is an independent set. Hence for each function σ there is a unique maximal independent set in G with shade σ ; denote it by A_σ . In order to have $v \in A_\sigma$, where $v = (i_1, \dots, i_d)$, the random variables $X(v)_1, \dots, X(v)_d$ must satisfy $X(v)_\ell = \sigma(\ell; i_1, \dots, i_{\ell-1}, i_{\ell+1}, \dots, i_d)$. Hence each v lies in A_σ with probability k^{-d} .

As a result, the expected size of A_σ is $(N/k)^d$. Since the variables $X(v)_\ell$ are independent for all v and ℓ , we can bound the probability that $|A_m| \geq \frac{N^d}{k^d - 0.5}$ using the Chernoff Bound (Proposition 3). Applied with $\delta = \frac{1}{2k^d - 1}$, this yields

$$\text{Prob} \left(|A_m| \geq \frac{N^d}{k^d - 0.5} \right) \leq e^{-\frac{N^d}{3(2k^d - 1)^2 k^d}} \leq e^{-\frac{N^d}{12k^{3d}}}.$$

Since there are $k^{dN^{d-1}}$ possible shades, the probability that some independent set has more than $\frac{N^d}{k^d - 0.5}$ vertices is at most $k^{dN^{d-1}} \cdot e^{-N^d/12k^{3d}}$, and we compute

$$k^{dN^{d-1}} \cdot e^{-N^d/12k^{3d}} = e^{\log kdN^{d-1} - N^d/12k^{3d}} \rightarrow 0.$$

If N is sufficiently large in terms of k and d , then the bound is less than 1, and there exists such a graph G with no independent set of size at least $\frac{N^d}{k^d - 0.5}$. \square

4 Products of single-edge graphs

In this section, we prove the lower bound for the d -fold grid over the class of graphs with at most one edge.

Theorem 7. *For $d \geq 2$, there exists $G \in \mathcal{S}^d$ such that $\chi(G) \geq \left\lfloor \sqrt{\frac{d}{6 \log d}} \right\rfloor$.*

Proof. Let $k = \left\lfloor \sqrt{\frac{d}{6 \log d}} \right\rfloor$. For $k \leq 2$, the conclusion is immediate. Hence, we assume $k \geq 3$. We generate a graph G with vertex set $[2k]^d$. For

$(\ell; i_1, \dots, i_{\ell-1}, i_{\ell+1}, \dots, i_d) \in [d] \times [2k]^{d-1}$, choose a random pair of distinct integers j and j' from $[2k]$, and make the vertices $(i_1, \dots, i_{\ell-1}, j, i_{\ell+1}, \dots, i_d)$ and $(i_1, \dots, i_{\ell-1}, j', i_{\ell+1}, \dots, i_d)$ adjacent in G . The choices of j and j' are independent for all elements of $[d] \times [2k]^{d-1}$.

Since $G \in \mathcal{S}^d$, its chromatic number is at most d . To show that $\chi(G)$ is at least k with positive probability, it suffices to show that with positive probability, G has no independent set of size at least $(2k)^d/k$.

Consider a set A in $V(G)$ with size $(2k)^d/k$; we bound the probability that A is an independent set in G . Again we think of an element z in $[d] \times [2k]^{d-1}$ as designating a line in $[2k]^d$ parallel to some axis. Let $A[z]$ be the intersection of A with this line. By the construction of G , the probability that some two vertices in $A[z]$ are adjacent in G is

$$1 - \binom{|A[z]|}{2} / \binom{2k}{2},$$

which is at most $1 - \binom{|A[z]|}{2} \frac{1}{2k^2}$. By applying Proposition 5 with $x = 1/2k^2$, $n = |A| \geq \frac{(2k)^d}{k} = 2(2k)^{d-1}$, and $m = (2k)^{d-1}$, we conclude that the probability of all subsets of A lying along lines in a particular direction being independent in G is at most

$$\prod_{z \in \{\ell\} \times [2k]^{d-1}} \left(1 - \binom{|A[z]|}{2} \frac{1}{2k^2} \right) \leq \left(1 - \frac{1}{2k^2} \right)^{(2k)^{d-1}}.$$

Let p be the probability that A is an independent set in G . Since the edges in each of the d directions are added to G independently,

$$p \leq \left(1 - \frac{1}{2k^2} \right)^{d(2k)^{d-1}} \leq e^{-\frac{d(2k)^{d-1}}{2k^2}} \leq e^{-2d(2k)^{d-3}} \leq 2^{-2d(2k)^{d-3}}.$$

We want to show that with positive probability, G has no independent set of size $(2k)^d/k$. Let M be the number of subsets of $V(G)$ with size $(2k)^d/k$. By Proposition 4,

$$M \leq 2^{\frac{(2k)^d}{k} \cdot (1 + \log k)} \leq 2^{2(2k)^{d-1} \cdot (1 + \log k)} \leq 2^{3(2k)^{d-1} \log k}.$$

Therefore, we bound the probability that G has an independent set of size $(2k)^d/k$ by the following computation:

$$2^{3(2k)^{d-1} \log k} \cdot 2^{-2d(2k)^{d-3}} = 2^{2(2k)^{d-3}(6k^2 \log k - d)} < 1.$$

The last inequality uses the fact that $6k^2 \log k - d$ is negative, by the choice of k . We conclude that some such graph G has no independent set of size at least $(2k)^d/k$. \square

5 Products of matchings

Finally, we consider the d -fold grid over the class \mathcal{M} of matchings.

Theorem 8. *For $d \geq 2$, there exists $G \in \mathcal{M}^d$ such that $\chi(G) \geq \left\lfloor \frac{d}{6 \log d} \right\rfloor$.*

Proof. As the proof is similar to the proof of Theorem 7, we will give less detail and focus on the differences between the proofs. Set $k = \left\lfloor \frac{d}{6 \log d} \right\rfloor$ and assume $k \geq 3$. We randomly generate a graph G with vertex set $[2k]^d$. As before an element z in $[d] \times [2k]^{d-1}$ designates a line in $[2k]^d$ parallel to some axis. We place a random perfect matching on the $2k$ vertices in each such line. Hence, the resulting graph G is d -regular. It suffices to show that with positive probability, G has no independent set of size at least $(2k)^d/k$.

Consider a set A in $V(G)$ with size $(2k)^d/k$; we bound the probability that A is an independent set in G . Let $A[z]$ be the intersection of A with a line designated by $z \in [d] \times [2k]^{d-1}$. Let X be the random variable that is the number of edges in G induced by $A[z]$. By the construction of G , we have $E(X) = \frac{1}{2k-1} \binom{|A[z]|}{2}$. When X is a nonnegative integer-valued random variable, $\text{Prob}[X \geq 1] \geq \frac{E(X)}{\max(X)}$. Since $A[z]$ cannot induce more than $|A[z]|/2$ edges, we bound the probability p that $A[z]$ contains an edge by computing

$$p \geq \frac{\frac{1}{2k-1} \binom{|A[z]|}{2}}{|A[z]|/2} = \frac{|A[z]| - 1}{2k - 1} \geq \frac{|A[z]| - 1}{2k}.$$

Let q_ℓ denote the probability that all subsets of A lying along lines in direction ℓ are independent (each such line consists of d -tuples that agree outside the

ℓ th coordinate). We compute

$$\begin{aligned} q_\ell &\leq \prod_{z \in \{\ell\} \times [2k]^{d-1}} \left(1 - \frac{|A[z]| - 1}{2k} \right) \leq \prod_{z \in \{\ell\} \times [2k]^{d-1}} e^{-\frac{|A[z]| - 1}{2k}} \\ &= e^{-\frac{(2k)^d/k - (2k)^{d-1}}{2k}} = e^{-(2k)^{d-2}}. \end{aligned}$$

The probability P that A is independent can now be bounded as follows:

$$P \leq e^{-d(2k)^{d-2}} \leq 2^{-d(2k)^{d-2}}.$$

Finally, an upper bound on the probability that G has an independent set of size $(2k)^d/k$ is obtained by multiplying the bound on P and the bound on the number of $(2k)^d/k$ -element subsets of vertices from Proposition 4.

$$2^{3(2k)^{d-1} \log k} \cdot 2^{-d(2k)^{d-2}} = 2^{(2k)^{d-2}(6k \log k - d)} < 1.$$

□

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