TAME PARTS OF FREE SUMMANDS IN COPRODUCTS OF PRIESTLEY SPACES

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ABSTRACT. It is well known that a sum (coproduct) of a family $\{X_i : i \in I\}$ of Priestley spaces is a compactification of their disjoint union, and that this compactification in turn can be organized into a union of pairwise disjoint order independent closed subspaces X_u, indexed by the ultrafilters u on the index set I. The nature of those subspaces X_u indexed by the free ultrafilters u is not yet fully understood.

In this article we study a certain dense subset $X_u^\partial \subseteq X_u$ satisfying exactly those sentences in the first-order theory of partial orders which are satisfied by almost all of the X_i's. As an application we present a complete analysis of the coproduct of an increasing family of finite chains, in a sense the first non-trivial case which is not a Čech-Stone compactification of the disjoint union $\bigcup_I X_i$. In this case, all the X_u's with u free turn out to be isomorphic under the Continuum Hypothesis.

1. Introduction

A Priestley space is a compact ordered topological space satisfying a certain separation condition. Although it may not be immediately clear that coproducts of arbitrary families exist in the ensuing category **PSp**, this is a consequence of the famous Priestley duality between **PSp** and the category of bounded distributive lattices, in light of the fact that the latter obviously has all products. But the question of what the coproducts of Priestley spaces actually look like is not so easily answered. A coproduct does contain the order-and-topologically disjoint union of the members of the family, as one would expect, but only as a dense subspace. Because it is compact, it must also contain other points, the

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points of the so-called remainder, and it is this remainder which is not yet completely understood.

What is known is that the coproduct $\coprod_{i\in I} X_i \equiv X$ of an infinite family can be divided into a union $\bigcup_u X_u$ of pairwise disjoint order-independent closed subspaces X_u, indexed by the ultrafilters u on I. The given Priestley spaces X_i can be identified with the subspaces indexed by the fixed ultrafilters, i.e., those of the form $i^* = \{J \subseteq I : i \in J\}$, $i \in I$. We call these the "fixed summands" of X. But one also has the other subspaces X_u, indexed by the free ultrafilters $u \in \beta I \smallsetminus I$, which together make up the remainder of X. The exact nature of these "free summands" is not apparent. Recently there has been some progress made in the study of their order structure, and, in particular, it has been shown that they can contain finite configurations that are not present in any of the fixed summands.

In the present paper we discuss a certain specific subspace X_u^∂ of X_u, referred to as its tame part. These tame parts are very well behaved. On the one hand they are dense, and, in the case of discrete X_i, they are constituted precisely of the isolated points of X_u. Moreover, any rooted tree which embeds in an X_u embeds also in its tame part X_u^∂. Further still, the order on the tame part X_u^∂ satisfies precisely those first order formulas in the theory of partial orders which are satisfied by almost all of the X_i's. Combining the latter two insights provides a second proof of a result of [1]: a rooted tree embeds in X iff it embeds in some X_i.

The facts about the embeddings $X_u^\partial \subseteq X_u$ are then used to analyze the case of increasing chains. We think this is an important case for understanding the nature of coproducts since, by the result in [8], if the fixed summands are not bounded in height, the behavior of the coproduct is non-standard insofar as the topology is concerned. (This is not to speak of the order, which can be wild even in the bounded case.) Thus, this constitutes the least complicated non-trivial situation. It turns out to be not quite simple, but the analysis based on the embeddings of the tame parts makes it transparent.

The paper is divided into four sections, of which the first is introductory and the second contains preliminaries. The meat of the paper is the third section, culminating in the definition and development of the tame part. In Section 4 we discuss the coproducts of chains and present a complete analysis of the case of finite summands.

2. Preliminaries

2.1. **Posets.** For a subset M of a poset $(X, \leq)$ we write, as usual, ${\downarrow}M = \{x : x \leq m \in M\}$ (${\uparrow}M = \{x : x \geq m \in M\}$), and abbreviate ${\downarrow}\{x\}$ to ${\downarrow}x$ (${\uparrow}\{x\}$ to ${\uparrow}x$). The sets $M \subseteq X$ such that ${\downarrow}M = M$ (${\uparrow}M = M$) are called *down-sets* (*up-sets*).

The immediate precedence of x before y, or the immediate succession of y after x, is indicated by

$$x \prec y \quad \text{or} \quad y \succ x.$$

Linearly ordered posets are referred to as *chains.*

2.2. **Priestley duality.** A *Priestley space* is an ordered compact space X such that for any $x \nleq y$ in X there is a clopen up-set U such that $y \notin U \ni x$. The monotone continuous maps are called *Priestley maps*, and the resulting category is designated **PSp**.

Recall the famous *Priestley duality* (see, e.g., [11], [12]) between **PSp** and **DLat**, the category of bounded distributive lattices. The equivalence functors $\mathcal{P} : \mathbf{DLat} \to \mathbf{PSp}^{\mathrm{op}}$, $\mathcal{U} : \mathbf{PSp} \to \mathbf{DLat}^{\mathrm{op}}$ can be given as

$$\mathcal{P}(A) = \{x \subseteq L : x \text{ a proper prime filter}\}, \ \mathcal{P}(h)(x) = h^{-1}[x],$$
$$U(X) = \{U \subseteq X : U \text{ a clopen up-set}\}, \ \mathcal{U}(f)(U) = f^{-1}[U]$$

with the lattice structure of $\mathcal{U}(X)$ given by inclusion, and the topology of $\mathcal{P}(A)$ induced by the topology of the product $\mathbf{2}^A$, with the prime filters viewed, for the moment, as the corresponding maps $A \to \mathbf{2}$. Thus the topology is determined by the basis

$$C(a,b) = \{x \mid b \notin x \ni a\}, \ a, b \in A.$$

Since **DLat** has all products, **PSp** has all coproducts. They are specific compactifications of the topological sum (disjoint union) of the spaces in question. For the facts about the structure of coproducts necessary for what follows, see 2.1 below; for a more thorough treatment see [8]. Some aspects of their order structure have been recently studied, e.g., in [1], [2], [3], and [4].

We will need, to start with, the coproduct of I many one-point spaces. Since the Priestley dual $\mathcal{P}(\cdot)$ of the one-point space is the two-point lattice $\mathbf{2}$, the coproduct is the Čech-Stone compactification of the discrete space I,

$$\beta I = \mathcal{P}(\mathbf{2}^I) = \{u : u \text{ an ultrafilter on } I\}.$$

2.3. **Finite poset conventions.** Finite posets are automatically Priestley spaces and will be viewed as such. Connected finite posets will be referred to as *configurations.* A *tree*, or more precisely a *rooted tree*, is a configuration T with the feature that $\uparrow x$ is a chain for each $x \in T$. A *co-tree* is a poset P such the opposite poset P^{op} is a tree. A *forest* is a disjoint union of trees; similarly we speak of *co-forests.* Note that forests are characterized by the non-existence of an induced poset isomorphic to $\{0 < 1, 2\}$.

A *combinatorial tree* is a tree as in combinatorics, i.e., an acyclic configuration, or more precisely, a configuration whose Hasse diagram, viewed as a graph, contains no cycle. Note that combinatorial trees are much more general than rooted trees and co-trees. Disjoint sums of combinatorial trees will be called *combinatorial forests*. In this article, trees, forests, etc., will always be finite.

The embedding, or, rather, embeddability of a poset P into a Priestley space will be indicated by

$$P \hookrightarrow X.$$

That is, $P \hookrightarrow X$ indicates the existence of a mapping $f : P \to X$ such that $x \le y$ iff $f(x) \le f(y)$; such a mapping is called a *copy* of P in X. We write

$$P \not\hookrightarrow X$$

if X contains no copy of P.

2.4. **Prime filters and ideals.** We will need an extension of the Birkhoff prime filter lemma. This result, Proposition 2.4.2, is a refinement of, and is foreshadowed by, Section 3 of [1].

Suppose we are given a bounded distributive lattice L, a finite tree T with root t_0, and two maps from T into L, $t \longmapsto a_t$ and $t \longmapsto b_t$. Using the a_t's and b_t's as parameters and c as a free variable, we define formulas $\psi_t(c)$, $t \in T$, in the first order language of bounded distributive lattices. The definition is inductive, starting with minimal elements and working upward through the tree. For $t \in \min(T)$ we define $\psi_t(c)$ to be

$$c \vee \bigvee_{s \not\le t} a_s \vee \bigvee_{s \ge t} b_s \ge a_t \wedge \bigwedge_{s \not\ge t} b_s,$$

and for $t \in T \smallsetminus \min(T)$ we define $\psi_t(c)$ to be

$$\mathop{\exists}_{s<t} c_s \left(\mathop{\&}_{s<t} \psi_s(c_s) \;\;\&\;\; \left(c \vee \bigvee_{s \not\le t} a_s \vee \bigvee_{s \ge t} b_s \ge \bigwedge_{s<t} c_s \wedge \bigwedge_{s \le t} a_s \wedge \bigwedge_{s \not\ge t} b_s \right)\right).$$

(Here & stands for logical conjunction.) Let F_t designate the set of those elements of L which satisfy ψ_t.

Lemma 2.4.1. *The following hold for all* $s, t \in T$.

1. F_t *is a filter containing* a_s *for* $s \le t$ *and* b_s *for* $s \not\ge t$.
2. *If* $s \le t$ *then* $F_s \subseteq F_t$. *Hence* F_{t_0} *is proper iff all the* F_t*'s are proper.*
3. *If* F_{t_0} *is proper then* $a_s \in F_t$ *iff* $s \le t$ *and* $b_s \in F_t$ *iff* $s \not\ge t$.
4. *If* $c \in F_t$ *and* $d \vee \bigvee_{s \not\le t} a_s \vee \bigvee_{s \ge t} b_s \ge c$ *then* $d \in F_t$.

Proof. Parts (1) and (2) yield to a simple bottom-up induction on T, while part (3) yields to a simple top-down induction. Part (4) is likewise easy to check. □

Proposition 2.4.2. *Let L be a bounded distributive lattice with Priestley space X, let T be a finite tree with root t_0, and let $t \longmapsto a_t$ and $t \longmapsto b_t$ be two maps from T into L. Then the following are equivalent.*

(1) *There is a copy $t \longmapsto x_t$ of T in X such that, for all $s, t \in T$, $a_s \in x_t$ iff $s \leq t$, and $b_s \in x_t$ iff $s \not\geq t$.*
(2) *F_{t_0} is proper, i.e., $L \models \neg\psi_{t_0}(0)$.*

Proof. Suppose $t \longmapsto x_t$ satisfies (1), and define filters F_t as above. Then a routine bottom-up induction on T establishes that $F_t \subseteq x_t$ for all t, and since x_{t_0} is a proper prime filter on L, (2) follows.

Assuming that (2) holds, we define a copy $t \longmapsto x_t$ of T in X inductively, this time working down from the top of the tree. Since F_{t_0} is proper, it must omit

$$\bigvee_{s \not\leq t_0} a_s \vee \bigvee_{s \geq t_0} b_s = b_{t_0}.$$

Let x_{t_0} be any prime filter containing F_{t_0} and omitting b_{t_0}. Now suppose that a prime filter $x_t \supseteq F_t$ has been defined so that $a_s \in x_t$ iff $s \leq t$ and $b_s \in x_t$ iff $s \not\geq t$, and consider $r \prec t$. We claim that the ideal generated by

$$(L \smallsetminus x_t) \cup \{\bigvee_{s \not\leq r} a_s, \bigvee_{s \geq r} b_s\}$$

is disjoint from F_r. For otherwise there would be lattice elements $c \notin x_t$ and $c_r \in F_r$ such that

$$c \vee \bigvee_{s \not\leq r} a_s \vee \bigvee_{s \geq r} b_s \geq c_s,$$

a circumstance which would force c into F_r by Lemma 2.4.1(4), and, since $F_r \subseteq F_t$, would contradict the assumption that $c \notin x_t \supseteq F_t$. Let x_r be any prime filter separating the aforementioned ideal from F_r, and continue the induction. □

2.5. **A set-theoretical assumption.** At one place we will need the continuum hypothesis. It will be indicated, as usual, by (CH).

The reader wishing for more information on posets can consult [5] and [7]. From category theory we need, in fact, only the basic terminology as introduced, e.g., in [10].

3. Free summands and their tame parts

Given Priestley spaces X_i, $i \in I$, we will represent their coproduct as

$$X = \coprod_{i \in I} X_i = \mathcal{P}(A),$$

where $A \equiv \prod_{i\in I} A_i$ and $A_i \equiv \mathcal{U}(X_i)$.

3.1. **The Koubek-Sichler analysis.** The structure of X is greatly elucidated by the penetrating analysis of Koubek and Sichler in [8]. Two insights play a crucial role in the investigation at hand. The first is a particular decomposition of X into convenient disjoint closed subspaces. *This decomposition, and the notational conventions needed to express it, will be assumed in what follows.*

Consider the natural embedding $\iota : \mathbf{2}^I \to A$ defined by

$$\iota(J)(i) = \begin{cases} 1 & \text{if } i \in J \\ 0 & \text{if } i \notin J \end{cases}, \ i \in J,\ J \subseteq I,$$

and the corresponding Priestley map $\varepsilon = \mathcal{P}(\iota) : X \to \beta I$, explicitly given by the formula $\varepsilon(x) = \iota^{-1}(x)$, $x \in X$. This divides X into disjoint closed subspaces

$$X_u \equiv \varepsilon^{-1}\{u\},\ u \in \beta I,$$

and, since βI is order trivial, these subspaces are order independent. We refer to the subspaces X_u, $u \in \beta I$, as *summands* of the coproduct, *fixed* or *free* depending on whether u is a fixed (meaning of the form $i^* \equiv \{J \subseteq I : i \in J\}$, $i \in I$) or a free (meaning not fixed) ultrafilter.

The fixed summands play a special role since, for $i \in I$, the coproduct insertion $\rho_i : X_i \to X$ is defined by

$$\rho_i(x) = \{a : x \in a(i)\}, \quad x \in X_i,$$

and this map, which can readily be seen to be injective, takes X_i onto X_{i^*}. When each X_i is identified with X_{i^*}, the (topological) sum $\bigcup_i X_i$ is then dense in $\coprod_I X_i$.

The free summands are even more interesting. Let u be a free ultrafilter in I. Consider the ultraproduct A_u, constructed, as usual, as follows. For $a, b \in A$, set $a \sim_u b$ iff $a\,(i) = b\,(i)$ almost everywhere, i.e., iff there is some $J \in u$ such that $a(i) = b(i)$ for all $i \in J$. In other words,

$$a \sim_u b \quad \text{iff} \quad a \wedge \iota(J) = b \wedge \iota(J) \text{ for some } J \in u.$$

Then $A_u = A/\sim_u$.

Lemma 3.1.1. *Let u be a free ultrafilter on I. Then for any $x \in X$,*

$$x \in X_u \quad \textit{iff} \quad \forall\, a, b \in A\, ((a \in x \;\&\; b \sim_u a) \Rightarrow b \in x).$$

Consequently, X_u is isomorphic to $\mathcal{P}\,(A_u)$.

Proof. If $x \in X_u$ then $\varepsilon\,(x) = \iota^{-1}\,(x) = u$, so that if $a \in x$ and if $a \wedge \iota\,(J) = b \wedge \iota\,(J)$ for some $J \in u$ then, since $\iota\,(J) \in x$, we have

$$b \geq b \wedge \iota\,(J) \in x \implies b \in x.$$

On the other hand, suppose that x is a point of X such that, for all $a, b \in A$, $b \in x$ whenever $b \sim_u a \in x$. Then for any $J \in u$,

$\iota(J) \sim_u \iota(I) \in x$ implies $\iota(J) \in x$, i.e., $J \in \varepsilon(x)$. But, since u is an ultrafilter, $u \subseteq \varepsilon(x)$ implies $u = \varepsilon(x)$, i.e., $x \in X_u$.

To establish the second statement, simply note that the lattice surjection $p : A \to A_u$ given by $a \longmapsto \{b : a \sim_u b\}$, $a \in A$, has an injective dual $\mathcal{P}(p) = p^{-1}$ whose range is precisely the set of those prime filters x on A which are closed under $\sim_u$, i.e., X_u. □

The second important insight of Koubek and Sichler has to do with the topology on X.

Proposition 3.1.2 ([8])**.** *The topology of $X = \coprod_I X_i$, I infinite, is that of the Čech-Stone compactification of the topological sum (disjoint union $\bigcup_I X_i$) iff the heights of all but finitely many of the X_i's are bounded by a fixed nonnegative integer.*

3.2. **Embedding a configuration in a free summand.** Recall [1] and [3]. For every combinatorial forest P, and in particular for every forest or co-forest, there is a sentence ψ_P in the first-order theory of bounded distributive lattices such that, for every Priestley space Z with lattice $A \equiv \mathcal{U}(Z)$,

$$P \hookrightarrow Z \text{ iff } A \models \psi_P.$$

Since first order sentences are preserved by ultraproducts by Łóś's Theorem ([9]), we have the following consequence.

Proposition 3.2.1. *Let u be a free ultrafilter on I. For a combinatorial forest P, and in particular for a forest or co-forest,*

$$P \hookrightarrow X_u \text{ iff } P \hookrightarrow X_i \text{ for almost all } i,$$

i.e., iff there is some $J \in u$ such that $P \hookrightarrow X_i$ for all $i \in J$.

Remark 3.2.2. Note that, for a fixed configuration P, "$P \hookrightarrow Z$" can be expressed by a sentence in the first order theory of *partial orders.* (This is not to be confused with the sentence ψ_P in the first order theory of *bounded distributive lattices.*) Proposition 3.2.1 states that *some* first order sentences that hold in the posets X_i are preserved in the free summands X_u. But not all are; for instance, the sentences expressing "$P \hookrightarrow Z$" with cyclic P are not; see, e.g., [2] and [4].

3.3. **The tame part of X_u.** Let $Y \equiv \prod_i X_i$. For $y \in Y$, define the bounded lattice homomorphism $\phi_y : A \to 2^I$ by

$$\phi_y(a) \equiv \{i : y(i) \in a(i)\},\ a \in A.$$

Let τ'_y stand for $\mathcal{P}(\phi_y) : \beta I \to X$, so that

$$\tau'_y(u) = \{a : \phi_y(a) \in u\},$$

One readily checks that $\phi_y \iota$ is the identity map on 2^I, in consequence of which $\varepsilon \tau'_y$ is the identity on βI, so that, for $u \in \beta I$,

$$\tau'_y(u) \in X_u.$$

Moreover, for $i \in I$,

$$\tau'_y(i^*) = \{a : \phi_y(a) \in i^*\} = \{a : y(i) \in a(i)\} = \rho_i(y(i)).$$

Thus we obtain the map $\tau' : Y \times \beta I \to X$ defined by

$$\tau'(y, u) = \tau'_y(u).$$

This map is continuous in the second coordinate, i.e., when y is held constant, but not in the first. Nevertheless, our interest lies in fixing the second coordinate and letting the first vary. *Therefore in this section u will represent a specified free ultrafilter in βI.*

Define $\tau'_u : Y \to X_u$ by setting

$$\begin{aligned}\tau'_u(y) \equiv \tau'(y, u) = \tau'_y(u) &= \{a : \phi_y(a) \in u\} \\ &= \{a : y(i) \in a(i) \text{ almost everywhere}\}.\end{aligned}$$

Lemma 3.3.1. *For $y_i \in Y$, $\tau'_u(y_1) = \tau'_u(y_2)$ iff $y(i) = y'(i)$ for almost all i, i.e., iff $\{i : y(i) = y'(i)\} \in u$.*

Proof. Set $K = \{i : y_1(i) = y_2(i)\}$. Let $\tau'_u(y_1) = \tau'_u(y_2)$, so that, for all $a \in A$,

$$\phi_{y_1}(a) \in u \quad \text{iff} \quad \phi_{y_2}(a) \in u$$

Suppose that $K \notin u$, so that either $J_1 \equiv \{i : y_1(i) \nleq y_2(i)\}$ or $J_2 \equiv \{i : y_2(i) \nleq y_1(i)\}$ lies in u, say J_1. For $i \in J_1$, use the total order disconnectivity of X_i to find $a(i) \in A_i$ such that $y(2) \notin a(i) \ni y_1(i)$, and otherwise choose $a(i)$ arbitrarily. Then $\phi_{y_1}(a) \supseteq J_1$ is in u while $\phi_{y_2}(a) \subseteq I \setminus J_1$ is not, contrary to the condition displayed above.

Now suppose that $K \in u$. If $\phi_{y_1}(a) \in u$ then $\phi_{y_2}(a) \supseteq \phi_{y_1}(a) \cap K$ is in u as well, and similarly $\phi_{y_2}(a) \in u$ implies $\phi_{y_1}(a) \in u$. □

Define an equivalence $\simeq_u$ on $Y = \prod_i X_i$ by declaring $y_1 \simeq_u y_2$ iff $y_1(i) = y_2(i)$ for almost all i, write $\widehat{y}$ for the equivalence class of $y \in Y$, and let

$$Y_u \equiv Y/\simeq_u = \{\widehat{y} : y \in Y\}$$

stand for the set of all equivalence classes, partially ordered by declaring $\widehat{y}_1 \geq \widehat{y}_2$ iff $y_1(i) \geq y_2(i)$ for almost all i. In sum, Y_u is the ultraproduct $\prod_u X_i$, so that we have the following by Łoś's Theorem [9].

Lemma 3.3.2. *For any sentence ϕ in the first order theory of partial orders, $Y_u \models \phi$ iff $X_i \models \phi$ for almost all i. In particular, we have the following.*

1. *Y_u has height (width) at most n iff almost all X_i's have height (width) at most n.*
2. *Y_u is finite iff almost all X_i's have height and width at most n for some fixed n. In this case Y_u is isomorphic to almost all X_i's.*
3. *For any configuration P, $P \hookrightarrow Y_u$ iff $P \hookrightarrow X_i$ for almost all i.*

Thus we can define a mapping $\tau_u : Y_u \longrightarrow X_u$ by setting

$$\tau_u(\widehat{y}) \equiv \tau'_u(y).$$

This mapping is one-one and satisfies $\tau_{i^*}(\widehat{y}) = \rho_i(y(i))$.

Lemma 3.3.3. *τ_u is an order embedding, that is,*

$$\tau_u(\widehat{y}_1) \leq \tau_u(\widehat{y}_2) \quad \textit{iff} \quad \widehat{y}_1 \leq \widehat{y}_2.$$

Proof. Set $K \equiv \{i : y_1(i) \leq y_2(i)\}$, so that $\widehat{y}_1 \leq \widehat{y}_2$ is equivalent to $K \in u$. If $K \in u$, and if $a \in \tau_u(\widehat{y}_1) = \tau'_u(y_1)$, then $\phi_{y_1}(a) \in u$, hence $\phi_{y_2}(a) \supseteq \phi_{y_1}(a) \cap K$ is in u again, and we have $a \in \tau'_u(y_2) = \tau_u(\widehat{y}_2)$. On the other hand, assume $\tau'_u(y_1) \subseteq \tau'_u(y_2)$, which is to say that, for all $a \in A$,

$$\phi_{y_1}(a) \in u \Rightarrow \phi_{y_2}(a) \in u.$$

Suppose $K \notin u$, so that $J = I \setminus K$ is in u. For each $i \in J$ we can choose $a(i) \in A_i$ such that $y_2(i) \notin a(i) \ni y_1(i)$ and, after defining $a(i)$ arbitrarily for $i \notin J$, we obtain $\phi_{y_1}(a) \in u$ while $\phi_{y_2}(a) \notin u$. □

We denote the image of Y_u under τ_u by

$$X_u^{\partial} \equiv \tau_u[Y_u] = \tau'_u[Y],$$

and we refer to X_u^{∂} as the *tame part* of the free summand X_u. .

Proposition 3.3.4. *The tame part X_u^{∂} is dense in X_u.*

Proof. Recall from Subsection 2.2 that the basic sets are of the form $C(a,b) = \{x \mid b \notin x \ni a\}$, $a, b \in A$. We will prove that whenever $C(a,b) \cap X_u \neq \emptyset$ then there is a $y \in Y = \prod X_i$ such that $\tau'_u(y) \in C(a,b)$.

Take an $x_0 \in C(a,b) \cap X_u$, so that $b \notin x_0 \ni a$. This implies that

$$J \equiv \{i : a(i) \not\leq b(i)\} \in u,$$

since otherwise

$$\{i : a(i) \leq b(i)\} = \{i : a(i) = (a \wedge b)(i)\} \in u,$$

which would imply $a \wedge b \in x_0$ by Lemma 3.1.1, resulting in the contradiction $b \in x_0$. For $i \in J$ choose $y(i) \in a(i) \setminus b(i)$, and for $i \notin J$ choose $y(i)$ arbitrarily. Now

$$\tau'_u(y) = \tau'_y(u) = \{c : \phi_y(c) \in u\} = \{c : \{i : y(i) \in c(i)\} \in u\},$$

and hence $a \in \tau'_u(y)$ since $\{i : y(i) \in a(i)\} \supseteq J$, and $b \notin \tau'_u(y)$ since $\{i : y(i) \in b(i)\} \subseteq I \smallsetminus J \notin u$. □

Moreover, not only are the points of X_u^{∂} dense in X_u, but whole copies of rooted trees are simultaneously dense in X_u, in a sense made precise as follows.

Proposition 3.3.5. *Let T be a rooted tree, and let $t \mapsto x_t$ be a copy of T in X_u. Let U_t be an arbitrary neighborhood of x_t for each $t \in T$. Then there exists a copy $t \longmapsto x'_t$ of T in X_u^{∂} such that $x'_t \in U_t$ for each $t \in T$.*

Proof. For each $t \in T$, pick $a'_t \in x_t$ and $b'_t \notin x_t$ such that

$$x_t \in C(a'_t, b'_t) \subseteq U_t.$$

For $s \nleq t$ choose $c(s,t) \in x_s \smallsetminus x_t$, and set

$$a_s \equiv a'_s \wedge \bigwedge_{s \nleq t} c(s,t),\ b_s \equiv b'_s \vee \bigvee_{s \ngeq t} c(t,s).$$

For $s, t \in T$ we evidently have $a_s \in x_t$ iff $s \leq t$, and $b_s \in x_t$ iff $s \ngeq t$. Now use the a_t's and b_t's to form formulas $\psi_t(c)$, and define their corresponding filters F_t, $t \in T$, as in Lemma 2.4.1. In light of the fact that $X_u \cong \mathcal{P}(A_u)$ by Lemma 3.1.1, we then have that $A_u \models \neg\psi_{t_0}(0)$, courtesy of Proposition 2.4.2. Applying Łoś's Theorem ([9]), we deduce the existence of a subset $I \supseteq J \in u$ such that $A_i \models \neg\psi_{t_0}(0)$ for all $i \in J$. But this means that $T \hookrightarrow X_i$ for all $i \in J$, again courtesy of Proposition 2.4.2, and we can use these copies of T to form a copy of T in X_u, as follows.

For each $i \in J$, Proposition 2.4.2 provides a copy $t \longmapsto y_t(i)$ of T in X_i such that $a_s(i) \in y_t(i)$ iff $s \leq t$ and $b_s(i) \in y_t(i)$ iff $s \ngeq t$. If we define $y_t(i)$ arbitrarily for $i \in I \smallsetminus J$, we get a copy $t \longmapsto \widehat{y}_t$ of T in Y_u, so that $t \longmapsto x'_t \equiv \tau_u(\widehat{x}_t)$ is a copy of T in X_u. This is the copy we seek, since it has the feature that $a_s \in x'_t$ iff $s \leq t$ and $b_s \in x'_t$ iff $s \ngeq t$, with the result that

$$x'_t \in C(a_t, b_t) \subseteq C(a'_t, b'_t) \subseteq U_t$$

for all $t \in T$. □

By a more involved argument imitating the procedure from [3], one can extend Proposition 3.3.5 to general combinatorial trees.

3.4. The case of discrete summands X_i.

Lemma 3.4.1. *If almost all X_i's are finite, then every point of X_u^∂ is isolated in X_u.*

Proof. Consider an arbitrary point $x \in X_u^\partial$, say $x = \tau'_u(y)$ for $y \in Y$, and fix a subset $I \supseteq J \in u$ such that X_i is finite for all $i \in J$. Define $a, b \in A$ by setting $a(i) \equiv \uparrow y(i)$ and $b(i) \equiv X_i \smallsetminus \downarrow y(i)$ for $i \in J$, and by defining $a(i) = b(i) = 0$ for $i \in I \smallsetminus J$. We claim that $C(a,b) \cap X_u = \{x\}$. For surely $x \in C(a,b)$, i.e., $b \notin x \ni a$, since

$$\{i \in I : a(i) \ni y(i) \notin b(i)\} \supseteq J \in u.$$

Consider $x' \in C(a,b) \cap X_u$, say $x' = \tau'(y')$ for $y' \in Y$. The fact that $b \notin x' \ni a$ implies that

$$\{i \in I : a(i) \ni y'(i) \notin b(i)\} \equiv K \in u,$$

and, since $K \subseteq J$, it follows that $y'(i) = y(i)$ for all $i \in K$, which is to say that $y \sim_u y'$. The upshot is that $x = \tau'_u(y) = \tau'_u(y') = x'$. This completes the proof of the claim and of the lemma. □

We summarize the situation.

Theorem 3.4.2. *Let $\{X_i : i \in I\}$ be an infinite family of non-pairwise-isomorphic finite Priestley spaces. Then the following hold for each free summand X_u.*

(1) X_u^∂ *is the set of isolated points of* X_u.
(2) X_u^∂ *is discrete, open, and dense in* X_u.
(3) $\left|X_u^\partial\right| = 2^\omega$.
(4) $X_u \smallsetminus X_u^\partial$ *is non-void and compact.*

Proof. X_u^∂ is dense by Proposition 3.3.4 and discrete, hence open, by Lemma 3.4.1. It follows that all isolated points of X_u lie in X_u^∂, and that $X_u \smallsetminus X_u^\partial$ is closed and hence compact. Since the X_i's are non-isomorphic, there are only finitely many X_i's of size $\leq n$ for each positive integer n, and from this it follows that I is countable. We claim that X_u^∂ is infinite. That is because there is, for each positive integer n, a sentence ψ_n in the first-order theory of partially ordered sets such that, for any poset Z, $Z \models \psi_n$ iff $|Z| \geq n$, and since $X_i \models \psi_n$ for almost all i, $X_u \models |X_u|$ hence $|X_u| \geq n$. The claim proves that $X_u \smallsetminus X_u^\partial \neq \emptyset$, but it also implies by [6] that $|X_u^\partial| \geq 2^\omega$. But since I is countable, the ultraproduct X_u^∂ cannot be bigger than 2^ω. $\square$

4. Coproducts of chains

By Proposition 3.1.2, the topology of $X = \coprod_{i\in I} X_i$, I infinite, is that of the Čech-Stone compactification of the disjoint sum $\bigcup X_i$ iff the heights of the X_i's are bounded by a fixed nonnegative integer. Thus the simplest coproduct with a non-standard topology is the coproduct of a family of increasing finite chains. This section is devoted to a complete analysis of this situation, using the facts from the preceding sections. It turns out that the situation is not quite simple, but is nevertheless transparent.

To fix notation, let

$$X_i \equiv \left\{x_0^i < x_1^i < x_2^i < \ldots < x_i^i\right\},\ i \in \mathbb{N}.$$

As before, set $A_i \equiv \mathcal{U}(X_i)$, $A \equiv \prod_{\mathbb{N}} A_i$, and $X \equiv \mathcal{P}(A)$.

4.1. Y_u **is independent of** u**.** Let u represent a free ultrafilter on $\mathbb{N}$, let $Y \equiv \prod_{\mathbb{N}} X_i$, and let $Y_u \equiv Y/\simeq_u$.

Lemma 4.1.1. *Y_u is a chain with greatest element $\widehat{y}_\top$ and least element $\widehat{y}_\perp$ given by*

$$y_\top(i) = x_i^i \text{ and } y_\perp(i) = x_i^0,\ i \in \mathbb{N}.$$

Furthermore, each element of Y_u except the greatest (least) has a successor (predecessor).

Proof. This follows immediately from Lemma 3.3.2, since the properties mentioned are all first order. $\square$

We next show that Y_u has some of the features of an η_1-set. An η_1-set is a chain C of uncountable cofinality and coinitiality such that any two countable subsets $A < B$ (meaning $a < b$ for all $a \in A$ and $b \in B$) possess an intermediate element $c \in C$ with $A < c < B$ (meaning $a < c < b$ for all $a \in A$ and $b \in B$). The concept was introduced and developed by Hausdorff; a good background reference is [7].

Lemma 4.1.2.

(1) *For any strictly increasing sequence* $\{\widehat{y}_n\} \subseteq Y_u$ *(strictly decreasing sequence* $\{\widehat{z}_n\} \subseteq Y_u$*) there exists a point* $\widehat{w} \in Y_u$ *such that* $\widehat{y}_n < \widehat{w} < \widehat{y}_\top$ *(*$\widehat{z}_n > \widehat{w} < \widehat{y}_\bot$*) for all* n.

(2) *For any sequences* $\{\widehat{y}_n\}, \{\widehat{z}_n\} \subseteq Y_u$ *such that* $\widehat{y}_n < \widehat{y}_{n+1} < \widehat{z}_{n+1} < \widehat{z}_n$ *for all* n *there exists a point* $\widehat{w} \in Y_u$ *such that* $\widehat{y}_n < \widehat{w} < \widehat{z}_n$ *for all* n.

Proof. We prove (2), the proof for (1) being similar. Set $J_0 \equiv \mathbb{N}$ and

$$J_1 \equiv \{i \in \mathbb{N} : y_1(i) < z_1(i)\},$$
$$J_{n+1} \equiv \{i \in \mathbb{N} : y_j(i) < y_{j+1}(i) < z_{j+1}(i) < z_j(i), 1 \le j \le n\}, \ n \in \mathbb{N}.$$

Then $J_i \supseteq J_{i+1}$ and $J_i \in u$ for all i, and $\bigcap J_i = \emptyset$ since all the X_i's are finite. For each index $k \in \mathbb{N}$, let $j(k)$ be the least integer i for which $k \in J_i$. Define $w \in Y$ by setting

$$w(i) \equiv \begin{cases} x_i^0 & \text{if } j(i) = 0, \\ y_{j(i)}(i) & \text{if } j(i) > 0. \end{cases}$$

By construction, $y_n(i) < w(i) < z_n(i)$ for all $i \in J_{n+1} \in u$, hence $\widehat{y}_n < \widehat{w} < \widehat{z}_n$ for all n. □

We hasten to point out that Y_u is no η_1-set, for two reasons: every element has either a predecessor or a successor, and Y_u has both a greatest and a least element. But these are the only reasons it is not; if we simply identify every element with its predecessor and successor, and ignore the top and bottom elements, the result is an η_1-set in a sense we now make precise.

Let us impose an equivalence relation on Y_u by declaring $\widehat{y}_1 \sim \widehat{y}_2$ iff

$$[\widehat{y}_1, \widehat{y}_2] = \{\widehat{y} \in Y_u : \widehat{y}_1 \le \widehat{y} \le \widehat{y}_2\}$$

is finite. Let $\widetilde{y} \equiv \{\widehat{y}_1 : \widehat{y}_1 \sim \widehat{y}\}$ designate the equivalence class of $\widehat{y} \in Y_u$. Then the structure of these equivalence classes is evident.

Lemma 4.1.3. $\widetilde{y}_\bot$ *is order isomorphic to* $\mathbb{N}$, $\widetilde{y}_\top$ *is order isomorphic to* $\mathbb{N}^{op}$, *and all other equivalence classes are isomorphic to* $\mathbb{Z}$.

Since these equivalence classes are order convex, i.e., since $\widehat{y}_1 \ge \widehat{y} \ge \widehat{y}_2$ and $\widehat{y}_1 \sim \widehat{y}_2$ imply $\widehat{y} \sim \widehat{y}_i$ for $\widehat{y}, \widehat{y}_i \in Y_u$, the set

$$\widetilde{Y}_u \equiv \{\widetilde{y} : \widehat{y} \in Y_u\}$$

inherits a total order which makes the projection map $\widehat{y} \longmapsto \widetilde{y}$ order preserving. Finally, set

$$C_u \equiv \widetilde{Y}_u \smallsetminus \{\widetilde{y}_\bot, \widetilde{y}_\top\}.$$

Lemma 4.1.4. *C_u is an η_1-set.*

Proof. This follows from Lemma 1 by a straightforward argument in several cases. We leave the details to the reader. □

Let C and D be disjoint chains. Their *ordered sum*, written $C \oplus D$, is their disjoint union $C \cup D$ ordered by declaring that $c < d$ for all $c \in C$ and $d \in D$. Their *lexicographic product*, written $C \vec{\times} D$, is their Cartesian product $C \times D$ ordered by declaring

$$(c_1, d_1) \geq (c_2, d_2) \text{ iff } (c_1 > c_2 \text{ or } (c_1 = c_2 \text{ and } d_1 \geq d_2)).$$

Proposition 4.1.5 (CH)**.** *Let u and v be free ultrafilters on $\mathbb{N}$. Then Y_u is order isomorphic to Y_v. Both are order isomorphic to*

$$\mathbb{N} \oplus \left(S \vec{\times} \mathbb{Z}\right) \oplus \mathbb{N}^{op},$$

where S is an η_1-set of cardinality $\aleph_1$.

Proof. Any two η_1-sets of cardinality $\aleph_1$ are order isomorphic, and, under the Continuum Hypothesis, this applies to both C_u and C_v. Since Y_u and Y_v are obtained by simply replacing the points of C_u and C_v by equivalence classes whose order types are spelled out in Lemma 4.1.3, the result follows. □

4.2. **The topology on X_u is the interval topology.** The analysis begins with an observations based on general principles. We make use of the fact that a bounded lattice is totally ordered iff its Priestley space is.

Lemma 4.2.1. *X_u is totally ordered.*

Proof. Since the property of being totally ordered is expressible by a formula ψ in the first-order theory of bounded lattices, $A_i \models \psi$ for all $i \in \mathbb{N}$, hence $A_u \models \psi$ by Łoś's Theorem ([9]) and A_u is totally ordered, hence X_u is totally ordered. □

By Lemma 4.1.1, X_u^∂ has least and greatest elements

$$x_\bot \equiv \tau'_u(y_\bot) = \left\{a \in A : x_0^i \in a(i) \text{ for almost all } i\right\} \text{ and}$$
$$x_\top \equiv \tau'_u(y_\top) = \left\{a \in A : x_i^i \in a(i) \text{ for almost all } i\right\},$$

and these elements have the same status in X_u.

Lemma 4.2.2. *$x_\bot$ and $x_\top$ are the least and greatest elements, respectively, of X_u.*

Proof. Consider an arbitrary $x \in X_u$. Since x is a proper prime filter on A, it must contain the greatest element but cannot contain the least element of A, these elements being defined by

$$0_A(i) = \emptyset,\ 1_A(i) = X_i,\ i \in \mathbb{N}.$$

By Lemma 3.1.1, x must contain every element $a \in A$ such that $a \sim_u 1_A$ but cannot contain any element $a \in A$ such that $a \sim_u 0_A$. From this it follows that $x_\perp \le x \le x_\top$. □

Again by Lemma 4.1.1, each element of X_u^∂ except the greatest (least) has a successor (predecessor), and this element is a successor (predecessor) in X_u as well.

Lemma 4.2.3. *If $x \prec x'$ in X_u^∂ then $x \prec x'$ in X_u.*

Proof. Suppose $x = \tau_u'(y)$ and $x' = \tau_u'(y')$ for $y, y' \in Y$. To say that $x \prec x'$ in X_u^∂ is to say that $y(i) \prec y'(i)$ for almost all i. To show that $x \prec x'$ also in X_u, consider an arbitrary $x'' = \tau_u'(y'') \in X_u$ such that $x \le x'' \le x'$. Since $y(i) \le y''(i) \le y(i)$ for almost all i, it follows that either $y''(i) = y(i)$ for almost all i, or $y''(i) = y'(i)$ for almost all i. That is to say that either $x'' = x$ or $x'' = x'$. □

We now investigate the basic open subsets $C(a,b) \cap X_u$ of X_u. For $a \in A$ such that $a \nsim_u 0_A$, define $y_a \in Y$ so that $y_a(i)$ is the least element of $a(i)$ if $a(i) \ne \emptyset$, and define $y_a(i)$ arbitrarily otherwise. Set $x_a \equiv \tau_u'(y_a)$.

Lemma 4.2.4. *For $a \in A$ such that $a \nsim_u 0_A$, and for $x \in X_u$,*

$$a \in x \text{ iff } x \ge x_a.$$

Proof. Upon reflecting that

$$\begin{aligned} x_a \equiv \tau_u'(y_a) &= \{a' \in A : y_a(i) \in a'(i) \text{ for almost all } i\} \\ &= \{a' \in A : a(i) \le a'(i) \text{ for almost all } i\}, \end{aligned}$$

the truth of the displayed condition becomes clear. □

Lemma 4.2.5. *For $a, b \in A$ such that $a \nsim_u 0_A$,*

$$C(a,b) \cap X_u = [x_a, x_b) = \{x \in X_u : x_a \le x < x_b\}.$$

$C(a,b) \cap X_u = \emptyset$ if $a \sim_u 0_A$.

Proof. This follows directly from Lemma 4.2.4. □

Lemma 4.2.6. *The Priestley topology on X_u coincides with its order topology.*

Proof. Lemma 4.2.5 demonstrates that a basic open set in the Priestley topology on X_u is an interval, and this interval is open in the interval topology in light of Lemma 4.2.3, since x_a either has a predecessor in X_u or is its least element. On the other hand, consider an open interval

$$(x, x') = \{x'' \in X_u : x < x'' < x'\}$$

of X_u. If $x'' \in (x, x')$ then $x < x'' < x'$, and by choosing $a \in x'' \smallsetminus x$ and $b \in x' \smallsetminus x''$, we get

$$x'' \in C(a, b) \cap X_u \subseteq (x, x').$$

This shows that (x, x') is open in the Priestley topology on X_u. □

Proposition 4.2.7. *X_u is the Dedekind-MacNeille completion (by cuts) of X_u^∂.*

Proof. It is a general fact that, if X is order-dense in a chain Y, and if Y is compact in the interval topology, then Y is the Dedekind-MacNeille completion of X. □

4.3. X_u is independent of u.

Theorem 4.3.1 (CH)**.** *Let u and v be free ultrafilters on $\mathbb{N}$. Then X_u is order isomorphic to X_v. Both are isomorphic to the Dedekind-MacNeille completion of*

$$\mathbb{N} \oplus \left(S \vec{\times} \mathbb{Z}\right) \oplus \mathbb{N}^{op},$$

where S is an η_1-set of cardinality $\aleph_1$.

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