# Many Facets of Dualities

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**Abstract.** In this paper we survey results related to homomorphism dualities for graphs, and more generally, for finite structures. This is related to some of the classical combinatorial problems, such as colorings of graphs and hypergraphs, and also to recently intensively studied Constraint Satisfaction Problems. On the other side dualities are related to the descriptive complexity and First Order definability as well as to universal graphs. And in yet another context they can be expressed as properties of the homomorphism order of structures. In the contemporary context homomorphism dualities are a complex area and it is our aim to describe some of the main ideas only. However we introduce the four conceptually different proofs of the existence of duals thus indicating the versatility of this notion. Particularly we describe setting of restricted dualities and the role of bounded expansion classes.

## 1 Introduction

Think of 3-colorability of a graph $G$. This is a well known hard (and a canonical NP-complete) problem. From the combinatorial point of view there is a standard way how to approach this problem (and monotone properties in general): we investigate minimal (usually called 4-critical) graphs without this property (i.e. which are not 3-colorable), we then denote by $\mathcal{F}$ the set (or language) of all such critical graphs and we define the set $Forb(\mathcal{F})$ of all structures which do not "contain" any $F \in \mathcal{F}$. Then the language $Forb(\mathcal{F})$ coincides with the language of 3-colorable graphs.

Unfortunately, in this case the set $\mathcal{F}$ is infinite and this seems to be a general phenomenon: in most "interesting" cases the set of minimal forbidden graphs seems to be infinite (and mostly even not "enough" structured). Yet in this paper we study exactly those cases which have finitely many obstructions. Such cases are called *finite homomorphism dualities.* And we shall see that nevertheless finitely many obstructions exists for a

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wide range of problems. However this holds not on the very elementary level. (It has been proved already in [43] that, except in the single trivial case, no coloring problem for undirected graphs admits finitely many minimal obstructions.) We have to generalize and this may seem to be a leitmotiv of this paper. We have to generalize not for its own sake but in order to find a proper setting for concrete problems (like the 3-colorability, or, more generally, Constraint Satisfaction Problems).

There are three main ingredients in our approach:

1. The use of relational structures and their homomorphisms (i.e. we deal with the *category* of graphs and structures).
2. The use of existential statements in the form of lifts and shadows.
3. Restriction of the obstruction characterization to a particular class of structures (such as planar graphs, or proper minor closed classes, or classes with bounded expansion as defined by P. Ossona de Mendez and author, e.g. [48]).

In this paper we survey the development in all 3 directions by giving concrete examples. Accordingly, the paper has 4 sections. In Section 2 we deal with dualities and their characterizations. In Section 3 we survey the results related to lifts and shadows, their descriptive complexity in the context of Constraint Satisfaction Problems (CSP). In Section 4 we relate dualities to classical model theoretic problems related to universal objects. In Section 5 we demonstrate the richness of Restricted Dualities and in the final section we summarize the main results together with some remarks and open problesm.

In the whole paper we deal not only with graphs but also with relational structures. This language is essential for questions which will be considered in this paper. It can be briefly introduced as follows:

A type $\Delta$ is a sequence $(\delta_i; i \in I)$ of positive integers. A *relational structure* $\mathbf{A}$ of *type* $\Delta$ is a pair $(X, (R_i; i \in I))$ where $X$ is a set and $R_i \in X^{\delta_i}$; thus $R_i$ is a $\delta_i$-ary relation on $X$. In this paper we shall always assume that $X$ is a finite set (thus we consider finite relational structures only). relational structures (of type $\Delta$) will be denoted by capital letters $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$.. A relational structure of type $\Delta$ is also called a $\Delta$-*structure* (or just a *structure*). If $\mathbf{A} = (X, (R_i : i \in I))$ we also denote the base set $X$ as $X(\mathbf{A})$ and the relation $R_i$ by $R_i(\mathbf{A})$. We denote by $Rel(\Delta)$ the class of all $\Delta$-type relational structures. The class $Rel(\Delta)$ will be considered as category endowed with all homomorphisms, which are just all relations preserving mappings. To be explicite, for relational structures $\mathbf{A}, \mathbf{B} \in Rel(\Delta)$ a mapping $f : X(\mathbf{A}) \longrightarrow X(\mathbf{B})$ is a *homomorphism*$\mathbf{A} \longrightarrow \mathbf{B}$ if for every relational symbol $R \in \Delta$ and for every tuple $(x_1, \dots, x_t) \in R(\mathbf{A})$ holds $(f(x_1), \dots, f(x_t)) \in R(\mathbf{B})$. The existence of such a homomorphism will be denoted by $\mathbf{A} \longrightarrow \mathbf{B}$ and its non-existence by $\mathbf{A} \not\longrightarrow \mathbf{B}$.

Natural examples of relational structures are abundant: graphs, hypergraphs, graphs with colored edges, ordered graphs, triples modelling betweeness, etc. A rich source and strong motivation is the theory of database queries, e.g. [12, 2, 17, 24, 20] and Constraint Satisfaction Problems [12, 20, 37, 24]. Many of these questions are best formulated in the categorical language of graphs and their homomorphisms. For data systems this goes back to [5], for combinatorial and algebraic setting see e.g.[61, 35, 20]. The notion of homomorphism now plays a role in many problems in areas as diverse as statistical physics, extremal theory, limit structure theorems, see e.g. [3, 20, 52].

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## 2 Finite Dualities

What can be better than finitely many obstructions? Yes, sure, but is this a realistic goal which has some interesting instances? In the undirected graph case the answer is negative [43]. However the properties characterized by a finite set $\mathcal{F}$ are very interesting if we consider them for more complicated structures than undirected graphs only. Towards this end we define the notion of *finite (homomorphism) duality.*

**Definition 1.** *Let $\mathcal{F}$, $\mathcal{D}$ be finite sets of structures (in a fixed class $Rel(\Delta)$). We say that sets $\mathcal{F}$ and $\mathcal{D}$ establish* finite duality *if the following holds for every structure $\mathbf{A} \in Rel(\Delta)$:*

$$\mathbf{F} \not\longrightarrow \mathbf{A} \textit{ for every } \mathbf{F} \in \mathcal{F} \Longleftrightarrow \mathbf{A} \longrightarrow \mathbf{D} \textit{ for some } \mathbf{D} \in \mathcal{D}.$$

*In this case we say that $(\mathcal{F}, \mathcal{D})$ is* dual pair*, and that $\mathcal{D}$ is* dual set *of $\mathcal{F}$.*

The simplest non-trivial instance of dualities is for oriented graphs (again: for undirected graphs we have trivial examples only; see [43] where the term *homomorphism duality* was introduced) and it is usually expressed in terms of orientations of graphs. The connection between chromatic number and orientations is not new and goes back to Gallai and Roy [16, 63]. These pioneering works provided a name ("Gallai-Roy" theorem) for this result although both of these papers are anticipated by M. Hasse [18] and L. M. Vitaver [67] where the same thing is proved (in the more algebraic language). Note that another influential connection between orientations and chromatic number is given by Minty [40] but that goes in a different direction (flows and matroids). For our purposes Gallai-Hasse-Roy-Vitaver theorem takes the following compact form:

**Theorem 1.** *For any directed graph $G$ the following holds:*

$$P_k \not\to G \iff G \to T_k$$

The undefined notions have the following meaning: $P_k$ denotes the directed path of length $k$ (i.e. with $k+1$ vertices) and $T_k$ denotes the transitive tournament with $k$ vertices.

It may be seen easily that for undirected graph this has the following consequence.

**Corollary 1.** *For an undirected graph $G$ the following statements are equivalent:*

1. $\chi(G) \leq k$ *(which is equivalent to $G \to K_k$);*
2. *There exists an orientation $\boldsymbol{G}$ of $G$ such that $\boldsymbol{G} \to T_k$;*
3. *There exists an orientation $\boldsymbol{G}$ of $G$ such that $P_k \not\to \boldsymbol{G}$.*

This particular result was the starting point ([43]) for the following result which characterizes homomorphisms dualities in general classes of relational structures [56, 14]. This characterization involves notion of relational tree and relational forest. These intuitive notions can be defined similarly as for graphs but perhaps the easiest way is to reduce them to graphs by means of the notion of *incidence graph $G_{\mathbf{A}}$ of a structure* $\mathbf{A} = (X, (R_i : i \in I))$. We include this construction for the sake of completeness. The vertex set of $G_{\mathbf{A}}$ is the set $X$ together with all tuples of relations $R_i : i \in I$. ($G_{\mathbf{A}}$ is a bipartite graph, $X$ is on the one side, tuples are on the other side of bipartition.) The edges will be formed by all incidencies between an element and a tuple (such as $x$ and $(x, y, z)$; $G_{\mathbf{A}}$ may have multiple edges such as between $x$ and $(x, x, y)$.Now we can say that a structure $\mathbf{A}$ is a *relational tree* (*relational forest*, respectively) if the incidence graph $G_{\mathbf{A}}$ is a tree (forest, respectively). First we state the theorem for singleton sets only.

**Theorem 2 (Singleton Homomorphism Dualities [56]).**

1. *For every relational tree* $\mathbf{T}$ *there exists a structure* $\mathbf{D_T}$ *(called the* dual of $\mathbf{T}$*) such that the following holds (for every structure* $\mathbf{A}$*):*

$$\mathbf{T} \not\to \mathbf{A} \Longleftrightarrow \mathbf{A} \to \mathbf{D_T}.$$

2. *Up to a homomorphism equivalence there are no other dual pairs (of singleton structures).*

This result was proved for oriented graphs in [25] and in the full generality (and by different methods) in [56]. The characterization of singleton dualities is the basis of the characterization of dual pairs of sets and of finite dualities:

**Theorem 3 (Finite Homomorphism Dualities [56, 14]).**

1. *For every finite set of relational forests* $\mathcal{T}$ *there exists a dual set of structures* $\mathcal{D}_\mathcal{T}$ *such that the following holds (for every structure* $\mathbf{A}$*):*

$$\mathcal{T} \not\to \mathbf{A} \Longleftrightarrow \mathbf{A} \to \mathcal{D}_\mathcal{T}.$$

2. *Up to a homomorphism equivalence there are no other dual pairs (of finite sets of structures).*

(Here we of course write $\mathcal{T} \not\to \mathbf{A}$ if $\mathbf{T} \not\to \mathbf{A}$ for every $\mathbf{T} \in \mathcal{T}$. Similarly, we write $\mathbf{A} \to \mathcal{D}_\mathcal{T}$ if $\mathbf{A} \to \mathbf{D}$ for some $\mathbf{D} \in \mathcal{D}_\mathcal{T}$.)

Theorems 2, 3 are nontrivial in both directions and the existence of duals is established by various means. This will be briefly reviewed. Perhaps the variety of techniques will convince an interested reader that these are interesting objects.

### 2.1 Existence of Duals I. - Explicite Construction

We review here the recent simple construction which appeared in [58, 59]. For the notation simplicity we consider oriented graphs only (i.e. our type is $\Delta = (2)$).

Let $T = (V, E)$ be an oriented tree. We define its dual $D_T = (V', E')$ as follows: The set $V'$ is the set of all mappings $f : V \longrightarrow V$ satisfying that $f(v)$ is adjacent to $v$ for every $v \in V$ (such mappings are called *neighbourly*). For neighbourly mappings $f, g$ we put $(f, g) \in E'$ if $(v, u) \neq (f(u), g(v))$ for every arc $(u, v) \in E$ . The later condition simply means that no arc is jointly "flipped" by the pair $(f, g)$.

*Proof (a rough sketch).* It is easy to see that in order to show that the graph $D_T$ is indeed the dual of the tree $T$ we have to prove two facts:

(i) $T \not\longrightarrow D_T$,
(ii) $T \not\longrightarrow G \Longrightarrow G \longrightarrow D_T$.

(i) is proved by a contradiction: The existence of a homomorphism $\phi : T \longrightarrow D_T$ implies the existence of an infinite walk on $T$ defined as sequence

$$v_0,\ \phi(v_0)(v_0) = v_1,\ \phi(v_1)(v_1) = v_2,\ \dots$$

This walk has to eventually return. Thus there exists $i$ such that $\phi(v_i)(v_i) = v_{i+1}$, $\phi(v_{i+1})(v_{i+1}) = v_i$ and either $(v_i, v_{i+1})$ or $(v_{i+1}, v_i)$ is an arc of $T$. Assume e.g. that $(v_i, v_{i+1})$ is an arc. Then $(\phi(v_i), \phi(v_{i+1}))$ is an arc of $D_T$. However $(\phi(v_i)(v_i), \phi(v_{i+1})(v_{i+1})) = (v_{i+1}, v_i)$, which a contradiction (a flip).

(ii) is proved constructively and the homomorphism $\phi : G \longrightarrow D_T$ is given by the formula

$$\phi(x)(u) = v,$$

where $x \in V(G), u \in V$, $v$ is adjacent to $u$ and $v$ determines a branch $B$ of $T$ at $u$ for which there is no homomorphism $(B, u) \not\longrightarrow (G, x)$. (Such a branch obviously exists by the "freeness" of the tree $T$.) If there are more such branches then we take the first one in a depth first order.

Note that this construction of a dual uses $2^{n \log(n)}$ vertices. Up to logarithmic factor this is optimal as shown in [58, 59]. Indeed, G. Kun and C. Tardif showed recently [33] that almost all oriented paths have exponentially large duals.

## 2.2 Existence of Duals II. - Homomorphism Order

Let us consider the class $Rel(\Delta)$ of all finite relational structures of type $\Delta$. The existence of the homomorphism (i.e. the simplification of the corresponding category) is a quasiorder which becomes a partial order if we factorize by the relation of the homomorphism equivalence. (Structures $\mathbf{A}, \mathbf{B}$ are said to be *homomorphism equivalent* if $\mathbf{A} \longrightarrow \mathbf{B}$ and $\mathbf{B} \longrightarrow \mathbf{A}$.) This partial order is called *homomorphism order* and it is denoted by $\mathcal{C}_\Delta$. It is important that the homomorphism equivalence may be described more effectively by the notion of core: A *core of structure* $\mathbf{A}$ is a minimal retract of $\mathbf{A}$. Core of any finite structure is (up to an isomorphism) unique and thus we can speak about the core of $\mathbf{A}$. Two structures are then homomorphism equivalent if and only if they have isomorphic cores. $\mathcal{C}_\Delta$ is the the set of all non-isomorphic *core* structures ordered by the existence of a homomorphism, see [20] for an introduction to this area. The order of $\mathcal{C}_\Delta$ will be denoted by $\leq$ and its strict version by $<$.

The partial order $\mathcal{C}_\Delta$ has spectacular properties. It is not only a lattice, it is also Heyting poset ([44]) and it has many interesting global and local properties. For example:

- $\mathcal{C}_\Delta$ is (countably) universal, [61, 20],
- $\mathcal{C}_\Delta$ is dense for undirected graphs, [68].

The later property leads to the following:

**Definition 2 (Gaps and Density).** *Let* $\mathbf{A}, \mathbf{B}$ *be structures in* $\mathcal{C}_\Delta$ *and let* $\mathbf{A} < \mathbf{B}$. *If there is no* $\mathbf{C} \in \mathcal{C}_{\boldsymbol{\Delta}}$ *such that* $\mathbf{A} < \mathbf{C} < \mathbf{B}$ *then the pair* $(\mathbf{A}, \mathbf{B})$ *is called a* gap *in* $\mathcal{C}_\Delta$.

*Density problem asks for a characterization of all gaps in* $\mathcal{C}_\Delta$.

Density problem is solved for all classes $\mathcal{C}_\Delta$. The characterization, given in [56] rests on a surprising connection of gaps and dualities.

**Theorem 4.** *Every gap* $\mathbf{A}<\mathbf{B}$ *with* $\mathbf{B}$ *connected, yields a duality* $(\mathbf{B},\mathbf{A}^{\mathbf{B}})$.

**Theorem 5.** *Every (singleton) duality* $(\mathbf{F}, \mathbf{D})$ *yields a gap* $\mathbf{F} \times \mathbf{D} < \mathbf{F}$.

This one-to-one correspondence between (singleton) dualities and (connected) gaps was originally used to prove the existence of duals: We construct for every relational tree $\mathbf{F}$ its (up to the homomorphism equivalence unique) *predecessor* $\mathbf{P_T}$ such that $\mathbf{P_T} < \mathbf{T}$ is a gap and thus $(\mathbf{P_T})^{\mathbf{T}}$ is dual of $\mathbf{T}$. This connection was also used to prove the necessity in both Theorems 2 and 3, see [56].

**Proofs** of Theorems 4, 5 use categorial algebra (and the proofs can be generalized to Heyting posets, [44]): If $\mathbf{A} < \mathbf{B}$ is a gap with $\mathbf{B}$ connected then for arbitrary structure $\mathbf{C}$ consider the object $\mathbf{A} + (\mathbf{B} \times \mathbf{C})$. Clearly $\mathbf{A} \leq \mathbf{A} + (\mathbf{B} \times \mathbf{C}) \leq \mathbf{B}$ and thus either $\mathbf{B} \longrightarrow (\mathbf{B} \times \mathbf{C})$ (by the connectivity of $\mathbf{B}$) and thus also $\mathbf{B} \longrightarrow \mathbf{C}$, or $\mathbf{B} \not\longrightarrow \mathbf{A} + (\mathbf{B} \times \mathbf{C})$ and thus $(\mathbf{B} \times \mathbf{C}) \longrightarrow \mathbf{A}$. But then, (using the definition of the power $\mathbf{A}^{\mathbf{B}}$), we have $\mathbf{C} \longrightarrow \mathbf{A}^{\mathbf{B}}$). This proves Theorem 4.

In Theorem 5 we want to prove that $\mathbf{F} \times \mathbf{D} < \mathbf{F}$ is a gap. We proceed as follows: If $\mathbf{C}$ satisfies $\mathbf{F} \times \mathbf{D} \leq \mathbf{C} \leq \mathbf{F}$ then either $\mathbf{F} \longrightarrow \mathbf{C}$ or not. In the later case $\mathbf{C} \longrightarrow \mathbf{D}$ (by duality) and thus also $\mathbf{C} \longrightarrow \mathbf{F} \times \mathbf{D}$.

It is quite remarkable that the algorithmically motivated concept of the dualities relates so closely to a concept of partial order theory. But duals seem to be natural objects. This is also indicated by the fact that there are two more constructions of duals and they have a different flavour. One of them will be introduced in the next section.

## 3 CSP as Duality

Let us return to our example of 3-colorability. Instead of a graph $G = (V, E)$ we consider the graph $G$ together with three unary relations $C_1$, $C_2$, $C_3$ which *cover* the vertex set $V$; this structure will be denoted by $G'$ and called a *lift* of $G$ (thus $G'$ has one binary and three unary relations). There are 3 *forbidden substructures*: For each $i = 1, 2, 3$ the single edge graph $K_2$ together with cover $C_i = \{1, 2\}$ and $C_j = \emptyset$ for $j \neq i$ form structure $\mathbf{F}'_i$ (where the signature of $\mathbf{F}'_i$ contains one binary and three unary relations). The language of all 3-colorable graphs is just the language $\Phi(Forb(\mathbf{F}'_1, \mathbf{F}'_2, \mathbf{F}'_3))$, where $\Phi$ is the forgetful functor which transforms $G'$ to $G$. We then call $G$ the *shadow* of $G'$.

Clearly this situation can be generalized and one of the main results of paper [32] states that every problem in NP is polynomially equivalent to the membership problem for a class $\Phi(Forb(\mathcal{F}'))$. Here $\mathcal{F}'$ is a finite set of (vertex or pair)-colored digraphs, $Forb(\mathcal{F}')$ is the class of all lifted graphs $G'$ for which there is no homomorphism $F' \longrightarrow G'$ for an $F' \in \mathcal{F}'$. Thus $Forb(\mathcal{F}')$ is the class of all graphs $G'$ with *forbidden* homomorphisms from $\mathcal{F}'$. More presisely this can be done as follows:

Let $\Gamma$ denote a finite set we refer to as colors. A *$\Gamma$-colored structure* (shortly a *colored structure* is a structure together with either a coloring of its vertices or a coloring of all tuples (of arities from the type of the relational structure) of vertices by colors from $\Gamma$. Mostly it suffices to consider the coloring of vertices only. We denote colored relational structures by $\mathbf{A}', \mathbf{B}'$ etc. We call $\mathbf{A}'$ a *lift* of $\mathbf{A}$ and $\mathbf{A}$ is called the *shadow* of $\mathbf{A}'$. We also write $\Phi(\mathbf{A}') = \mathbf{A}$ and we think of $\Phi$ as a forgetful functor (which "forgets" the colors). (Note that both constructions of lifts and shadows are known in model theory and in this context they are called *extension* and *reduct*, see [21]). Our terminology is motivated by Computer Science applications of category theory; yet another approach is given in [37, 38].) Thus (vertex-) colored structures can also be described as *monadic* lifts (monadic meaning that only vertices are colored, only unary relations are added). A homomorphism of relational structures preserves all the edges (arcs). A *homomorphism* of colored relational structures preserves the color of vertices (and colors of tuples), too. We call a mapping between two (colored) digraphs a *full homomorphism* if in addition the preimage of an edge is an edge. Full homomorphisms have very easy structure, as every full homomorphism which is onto is a retraction. The other special homomorphisms we will be interested in are *injective* homomorphisms.

Let $\mathcal{F}'$ be a finite set of colored relational structures. By $Forb(\mathcal{F}')$ we denote the set of all colored relational structures $\mathbf{A}'$ satisfying $\mathbf{F}' \not\longrightarrow \mathbf{A}'$ for every $\mathbf{F}' \in \mathcal{F}'$. (If we use injective or full homomorphisms this will be denoted by $Forb_{inj}(\mathcal{F}')$ or $Forb_{full}(\mathcal{F}')$, respectively.)

Contrary to our common sense, the left side of dualities (i.e. classes of form $Forb(\mathcal{F})$) is more powerfull than the right side. We can prove that shadows of classes $\mathcal{F}'$ have computational power of the whole class NP. More precisely in [31, 32] we proved:

**Theorem 6.** *For every language $L \in NP$ there exist a finite set of colors $\Gamma$ and a finite set of $\Gamma$-colored digraphs $\mathcal{F}'$, where we color the pairs of vertices, such that $L$ is computationally equivalent to the membership problem for the set of all digraphs $G$ for which there exists a $\Gamma$ coloring graph $G'$ of the pairs of vertices in $G$ such that no $F' \in \mathcal{F}'$ is homomorphic to $G'$.*

*Symbolically, $\Phi(Forb(\mathcal{F}'))$ is the class whose membership problem is polynomial equivalent to $L$.*

Similar results hold also for classes $Forb_{inj}(\mathcal{F}')$ and $Forb_{full}(\mathcal{F}')$ (in these cases even with monadic lifts only!), see [31, 32].

Let us consider the right side of dualities: The *Constraint Satisfaction Problem* corresponding to the relational structure $\mathbf{D}$ is the membership problem for the class of all structures defined by $\{\mathbf{B} : \mathbf{B} \longrightarrow \mathbf{D}\}$. Similarly, for a finite set of colored relational structures $\mathcal{D}'$ we denote by $CSP(\mathcal{D}')$ the class of all colored structures $\mathbf{A}'$ satisfying $\mathbf{A}' \longrightarrow \mathbf{D}'$ for some $\mathbf{D}' \in \mathcal{D}'$. ($CSP(\mathcal{D}')$ is a finite union of $CSP(\mathbf{D}')$ for $\mathbf{D}' \in \mathcal{D}'$. This is sometimes denoted by $\rightarrow \mathcal{D}'$.) If the classes $Forb(\mathcal{F}')$ and $CSP(\mathcal{D}')$ are equal then we get a finite homomorphism duality (for the lifted category) which we introduced earlier. Explicitly, in this notation, a finite duality means that the following equivalence holds for every (colored) relational structure $\mathbf{A}'$:

$$\forall \mathbf{F}' \in \mathcal{F}' \quad \mathbf{F}' \not\longrightarrow \mathbf{A}' \iff \exists \mathbf{D}' \in \mathcal{D}' \quad \mathbf{A}' \longrightarrow \mathbf{D}'.$$

By $\Phi$ we denote the forgetful functor which associates to a $\Gamma$-colored relational structure the uncolored one, i.e. it forgets about the coloring. We will investigate classes of the form $\Phi(Forb(\mathcal{F}'))$. We call the pair $(\mathcal{F}', \mathcal{D})$ *shadow duality* if $\Phi(Forb(\mathcal{F}')) = CSP(\mathcal{D})$. An example of shadow duality is the language of 3-colorable graphs discussed in the introduction (or, as we shall see, any CSP problem in general).

We should add one remark. We of course do not only claim that every problem in NP can be polynomially *reduced* to a problem in any of these classes. This would only mean that each of these classes contains an NP-complete problem. What we claim is that these classes have the *computational power* of the whole NP class. More precisely, to each language $L$ in NP there exists a language $M$ in any of these three classes such that $M$ is *polynomially equivalent* to $L$, i.e. there exist *polynomial reductions* of $L$ to $M$ and $M$ to $L$. E.g. assuming P $\neq$ NP there is a language of form $\Phi(Forb(\mathcal{F}'))$ that is neither in P nor NP-complete, since there is such a language in NP by Ladner's celebrated result [34].

The expressive power of classes $\Phi(Forb(\mathcal{F}'))$ corresponds to many combinatorially studied problems and presents a combinatorial counterpart to the celebrated result of Fagin [11] who expressed every NP problem in logical terms by means of an Existential Second Order formula. The proof of Theorem 6 uses refinements of Fagin's theorem due to Feder and Vardi [12].

The fact that the membership problem for classes $\Phi(Forb(\mathcal{F}'))$ (and also their injective and full variants $\Phi(Forb_{inj}(\mathcal{F}'))$ and $\Phi(Forb_{full}(\mathcal{F}'))$, see [32]) have full computational power is pleasing from the combinatorial point of view as these classes cover well known examples of hard combinatorial problems: Ramsey type problems (where as in Theorem 6 we consider edge colored graphs), colorings of bounded degree graphs (defined by an injectivity condition) and structural partitions (studied e.g. in [13]). It follows that, in the full generality, one cannot expect dichotomies here. On the other side of the spectrum, Feder and Vardi have formulated the celebrated *Dichotomy conjecture* for all coloring problems (CSP).

The shadow dualities are related to the decision problems for classes $CSP(\mathbf{D})$.

The main result of [32, 31] presents an easy characterization of those languages $\Phi(Forb(\mathcal{F}'))$ which are coloring problems (CSP):

**Theorem 7.** *Consider the finite set of colors $\Gamma$ and the language $\Phi(Forb(\mathcal{F}'))$ for a finite set $\mathcal{F}'$ of vertex $\Gamma$-colored relational structures.*

*If no $\mathbf{F}' \in \mathcal{F}'$ contains a cycle then there is a finite set of relational structures $\mathcal{D}$ such that $\Phi(Forb(\mathcal{F}')) = CSP(\mathcal{D})$.*

*If one of the lifts $\mathbf{F}'$ in a minimal subfamily of $\mathcal{F}'$ contains a cycle in its core then the language $\Phi(Forb(\mathcal{F}'))$ is not a finite union of CSP languages.*

This can be viewed as an extension of the duality characterization theorem for structures [14]. However the proof given in [32] uses the Theorem 2,3 together with the homomorphism properties of structures not containing short cycles (i.e. with a large girth). This is a combinatorial problem studied intensively since times of P. Erdős. The following result has proved to be repeatedly useful in various applications. It is often called the *Sparse Incomparability Lemma*:

**Theorem 8.** *Let $k, \ell$ be positive integers and let $\mathbf{A}$ be a structure. Then there exists a structure $\mathbf{B}$ with the following properties:*

1. *There exists a homomorphism $f : \mathbf{B} \longrightarrow \mathbf{A}$;*
2. *For every structure $\mathbf{C}$ with at most $k$ points the following holds: there exists a homomorphism $\mathbf{A} \longrightarrow \mathbf{C}$ if and only if there exists a homomorphism $\mathbf{B} \longrightarrow \mathbf{C}$;*
3. *$\mathbf{B}$ has girth $\geq \ell$.*

This result was proved by probabilistic method in [55, 60], see also [20]. The polynomial time construction of $\mathbf{B}$ is possible, too: in the case of binary relations (digraphs) this was done in [39] and for relational structures in [30].

On a higher level Theorem 7 may be interpreted as stability of dualities for finite structures. While shadows of the classes $Forb(\mathcal{F}')$ are computationally equivalent to the whole NP, the shadow dualities are not bringing anything new: these are just shadows of dualities. This is interesting also from the point of view of descriptive complexity as one can show that the coloring problems in the class MMSNP (see [12]) are just shadow finite dualities. This holds for graphs as well for relational structures, see [31, 32] for details of these aspects of dualities.

### 3.1 Existence of Duals III. - Deletion Method

Inspired by the previous connection of lifts and shadows and CSP we can construct the dual structures by monadic lifts. We only sketch the construction which in its spirit goes back to Komárek [25] and it is implicit also in [12] (I thank to V. Dalmau and J. Foniok who informed me about this). Allow me here to mention a bit of history: Pavel Komárek was my student in 80ies and I directed his attention to dualities in the broad setting. I have been convinced that this is a good and elegant approach to *good characterizations* (in the sense of Edmonds) and from this point of view I also wrote Czech graph theory book [41]. With A. Pultr we also wrote [43] where we coined the term duality. We originaly [43] conjectured that Gallai-Hasse-Roy-Vitaver theorem is the only instance of duality for oriented graphs. Nearly 10 years later Komárek quickly found a new example and then infinitely many new examples which were reported in [26]. This revolutionized the scene and we conjectured a converse: that any oriented tree leads to a duality. This has been proved by Komárek in his thesis [25]. The proof has never been published and (unfortunately) Komárek himself did not pursue an academic career. In a different and general setting (and by different techniques) the theorem was proved in [56]. This was the start of the theory covered here.

The deletion method essentially uses monadic lifts. Let us sketch it at least briefly now again on the case of oriented trees (type $\Delta = (2)$). Let $F$ be a fixed (forbidden) tree. Let $(B_i, x_i) : i = 1, ..., t$ be the set consisting of all possible branches which appear in $F$. Thus every branch is determined by a vertex $x_i$ and an edge $e_i$ containg $x_i$. Let $X$ consists from all subsets $I$ of $\{1, 2, \ldots, t\}$ for which there is no homomorphism $F \longrightarrow B_I$ where $B_I$ denotes the disjoint union of all $(B_i, x_i), i \in I$, with all roots $x_i$ identified. $X$ will be the vertex set of our dual graph $D_F$.

The edges of $D_F$ will be defined in two steps: First, we consider all pairs $(I, J)$. And then we delete a pair $(I, J)$ if there exists $i \in I$ and $j \in J$ with an edge $e = (x_i, x_j) \in E(F)$ such that both branches $(B_i, x_i)$ and $(B_j, x_j)$ contain edge $e$.

Of course the language of both [25] and [12] is different and proofs and constructions more complicated So this is a good example of the use of lifts and shadows.

## 4 Universality and Existence of Duals

Homomorphism duality may be rephrased in yet another context. Consider a finite set of connected structures $\mathcal{F}$ and the class Forb($\mathcal{F}$). Then the dual object $\mathbf{D}_{\mathcal{F}}$ is the maximum (or greatest) element of the class Forb($\mathcal{F}$) in the homomorphism order $\mathcal{C}_{\Delta}$. Consequently Theorem 2 characterizes all the classes Forb($F$) which have maximum and Theorem 3 characterizes all classes of form Forb($\mathcal{F}$) which are bounded by a finitely many maximal elements.

We can also say that $\mathbf{D}_{\mathcal{F}}$ is *hom-universal* object ([55]) for the class Forb($\mathcal{F}$). Hom-universal objects should be distinguished from embedding universality: Given a class $\mathcal{K}$ of countable structures, an object $\mathbf{U} \in \mathcal{K}$ is called *(embedding) universal* for $\mathcal{K}$ if for every object $\mathbf{A} \in \mathcal{K}$ there exists an embedding $\mathbf{A} \longrightarrow \mathbf{U}$).

The characterization of those classes $\mathcal{K}$ which have an universal object is a well known open problem in model theory which was studied intensively, see e.g. [27, 28, 7, 4]. The whole area was inspired by the negative results (see [19, 9]): for example the class of graphs not containing $C_l$ (=cycle of length $l$) fails to be universal for any $l > 3$. Until [6] in fact there were not many classes known with universal objects. The strongest results in the positive direction were obtained by Cherlin, Shelah and Shi in [6]. Particularly, they proved the following

**Theorem 9.** *For every finite set $\mathcal{F}$ of finite connected graphs the class* $\mathrm{Forb}(\mathcal{F})$ *has an embedding universal object.*

This result was extended to relational structures in [8]. The proof of Theorem 9 given in [6] is based on techniques of model theory and it is possible to say that no explicit universal object is constructed. Using lifts and shadows with J. Hubička we recently gave an alternative and more explicit combinatorial proof of Theorem 9 for structures [22]. Along the lines above we can get universal structure for the classes $\mathrm{Forb}(\mathcal{F})$ in particularly easy way as the shadow $\mathbf{U}$ of the direct (Fraïssé) limit $\mathbf{U}$ of an explicitely defined lifted class $\mathrm{Forb}(\mathcal{F}')$ (which is an amalgamation class), see [22] for details.

### 4.1 Existence of duals IV - Generic Duals

Clearly every universal object is also hom-universal. This however does not hold conversely (as shown by examples of classes with bounded chromatic numbers: $K_4$ is hom-universal for the class of planar graphs by virtue of the 4-color theorem). As we already know, of special interests are classes $\mathrm{Forb}(\mathcal{F})$ which have finite hom-universal graph: these are just duals. The proof of Theorem 9 given in [22] gives more: In the case that $\mathcal{F}$ is a finite set of relational trees then the theorem is proved just by monadic lifts (similarly as in the above Construction III) and the resulting universal object $\mathbf{U}_{\mathcal{F}}$ has a finite retract $\mathbf{D}_{\mathcal{F}}$ which is consequently hom-universal. This implies yet another proof of the existence of duals.

Moreover, as the universal lifted structure $\mathbf{U}_{\mathcal{F}}$ may be chosen to be *generic* (meaning ultrahomogeneous and universal) then we see that we may think of duals as a retract of a generic object - *duals are generic*; see [22].

## 5 Restricted dualities

Finite dualities became much more abundant when we demand the validity of the duality formula just for all graphs from a given class $\mathcal{K}$. In such cases we speak about $\mathcal{K}$-*restricted duality.* It has been proved in [51] that so called *Bounded Expansion* classes (which include both proper minor closed classes and classes of graphs with bounded degree) have a restricted duality for every choice of $\mathcal{F}$. As a consequence of this we can show that the shadow $\Phi(Forb(\mathcal{F}))$ of a vertex colored class of structures $Forb(\mathcal{F})$ is always the restriction of a CSP language when restricted to a bounded expansion class (this notion generalizes bounded degree and proper minor closed classes) [48].

More explicitely, the following definition is the central definition of this section:

**Definition 3.** *A class of structures $\mathcal{K}$ admits* all restricted dualities *if, for any finite set of connected structures $\mathcal{F} = \{F_1, F_2, \ldots, F_t\}$, there exists a finite structure $\mathbf{D}_{\mathcal{F}}^{\mathcal{K}}$ such that $\mathbf{F}_i \not\longrightarrow \mathbf{D}_{\mathcal{F}}^{\mathcal{K}}$ for $i = 1, \ldots, t$ and for all $\mathbf{G} \in \mathcal{K}$,*

$$(\mathbf{F}_i \not\longrightarrow \mathbf{G}), i = 1, 2, \ldots, t, \iff (\mathbf{G} \longrightarrow \mathbf{D}_{\mathcal{F}}^{\mathcal{K}}). \tag{1}$$

*Any instance of (1) is called a* restricted duality *(for the class $\mathcal{K}$).*

To motivate this definition let us consider the following example.

The Grötzsch's celebrated theorem (see e.g. [65]) says that every triangle-free planar graph is 3-colorable. In the language of homomorphisms this says that for every triangle-free planar graph $G$ there is a homomorphism of $G$ into $K_3$. Using the partial order terminology, Grötzsch's theorem says that $K_3$ is an upper bound (in the homomorphism order) for the class $\mathcal{P}_3$ of all planar triangle-free graphs. The fact that $K_3 \notin \mathcal{P}_3$ suggests a natural question (first formulated in [42]): Is there yet a smaller bound? The answer, which may be viewed as a strengthening of Grötzsch's theorem, is positive: there exists a triangle free 3-colorable graph $H$ such that $G \longrightarrow H$ for every graph $G \in \mathcal{P}_3$. This has been proved in [45, 46] in a stronger version for minor-closed classes.

One can view these results as restricted dualities (which hold in the class of planar graphs). Restricted duality results have since been generalized not only to proper minor closed classes of graphs and but also to other forbidden subgraphs, in fact to any finite set of connected graphs thus yielding all restricted dualities for the class of planar graphs. This then implies that Grötzsch's theorem can be strengthened by a sequence of even stronger bounds and that the supremum (in the homomorphism order) of the class of all triangle free planar graphs does not exist, [47].

What is the proper setting for the restricted dualities? This is presently an open problem but the strongest result in this direction is the notion of a class with bounded expansion. Such a class may be defined in several (very) different ways, see [48, 51, 52].

It is important that this seemingly elusive global property (having an upper bound) has a localized version by means of the densities of *shallow minors*. We can proceed as follows ([49]):

The *maximum average degree* $\mathrm{mad}(G)$ of a graph $G$ is the maximum over all subgraphs $H$ of $G$ of the average degree of $H$, that is $\mathrm{mad}(G) = \max_{H\subseteq G} \frac{2|E(H)|}{|V(H)|}$. The *distance* $d(x,y)$ between two vertices $x$ and $y$ of a graph is the minimum length of a path linking $x$ and $y$, or $\infty$ if $x$ and $y$ do not belong to same connected component. Also we denote by $G[A]$ the subgraph of $G$ induced by a subset $A$ of its vertices.

We introduce several notations:

– The *radius* $\rho(G)$ of a connected graph $G$ is:

$$\rho(G) = \min_{r\in V(G)} \max_{x\in V(G)} \mathrm{d}(r,x)$$

– A *center* of $G$ is a vertex $r$ such that $\max_{x\in V(G)} \mathrm{d}(r,x) = \rho(G)$.

**Definition 4.** *Let $G$ be a graph. A* ball *of $G$ is a subset of vertices inducing a connected subgraph. The set of all the families of pairwise disjoint balls of $G$ is noted $\mathfrak{B}(G)$.*

*Let $\mathcal{P} = \{V_1, \dots, V_p\}$ be a family of pairwise disjoint balls of $G$.*

– *The* radius $\rho(\mathcal{P})$ *of $\mathcal{P}$ is $\rho(\mathcal{P}) = \max_{X\in\mathcal{P}} \rho(G[X])$.*
– *The* quotient $G/\mathcal{P}$ *of $G$ by $\mathcal{P}$ is a graph with vertex set $\{1, \dots, p\}$ and edge set $E(G/\mathcal{P}) = \{\{i,j\} : (V_i \times V_j) \cap E(G) \neq \emptyset$ or $V_i \cap V_j = \emptyset\}$.*
– *If $\rho(\mathcal{P}) \le r$ then graph $G/\mathcal{P}$ is called* shallow minor at depth $r$ of graph $G$.

The following invariants generalize maximum average degree:

**Definition 5.** *The* greatest reduced average density (grad) *of a graph $G$ with rank $r$ is*

$$\nabla_r(G) = \max \frac{|E(G/\mathcal{P})|}{|\mathcal{P}|},$$

*where maximum is taken over all $\mathcal{P} \in \mathfrak{B}(G)$ satisfying $\rho(\mathcal{P}) \le r$.*

The following is our key definition:

**Definition 6.** *A class of graphs $\mathcal{C}$ has* bounded expansion *if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every graph $G \in \mathcal{C}$ and every $r$,*

$$\nabla_r(G) \le f(r). \tag{2}$$

*$f$ is called the* expansion function.

The definition of bounded expansion can be carried over to general structures by means of incidence graphs. Thus we may speak about classes of structures with bounded expansion.

The definition of bounded expansion is very robust: it may be alternatively defined by means of forbidden shallow subdivisions [10], by means of special colorings of vertices [69, 52]. The definition is preserved by most local operations (for example by doubling of vertices). Proper minor closed classes have bounded expansion with the constant expanding function. Graphs with all vertices bounded by $d$ have exponential expansion function. Several geometrically defined graphs have polynomial expansion function. See [48–52] for many more examples. Despite of this generality we have the following [51]:

**Theorem 10.** *Any class of structures with bounded expansion has all restricted dualities.*

## 6 Remarks

**1.** The existence of a homomorphism from an oriented path $P$ to a graph $G$ may be sometimes conveniently tested by means of matrix multiplication: Let $G = (V, E), V = \{v_1, \dots, v_n\}$ and let $A = (a_{ij})$ be the adjacency matrix (we put $a_{ij} = 1$ iff $(v_i, v_j) \in E$. For a path $P$ with $k + 1$ vertices (and thus of $k$ arcs) we consider the product $B$ of matrices $B_1, B_2, \dots, B_k$ where $B_i = A$ if the $k$-th arc of $P$ is forward and $B_i = A^T$ if the $k$-th arc of $P$ leads backwards. Then $b_{ij} = 1$ if and only if there exists a homomorphism from $P$ to $G$ mapping the first vertex of $P$ to $v_i$ and the last vertex to $v_j$. Thus $P \longrightarrow G$ if and only if the matrix $B \neq 0$.

This connection (which is already made in [67]) can be used for an effective testing of large (recursiveky defined) paths, see [57]. In this context it is fitting to note that the fastest algorithm for testing the existence of a homomorphism $G \longrightarrow H$ for a *fixed* $G$ is based on the fast matrix multiplication, [54].

**2.** We proved that shadow dualities and lifted monadic dualities are in $1 - 1$ correspondence. This abstract result has several consequences and streamlines some earlier results in descriptive complexity theory (related to MMSNP and CSP classes) [37, 38]. The simplicity of this approach suggests some other problems. It is tempting to try to relate Ladner's diagonalization method [34] in this setting (as it was pioneered by Lovász and Gács [15] for NP∩coNP in a similar context). The characterization of Lifted Dualities is beyond reach but particular cases are interesting as they generalize results of [56, 14] and as the corresponding duals present polynomial instances of CSP.

But perhaps more importantly, our approach to the complexity subclasses of NP is based on lifts and shadows as a combination of algebra, combinatorics and logic. We believe that it has further applications and that it forms a useful paradigm.

**3.** Let us finish this paper by listing the characterization theorem for finite dualities. (We say that a class of structures $\mathcal{K}$ is *homomorphism closed* if $\mathbf{A} \in \mathcal{K}, \mathbf{A} \longrightarrow \mathbf{B}$ implies $\mathbf{B} \in \mathcal{K}$. We also denote by $\mathcal{F} \longrightarrow$ the class of all structures $\mathbf{A}$ for which there exists $\mathbf{F} \in \mathcal{F}$ such that $\mathbf{F} \longrightarrow \mathbf{A}$. The class $\mathcal{F} \longrightarrow$ is the complementary class of $\mathrm{Forb}(\mathcal{F})$. By a combination of results [1, 62, 56, 14] we obtain

**Theorem 11.** *Let $\mathcal{K}$ be a class of structures closed under homomorphisms. For $\mathcal{K}$ are the following statements equivalent:*

– *$\mathcal{K}$ is first order definable class;*
– *$\mathcal{K} = \mathcal{F} \longrightarrow$ for a finite set $\mathcal{F}$ of structures.*

*It follows that any first order definable class $\mathcal{K} = CSP(\mathcal{D})$ is defined by a finite duality (and thus the corresponding set $\mathcal{F}$ is a set of finite relational trees].*

By a combination of [2, 51, 52] we have a surprisingly strong relativized version of this result:

**Theorem 12.** *Let $\mathcal{K}$ be a bounded expansion class. For a homomorphism closed subclass $\mathcal{L}$ of $\mathcal{K}$ are the following statements equivalent:*

– *$\mathcal{L}$ is first order definable in $\mathcal{K}$;*
– *$\mathcal{L} = (\mathcal{F}) \longrightarrow$ for a finite set $\mathcal{F}$ of structures;*
– *$\mathcal{L}$ is defined by a restricted finite duality.*

In [52] and [53] we developed further the connections of classes of sparse graphs to logic and descriptive complexity.

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