On Distance Local Connectivity and the Hamiltonian Index

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Abstract.

In this paper we introduce the concept of distance local connectivity of a graph. We give several sufficient conditions in terms of the independence number and of the vertex degrees, and we show a relation between the distance local connectivity and the hamiltonian index of a graph.

Keywords: Degree condition, Distance local connectivity, Hamiltonian index, Independence number

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1 Introduction.

By a graph we mean a simple undirected graph $G = (V(G), E(G))$. We use [3] for terminology and notation not defined here. For $x, y \in V(G)$, an $x, y-path$ is a path between vertices x and y in G and $\mathrm{dist}_G(x, y)$ denotes the *distance* between x and y in G, i.e. the length of a shortest x, y-path in G. Let $d_G(x)$ denote the degree of a vertex x in G, $\delta(G)$ the minimum degree of G, $\Delta(G)$ the maximum degree of G and $\alpha(G)$ the independence number of G. For a nonempty set $U \subseteq V(G)$, the induced subgraph on U is denoted by $\langle U \rangle$. For $x \in V(G)$, $G - x$ denotes the subgraph of G obtained by deleting x and all

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edges adjacent to x. If $e \in E(G)$, then $G - e$ denotes the subgraph of G such that $V(G-e) = V(G)$ and $E(G-e) = E(G) \setminus \{e\}$. If G_1, G_2 are graphs, then $G = G_1 \cup G_2$ is the graph $G = (V, E)$ such that $V = V(G_1) \cup V(G_2)$ and $E = E(G_1) \cup E(G_2)$, and $G_1 \triangle G_2$ denotes the graph $(G_1 \cup G_2) - (E(G_1) \cap E(G_2))$. We say that G is *claw-free* if it does not contain a copy of the graph $K_{1,3}$ as an induced subgraph. Let

$$\sigma_k(G) = \min\{\sum_{i=1}^{k} d_G(x_i) | \{x_1, \ldots, x_k\} \subset V(G), \text{ independent}\}.$$

The *square* of a graph G, denoted by G^2, is the graph in which $V(G^2) = V(G)$ and $E(G^2) = E(G) \cup \{\{u, v\}, | \, u, v \in V(G), \ \mathrm{dist}_G(u, v) = 2\}$. For $x \in V(G)$, we set $N_G(x) = \{y \in V(G), xy \in E(G)\}$ and $N_G[x] = N_G(x) \cup \{x\}$. We say that x is *locally connected* in G, if $\langle N_G(x) \rangle$ is connected. We say that G is a *locally connected graph*, if every vertex of G is locally connected.

Chartrand and Pippert proved the following Ore-type condition for local connectivity of graphs:

Theorem A [5]. *Let G be a connected graph of order n. If*

$$d_G(u) + d_G(v) > \frac{4}{3}(n-1)$$

for every pair of vertices u, $v \in V(G)$, then G is locally connected.

For $x \in V(G)$, let $N_2(x)$ be the subgraph induced by the set of edges uv, such that

$$\min\{\mathrm{dist}_G(x, u), \, \mathrm{dist}_G(x, v)\} = 1.$$

We say that x is an N_2-*locally connected* vertex in G, if $N_2(x)$ is connected. We say that G is N_2-*locally connected*, if every vertex of G is N_2-locally connected. Ryjáček in [12] proved an Ore-type condition for N_2-locally connected graphs. In Section 2 we introduce the concept of distance local connectivity and we extend these results to distance locally connected graphs.

The following result shows that the concept of local connectivity is closely related to hamiltonicity.

Theorem B [10]. *Let G be a connected locally connected claw-free graph of order at least 3. Then G is hamiltonian.*

Motivated by Theorem B, in Section 3 we will prove results on the hamiltonicity in iterated line graphs, whose preimages are distance locally connected or N_2-locally connected. As a motivation we give the following Theorem C.

A *closed trail* is a connected graph T such that every vertex of T has even degree. A *dominating closed trail* (DCT) in a graph G is a closed trail T such that T is a subgraph of G and each edge of G is *dominated* by T, i.e. has at least one end vertex on T. The trivial DCT, i.e., DCT containing only one vertex, is allowed too.

Harary and Nash-Williams proved the following:

Theorem C [7]. *Let G be a graph with at least three edges. Then G has a DCT if and only if the line graph $L(G)$ of G is hamiltonian.*

Paulraja proved the following:

Theorem D [11]. *Let G be a connected graph. If every edge of G belongs to a triangle, then G has a spanning eulerian subgraph.*

Theorem E [11]. *Let G be a connected claw-free graph such that every edge of G belongs to a cycle of length at most 5. Then G has a spanning eulerian subgraph.*

Paulraja [11] also conjectured that if every edge of a 2-connected graph G lies in a cycle of length at most 4 in G, then G has a DCT. Lai [8] showed that this conjecture is true.

Theorem F [8]. *Let G be a 2-connected graph such that every edge of G lies in a cycle of length at most 4. Then G has a DCT*

It is easy to see that a spanning eulerian subgraph is a DCT.

For an integer $k \geq 0$, the *k-th iterated line graph* $L^k(G)$ of a graph G, is defined recursively by $L^0(G) = G$ and $L^j(G) = L(L^{j-1}(G))$ for $j = 1, \ldots, k$. The *hamiltonian index* of a graph G, denoted $h(G)$, is the smallest integer k for which $L^k(G)$ is hamiltonian. Chartrand [4] showed that the hamiltonian index exists for every connected graph that is not a path.

For i a nonnegative integer, let $V_i(G) \subset V(G)$ denote the set of vertices such that $V_i(G) = \{x, d_G(x) = i\}$. A *branch* of G is a nontrivial path such that

- for every internal vertex v, if any, $v \in V_2(G)$,
- for every end vertex u, $u \notin V_2(G)$.

Let $B(G)$ denote the set of all branches of G, and let $B_1(G)$ denote the subset of $B(G)$ in which every branch has one end vertex in $V_1(G)$. For any subgraphs H_1, H_2 of G, let $\text{dist}_G(H_1, H_2) = \min\{\text{dist}_G(x, y),\ x \in V(H_1),\ y \in V(H_2)\}$. Liu and Xiong characterized graphs with hamiltonian index at most n.

Theorem G [9]. *Let G be a connected graph that is not a 2-cycle and let $n \geq 2$ be an integer. Then $h(G) \leq n$ if and only if $EU_n(G) \neq \emptyset$, where $EU_n(G)$ denotes the set of those subgraphs H of G which satisfy the following conditions:*

i) every vertex of H has even degree,

ii) $V_0(H) \subset \bigcup_{i=3}^{\Delta(G)} V_i(G) \subset V(H)$,

iii) $\text{dist}_G(H_1, H \setminus H_1) \leq n - 1$ for any component H_1 of H,

iv) $|E(b)| \leq n + 1$ for any branch $b \in B(G)$ with $E(b) \cap E(H) = \emptyset$,

v) $|E(b_1)| \leq n$ for any branch $b_1 \in B_1(G)$.

2 Distance local connectivity.

Let G be a graph, $x \in V(G)$, and let m be a positive integer. The N_1^m-*neighbourhood* of x, denoted by $N_1^m(x)$, is the set of all vertices $y \in V(G)$, $y \neq x$, such that $\text{dist}_G(x, y) \leq m$. A vertex x is called N_1^m-*locally connected* if $\langle N_1^m(x) \rangle$ is connected. A graph G is said to be an N_1^m-*locally connected graph* if every vertex of G is N_1^m-locally connected.

We prove the following proposition.

Proposition 1. *Let G be a graph and m be a positive integer. If G is a connected N_1^m-locally connected graph, then G is 2-connected.*

Proof. If G is not 2-connected, then G has a cutvertex x. Thus, $\langle N_1^m(x) \rangle$ is a subgraph of a disconnected graph $G - x$ with at least one vertex in each of the components of $G - x$. Hence x is not N_1^m-locally connected, a contradiction.

$\square$

We will concentrate on the family of 2-connected graphs only. Let C be a cycle, $x \in V(C)$, and let $\vec{C}$ be an orientation of C. Let $x^{-(i)}$ denote the i-th predecessor of x on C and $x^{+(i)}$ denote the i-th successor of x on C in the orientation $\vec{C}$.

The following lemma shows that an N_1^m-locally disconnected vertex belongs to a large induced cycle.

Lemma 1. *Let G be a 2-connected graph, $x \in V(G)$, and let m be a positive integer. If x is not N_1^m-locally connected, then there is an induced cycle C of length at least $2m+2$ such that, in an orientation of C,*

- *$dist_G(x^{-(i)}, x) = i$ and $dist_G(x^{+(i)}, x) = i$, $i = 1, \dots, m$,*
- *$dist_G(y, x) > m$, for every $y \in V(C) \setminus \{x, x^{-(1)}, \dots, x^{-(m)}, x^{+(1)}, \dots, x^{+(m)}\}$.*

Proof. Suppose $x \in V(G)$ is not N_1^m-locally connected. The subgraph $\langle N_1^m(x) \rangle$ consists of at least two components. Let G_1 be arbitrary component of $\langle N_1^m(x) \rangle$ and G_2 be the union of all the other components of $\langle N_1^m(x) \rangle$. Since G is 2-connected, there is a cycle C containing x, such that $x^{-(1)} \in G_1$ and $x^{+(1)} \in G_2$ in an orientation of C. Choose C shortest possible with this property. Since C is shortest possible, $\mathrm{dist}_G(x, x^{+(m)}) = \mathrm{dist}_G(x, x^{-(m)}) = m$. Since x is not N_1^m-locally connected, $x^{-(m)}x^{+(m)} \notin E(G)$, implying that $|V(C)| \geq 2m+2$. It is easy to see that C has the required properties since otherwise there is a shorter cycle. $\square$

2.1 Sufficient conditions for distance local connectivity involving independence number.

Chvátal and Erdős in [6] proved that every k-connected graph with $\alpha(G) \leq k$ is hamiltonian. Ainouche, Broersma and Veldman in [1] strenghtened the Chvátal-Erdős' theorem for claw-free graphs by showing that a k-connected claw-free graph G is hamiltonian if $\alpha(G^2) \leq k$. In this paragraph we prove similar conditions for distance local connectivity in 2-connected graphs.

Theorem 1. *Let G be a 2-connected graph, m be a positive integer. If $\alpha(G) \leq m$, then G is N_1^m-locally connected.*

Proof. Suppose that G is not N_1^m-locally connected, i.e., there is a vertex $x \in V(G)$ which is not N_1^m-locally connected. By Lemma 1 there is an induced cycle C in G of length at least $2(m+1)$ containing x. Clearly, the cycle C contains $\frac{2m+2}{2} = m+1$ independent vertices, which is a contradiction. $\square$

Now we give sufficient conditions for distance local connectivity in terms of the independence number of the square of a graph.

Theorem 2. *Let $m \geq 2$ be an integer and let G be a 2-connected graph such that, for every vertex x of G:*

1) *if $m \equiv 0 \pmod 3$, then $|M| \leq \frac{2}{3}m$ for every $M \subset N_1^{m+1}(x)$ such that M is independent in $(G-x)^2$,*
2) *if $m \equiv 1 \pmod 3$, then $|M| \leq \frac{2}{3}(m-1)$ for every $M \subset \left(N_1^{m+1}(x) \setminus N_1^1(x)\right)$ such that M is independent in $(G - N_G[x])^2$,*
3) *if $m \equiv 2 \pmod 3$, then $|M| \leq \frac{2}{3}(m-2)+1$ for each $M \subset N_1^m[x]$ such that M is independent in G^2.*

Then G is N_1^m-locally connected.

Proof. Suppose that there is a vertex $x \in V(G)$ which is not N_1^m-locally connected. By Lemma 1 there is an induced cycle C of length at least $2(m+1)$ in G containing the vertex x. Since x is not N_1^m-locally connected, the cycle C can be chosen such that $x^{-(1)}$ and $x^{+(1)}$ belong to different components of $\langle N_1^m(x) \rangle$.

Now we consider the following cases.

Case 1: $m \equiv 0 \pmod 3$. We choose the following set of vertices of G:

$$M = \left\{x^{+(1)}, x^{+(4)}, \ldots, x^{+(m-2)}, x^{-(1)}, x^{-(4)}, \ldots, x^{-(m-2)}, x^{+(m+1)}\right\}$$

Then no two vertices of M have a common neighbour in $G-x$, implying that M is independent in $(G-x)^2$. Since $|M| = \frac{2}{3}m+1$, we have

$$\alpha((G-x)^2) \geq \frac{2}{3}m+1,$$

a contradiction.

Case 2: $m \equiv 1 \pmod 3$. We choose the following set of vertices of G:

$$M = \left\{x^{+(2)}, x^{+(5)}, \ldots, x^{+(m-2)}, x^{-(2)}, x^{-(5)}, \ldots, x^{-(m-2)}, x^{+(m+1)}\right\}$$

The set M is independent in $(G - N_G[x])^2$, implying

$$\alpha((G - N_G[x])^2) \geq \frac{2}{3}(m-1)+1,$$

a contradiction.

Case 3: $m \equiv 2 \pmod 3$. We choose the following set of vertices of G:

$$M = \left\{x^{+(1)}, x^{+(4)}, \ldots, x^{+(m-1)}, x^{-(2)}, x^{-(5)}, \ldots, , x^{-(m)}\right\}$$

The set M is independent in G^2, implying

$$\alpha(G^2) \geq \frac{2}{3}(m-2) + 2,$$

a contradiction.

□

In fact, in the proof of the previous theorem, we have shown a little more: if x is not a distance-locally connected vertex in G, then there is a set $M \subset V(G)$ such that

- M is independent in the square of a specific subgraph of G,
- M is a subset of a distance neighbourhood of x.

This yields the following consequence.

Corollary 1. *Let $m \geq 2$ be an integer, G a 2-connected graph. If $x \in V(G)$ is not N_1^m-locally connected, then there is a set $M \subset V(G)$ such that*

1) *M is independent in $(G-x)^2$, $M \subset N_1^{m+1}(x)$ and $|M| \geq \frac{2}{3}m + 1$, if $m \equiv 0 (\mathrm{mod}\, 3)$,*
2) *M is independent in $(G - N_G[x])^2$, $M \subset \left(N_1^{m+1}(x) \setminus N_1^1(x)\right)$ and $|M| \geq \frac{2}{3}(m-1) + 1$, if $m \equiv 1 (\mathrm{mod}\, 3)$,*
3) *M is independent in G^2, $M \subset N_1^m[x]$ and $|M| \geq \frac{2}{3}(m-2) + 2$, if $m \equiv 2 (\mathrm{mod}\, 3)$.*

As an immediate consequence of Theorem 2 we have the following Theorem.

Theorem 3. *Let $m \geq 2$ be an integer and let G be a 2-connected graph such that, for every $x \in V(G)$,*

$$\begin{array}{lll} 1)\ \alpha((G-x)^2) \leq \frac{2}{3}m & \text{if } m \equiv 0 \pmod 3, \\ 2)\ \alpha((G-N_G[x])^2) \leq \frac{2}{3}(m-1) & \text{if } m \equiv 1 \pmod 3, \\ 3)\ \alpha(G^2) \leq \frac{2}{3}(m-2)+1 & \text{if } m \equiv 2 \pmod 3. \end{array} \tag{1}$$

Then G is N_1^m-locally connected.

Let G be a graph with the structure shown in Fig. 1, such that each of the sets $U_1, \ldots, U_m$, $V_1, \ldots V_m, W$ induces a clique of arbitrary order. It is easy to see that G is not N_1^m-locally connected and in each of the three cases considered in the proof of Theorem 3 there is an equality. Thus, the conditions given in Theorem 3 are sharp.

The following sufficient condition involving $\alpha(G^2)$ is an immediate consequence of Theorem 3.

Corollary 2. *Let G be a 2-connected graph, let $m \geq 2$ be an integer. If $\alpha(G^2) \leq \frac{2}{3}m - 1$, then G is N_1^m-locally connected.*

2.2 Degree conditions for distance local connectivity.

Bondy in [2] proved that if $d_G(x)+d_G(y) \geq |V(G)|$ for each pair of nonconsecutive vertices of a k-connected graph G, then $\alpha(G) \leq k$. Ore's degree condition for hamiltonicity can be found as an consequence of mentioned theorem of Chvátal and Erdős (see [6]) and the above mentioned theorem of Bondy. In a similar direction, we present in this paragraph some degree conditions for distance local connectivity in 2-connected graphs.

Theorem 4. *Let G be a 2-connected graph of order n and let m be a positive integer. If*

$$\sigma_{m+1}(G) \geq \frac{3}{2}n - m,$$

then G is N_1^m-locally connected.

Proof. Suppose that G is not N_1^m-locally connected. Then there is a vertex x such that $\langle N_1^m(x) \rangle$ consists of at least two components. Let G_1 denote one of the components, let G_2 denote the union of all the other components. Let $U_i = \{x_i \in V(G_1), \operatorname{dist}_G(x_i, x) = i\}$, $V_i = \{x_i \in V(G_2), \operatorname{dist}_G(x_i, x) = i\}$, $i = 1, \ldots, m$. Let $W = \{y \in V(G),\ \operatorname{dist}_G(x, y) > m\}$. Since G is 2-connected, the sets U_i, V_i are all nonempty. By Lemma 1, there is an induced cycle C such that $V(C) \geq 2m + 2$ and C can be chosen such that C contains a vertex of each of the sets $U_i, V_i, i = 1, \ldots, m$. Choose C shortest possible with this property. The structure of G is shown in Fig. 1. Note that elliptical parts are not necessarily cliques.

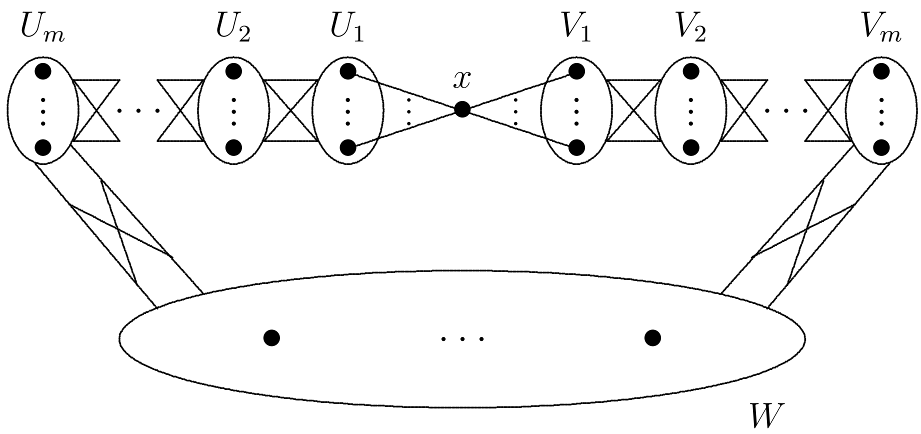

Fig. 1

Choose vertices $w \in W$, $u_i \in U_i$, $v_i \in V_i$, $i = 1, \ldots, m$ and set

$$A = \begin{cases} \{u_1, v_1, u_3, v_3, \ldots, u_m, v_m\} & \text{if } m \text{ is odd,} \\ \{u_1, v_1, u_3, v_3, \ldots, u_{m-1}, v_{m-1}, w\} & \text{if } m \text{ is even} \end{cases}$$

and

$$B = \begin{cases} \{x, u_2, v_2, u_4, v_4, \ldots, u_{m-1}, v_{m-1}, w\} & \text{if } m \text{ is odd,} \\ \{x, u_2, v_2, u_4, v_4, \ldots, u_m, v_m\} & \text{if } m \text{ is even.} \end{cases}$$

Then A, B are disjoint $(m+1)$-element independent sets on C. For any vertex $z \in V(G)$, z can be a common neighbour of at most two vertices from A, or from B respectively.

Obviously

$$\begin{array}{ll} d_G(u_1) \leq 1 + |U_1| - 1 + |U_2|, & d_G(v_1) \leq 1 + |V_1| - 1 + |V_2|, \\ d_G(u_2) \leq |U_1| + |U_2| - 1 + |U_3|, & d_G(v_2) \leq |V_1| + |V_2| - 1 + |V_3|, \\ d_G(u_3) \leq |U_2| + |U_3| - 1 + |U_4|, & d_G(v_3) \leq |V_2| + |V_3| - 1 + |V_4|, \\ \quad\vdots & \quad\vdots \\ d_G(u_m) \leq |U_{m-1}| + |U_m| - 1 + |W|, & d_G(v_m) \leq |V_{m-1} + |V_m| - 1 + |W|, \end{array}$$

Hence we have

$$\sum_{z\in A} d_G(z) \le \begin{cases} n-(m+1)+1+|U_2|+|V_2|+|U_4|+|V_4|+\ldots+ & \\ \quad +|U_{m-1}|+|V_{m-1}|+|W| & \text{if } m \text{ is odd,} \\ n-(m+1)+1+|U_2|+|V_2|+|U_4|+|V_4|+\ldots+ & \\ \quad +|U_m|+|V_m| & \text{if } m \text{ is even,} \end{cases}$$

Analogously we have

$$\sum_{y\in B} d_G(y) \le \begin{cases} n-(m+1)+1+|U_1|+|V_1|+|U_3|+|V_3|+\ldots+ & \\ \quad +|U_m|+|V_m| & \text{if } m \text{ is odd,} \\ n-(m+1)+1+|U_1|+|V_1|+|U_3|+|V_3|+\ldots+ & \\ \quad +|U_{m-1}|+|V_{m-1}|+|W| & \text{if } m \text{ is even.} \end{cases}$$

Since $1+|U_1|+|V_1|+|U_2|+|V_2|+\ldots+|U_m|+|V_m|+|W|=n$, we have

$$\sum_{z\in A} d_G(z) + \sum_{y\in B} d_G(y) \le 2(n-m-1)+n,$$

implying

$$\sigma_{m+1}(G) \le \frac{3}{2}n-m-1,$$

a contradiction. □

Alternatively, from Theorem 3 we can obtain the following degree conditions for N_1^m-locally connected graphs. Note that these conditions are incomparable, i.e., none of Theorems 4 and 5 implies the other one, as will be shown in Corollary 5.

Theorem 5. *Let $m \ge 2$ be an integer and let G be a 2-connected graph of order n such that*

1) $\sigma_{\frac{2}{3}m+1}(G) \ge n-\frac{2}{3}m+1$ *if* $m \equiv 0 \pmod 3$,
2) $\sigma_{\frac{2}{3}(m-1)+1}(G) \ge n-\frac{2}{3}(m-1)-1$ *if* $m \equiv 1 \pmod 3$,
3) $\sigma_{\frac{2}{3}(m-2)+2}(G) \ge n-\frac{2}{3}(m-2)-1$ *if* $m \equiv 2 \pmod 3$.

Then G is N_1^m-locally connected.

Proof. Suppose that G is not N_1^m-locally connected. There is a vertex x such that $\langle N_1^m(x)\rangle$ consists of at least two components. Let G_1 be the smallest component of $\langle N_1^m(x)\rangle$, let G_2 be the union of all the other components of $\langle N_1^m(x)\rangle$. Let $u_1 = |V(G_1) \cap N_G(x)|$ and $v_1 = |V(G_2) \cap N_G(x)|$. By Lemma 1 there is an induced cycle C of length at least $2(m+1)$ in G containing the vertex x. Since x is not N_1^m-locally connected, the cycle C can be chosen such that $x^{-(1)}$ and $x^{+(1)}$ belong to different components of $\langle N_1^m(x)\rangle$.

Case 1: $m \equiv 0 \pmod 3$. By Corollary 1 case 1), there is a set $M \subset V(G-x)$ such that $|M| = \frac{2}{3}m+1$ and M is independent in $(G-x)^2$. Let $t = |M|$. The set M can be chosen in the following way: $M = \{x_1, x_2 \dots x_t\}$, where $x_{2j-1} = x^{-(3j-2)}$, $x_{2j} = x^{+(3j-2)}$, $j = 1, \dots, \frac{m}{3}$, $x_t = x^{+(m+1)}$. Hence

$$N_1^1(u) \cap N_1^1(v) = \emptyset$$

for every pair $u, v \in M$. Obviously

$$\sum_{x_i \in M} d_{G-x}(x_i) \le (n-1) - \frac{2}{3}m - 1.$$

The vertex x can be adjacent to at most two vertices of M by the definition of M. Therefore

$$\sum_{x_i \in M} d_G(x_i) \le n - \frac{2}{3}m,$$

a contradiction.

Case 2: $m \equiv 1 \pmod 3$. By Corollary 1 case 2), there is a set $M \subset V(G - N_G[x])$ such that $|M| = \frac{2}{3}(m-1)+1$ and M is independent in $(G - N_G[x])^2$. Let $t = |M|$. The set M can be chosen in the following way: $M = \{x_1, x_2 \dots x_t\}$, where $x_{2j-1} = x^{-(3j-1)}$, $x_{2j} = x^{+(3j-1)}$, $j = 1, \dots, \frac{m}{3}$, $x_t = x^{+(m+1)}$. Hence

$$\sum_{x_i \in M} d_{G-N_G[x]}(x_i) \le (n-1-u_1-v_1) - \frac{2}{3}(m-1) - 1.$$

Each neighbour of x is adjacent to at most one vertex of M by the definition of M. This yields

$$\sum_{x_i \in M} d_G(x_i) \le n - \frac{2}{3}(m-1) - 2,$$

a contradiction.

Case 3: $m \equiv 2 \pmod 3$. By Corollary 1 case 3), there is a set $M \subset V(G)$ such that $|M| = \frac{2}{3}(m-2)+2$ and M is independent in G^2. Hence

$$\sum_{x_i \in M} d_G(x_i) \le n - \frac{2}{3}(m-2) - 2,$$

which is a contradiction again. □

Theorem 5 implies the following Dirac-type condition.

Corollary 3. *Let $m \geq 2$ be an integer and let G be a 2-connected graph of order n such that*

$$\delta(G) \geq \begin{cases} \frac{3n}{2m+3} - \frac{2m-3}{2m+3} & \text{if } m \equiv 0 \pmod 3, \\ \frac{3n}{2m+1} - 1 & \text{if } m \equiv 1 \pmod 3, \\ \frac{3n}{2m+2} - \frac{2m-1}{2m+2} & \text{if } m \equiv 2 \pmod 3. \end{cases}$$

Then G is N_1^m-locally connected.

However, from Theorem 4 we obtain the following minimum degree condition.

Corollary 4. *Let m be a positive integer and let G be a 2-connected graph such that*

$$\delta(G) \geq \frac{3n}{2m+2} - \frac{m}{m+1}.$$

Then G is N_1^m-locally connected.

Comparing Corollary 3 and Corollary 4 we finally obtain the following

Corollary 5. *Let $m \geq 2$ be an integer, let G be a 2-connected graph of order n such that*

$$\delta(G) \geq \begin{cases} \frac{3n}{2m+3} - \frac{2m-3}{2m+3} & \text{if } m \equiv 0 (\bmod 3) \quad \text{and } n \geq \frac{8}{3}m + 2, \\ \frac{3n}{2m+2} - \frac{m}{m+1} & \text{otherwise.} \end{cases}$$

Then G is N_1^m-locally connected.

Proof.

Case 1: $m \equiv 0 (\bmod 3)$. We have $\frac{3n-2m+3}{2m+3} < \frac{3n-2m}{2m+2}$. Thus we obtain $(3n - 2m + 3)(2m + 2) < (3n - 2m)(2m + 3)$, and hence $n > \frac{8}{3}m + 2$.

Case 2: $m \equiv 1 (\bmod 3)$. We have $\frac{3n-2m-1}{2m+1} > \frac{3n-2m}{2m+2}$. Then we obtain $(3n - 2m - 1)(2m + 2) > (3n - 2m)(2m + 1)$. and hence $n > \frac{4}{3}m + \frac{2}{3}$. Clearly, if G is 2-connected and $n < 2m$, then G is trivially N_1^m-locally connected.

Case 3: $m \equiv 2(\bmod 3)$. We have $\frac{3n-2m+1}{2m+2} > \frac{3n-2m}{2m+2}$ and hence $3n-2m+1 > 3n-2m$.

$\square$

The conditions in Theorem 4 and in Corollary 4 are sharp for $m \equiv 2 \,(\bmod 3)$. Let k be an arbitrary integer, let $m \equiv 2\,(\bmod 3)$ be a positive integer. Let $U_1,\ U_2,\ U_4,\ U_5,\ U_7 \dots U_m, V_1,\ V_2,\ V_4, V_5, V_7, \dots V_m$ be cliques of order k. Let $U_3, U_6, \dots, V_3, V_6, \dots, W$ be cliques of order 1. Construct a graph G by joining a new vertex x with all vertices of $U_1 \cup V_1$, each vertex of U_i with each vertex of U_{i+1}, each vertex of V_i with each vertex of V_{i+1}, $i = 1, \dots, m-1$, and each vertex of $U_m \cup V_m$ with the vertex of W. Then G is $2k$-regular, $n = |V(G)| = \frac{4}{3}(m+1)k + \frac{2}{3}(m+1)$ and $\sigma_{m+1}(G) = 2k(m+1)$. Clearly

$$2k(m+1) = \frac{3}{2}\left[\frac{4}{3}(m+1)k + \frac{2}{3}(m+1)\right] - (m+1) = \frac{3}{2}n - m - 1,$$

$$\delta(G) = 2k = \frac{3n}{2(m+1)} - 1$$

and the graph G is not N_1^m-locally connected.

The conditions in Theorem 5 and Corollary 3 are sharp for $m \equiv 2\,(\bmod 3)$ too. We consider the same graph G as in the previous example. Clearly

$$\sigma_{\frac{2}{3}(m-2)+2}(G) = \left(\frac{2}{3}(m-2)+2\right)2k = \frac{4}{3}(m+1)k = n - \frac{2}{3}(m+1) =$$

$$= n - \frac{2}{3}(m-2) - 2,$$

$$\delta(G) = \frac{3n}{2(m+1)} - 1$$

and the graph is not N_1^m-locally connected.

3 Hamiltonian index.

For any 2-connected graph G, let $\ell(G)$ denote the smallest integer ℓ such that every edge of G belongs to a cycle of length at most ℓ. Note that $\ell(G)$ is well-defined since if G is 2-connected then every edge of G is in a cycle.

The following theorem is the main result of this section.

Theorem 6. *Let G be a 2-connected graph. Then*

$$h(G) \leq \begin{cases} \left\lfloor \frac{\ell(G)-1}{2} \right\rfloor & \text{for } 3 \leq \ell(G) \leq 5, \\ \\ \left\lfloor \frac{\ell(G)}{2} \right\rfloor - 1 & \text{for } \ell(G) \geq 6. \end{cases}$$

Moreover we prove the following result, showing the relation between hamiltonian index and distance local connectivity.

Theorem 7. *Let m be a positive integer and let G be a connected N_1^m-locally connected graph. Then*

$$h(G) \leq \begin{cases} m & \text{for } m \in \{1, 2\}, \\ m-1 & \text{for } m \geq 3. \end{cases}$$

Moreover, if G is connected, N_2-locally connected, then $h(G) \leq 1$.

For the proofs of Theorem 6 and Theorem 7 we need some auxiliary statements.

Lemma 2. *Let G be a 2-connected N_2-locally connected graph, let x be a vertex of G. Then for every pair u, v of neighbouring vertices of x there is an u, v-path P such that*

i) $1 \leq dist_G(x, y) \leq 2$ *for every* $y \in V(P)$,

ii) there is no pair of consecutive vertices on P which are both at distance exactly 2 from x.

Proof. Since G is 2-connected, there cannot be a vertex of degree 1 in G, and hence every neighbour of x is in $N_2(x)$. By the definition of N_2-local connectivity, any two vertices of $N_2(x)$ are connected by a path in $N_2(x)$, and, by the definition of $N_2(x)$, each edge of $N_2(x)$ has at least one end-vertex at distance 1 from x. $\square$

Theorem 8. *Let G be a 2-connected N_2-locally connected graph. Then G has a DCT.*

Proof. If G is a 2-connected N_2-locally connected graph, then each edge of G lies in a cycle of length at most 4. Then, by Theorem F, G has a DCT. $\square$

The graph in Figure 2 shows that Theorem 8 does not hold if G is not 2-connected.

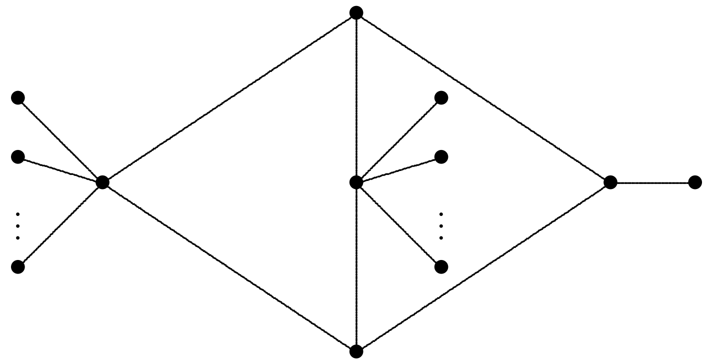

Fig. 2

Lemma 3. *Let $m \geq 3$ be an integer, let G be a 2-connected graph in which every edge belongs to a cycle of length at most $2m+1$. Then $h(G) \leq m-1$.*

Proof. Suppose that every edge of a 2-connected graph G belongs to a cycle of length at most $2m+1$. Let H be a subgraph of G such that

i) every vertex of H has even degree in H,

ii) H contains every vertex x_i of G with $d_G(x_i) \geq 3$,

iii) every vertex y of degree two in G has also degree two in H if $y \in V(H)$.

It is easy to see that such a graph H exists (consider e.g. $V(H) = \bigcup_{i=3}^{\Delta(G)} V_i(G)$ and $E(H) = \emptyset$). Suppose H is chosen such that

iv) $|\{v \in V(H) | d_H(v) > 0\}|$ is maximum,

v) the number of components of H is minimum.

We show that H satisfies all conditions of Theorem G.

a) Clearly H satisfies condition i) of Theorem G.

b) It is easy to see that $V_0(H) \subset \bigcup_{i=3}^{\Delta(G)} V_i(G)$ and $\bigcup_{i=3}^{\Delta(G)} V_i(G) \subset V(H)$. Hence H satisfies condition ii) of Theorem G.

c) We show that $\text{dist}_G(H_1, H \setminus H_1) \leq m-2$ for every component H_1 of H. Let, to the contrary, H_1 be a component of H such that $\text{dist}_G(H_1, H \setminus H_1) \geq m-1$. Since G is 2-connected, there are at least two vertex-disjoint paths P_1, P_2 between H_1 and $H \setminus H_1$. Choose P_1 shortest possible. Among all the paths which are vertex-disjoint with P_1 we choose the path P_2 shortest possible. Clearly P_1 and P_2 are both branches of G and they are both of length at least $m-1$. Let $x_i = V(H_1) \cap V(P_i)$, $y_i = V(H \setminus H_1) \cap P_i$, $i = 1, 2$. The paths P_1, P_2 can be chosen in such a way that the vertices y_1, y_2 belong to one component of $H \setminus H_1$. We consider the following cases:

- If H_1 and $H \setminus H_1$ are both trivial, then we can construct a subgraph H' by adding the paths P_1 and P_2 to H. Clearly H' satisfies all conditions $i), ii), iii)$ and H' contains more vertices of degree at least 2 than H, a contradiction with $iv)$.
- Suppose at least one of components of H is nontrivial. By symmetry we suppose $H \setminus H_1$ is nontrivial. The paths P_1 and P_2 can be chosen such that $y_1 \neq y_2$. If $\text{dist}_G(x_1, x_2) + \text{dist}_G(y_1, y_2) \geq 4$, then any cycle containing an edge of P_1 has length at least $2m+2$, since P_2 is shortest possible. This is a contradiction. Hence $\text{dist}_G(x_1, x_2) + \text{dist}_G(y_1, y_2) \leq 3$. Since H_1 is connected, there is a x_1, x_2-path P in H_1. Since y_1 and y_2 belong to one component of H, there is a y_1, y_2-path Q. For the case $x_1 = x_2$, or $y_1 = y_2$ respectively, we consider the trivial path P, or Q.

  Let $U_1 = \{v \in V(H);\ \ v \text{ is an internal vertex of } P \text{ or } Q \text{ with } d_G(v) = 2\}$. Since $\text{dist}_G(x_1, x_2) + \text{dist}_G(y_1, y_2) \leq 3$, $|U_1| \leq 2$. Let $H' = (P_1 \cup P_2 \cup (H \triangle (P \cup Q))) - U_1$. First suppose that none of the paths P, Q is trivial. Then $|U_1| < 2$. Since the number of all internal vertices of $P_1 \cup P_2$ of degree 2 is greater than $|U_1|$, the subgraph H' contains more vertices of degree at least two than H, a contradiction with $iv)$.

  Finally we suppose that one of the paths P, Q, say P, is trivial. If $|U_1| < 2$, then H' contains more vertices of degree at least two than H, a contradiction with $iv)$. Hence $|U_1| = 2$. Clearly $E(Q) \subset E(H)$. The subgraph H' contains the same number of vertices of degree at least two as subgraph H, but H' has less components than H, a contradiction with $v)$.

Hence we have shown that H satisfies condition *iii*) of Theorem G.

d) We show that $|E(b)| \leq m$ for any branch $b \in B_H$, where $B_H = \{b \in B_G | \, E(b) \cap E(H) = \emptyset\}$. Let to the contrary, P be a branch of length at least $m+1$ such that none of the edges of P belongs to H. Let x_1, x_2 denote the end-vertices of P. Let Q denote a shortest x_1, x_2-path in $G \setminus P$. If $|E(Q)| > m$, then any cycle containing an edge of P has length at least $2m+2$, since Q is shortest possible. This is a contradiction. Hence Q has length at most m. Let $G_1 = H \triangle Q$ and $U_1 = \{v \in V(H) | \, v \text{ is an internal vertex of } Q \text{ and } d_G(v) = 2\}$. Clearly $|U_1| < m-2$. Then $H' = (G_1 \cup P) - U_1$ contains more vertices of degree at least two than H, a contradiction with *iv*). Hence H satisfies condition iv) of Theorem G.

e) It is easy to see that there is no branch $b \in B_1$ in a 2-connected graph. Hence H satisfies condition v) of Theorem G.

The subgraph $H \in EU_n$ for $n = m-1$, hence $h(G) \leq m-1$. □

Proof of Theorem 6.

1) If $\ell(G) = 3$ then, by Theorem D, G has a DCT. Then, by Theorem C, $L(G)$ is hamiltonian.
2) If $\ell(G) \geq 7$ is an odd integer, then $h(G) \leq \left\lfloor \frac{\ell(G)}{2} \right\rfloor - 1$ by Lemma 3. If $\ell(G) \geq 6$ is an even integer, then every edge of G belongs to a cycle of length at most $\ell(G)+1$ and, by Lemma 3, $h(G) \leq \left\lfloor \frac{\ell(G)}{2} \right\rfloor - 1$ again.
3) If $\ell(G) = 5$, then the statement follows from Lemma 3 since every edge of G belongs to a cycle of length at most 7. Hence $h(G) \leq 2 = \left\lfloor \frac{\ell(G)-1}{2} \right\rfloor$.
4) If $\ell(G) = 4$, then the statement follows from Theorem F. □

Lemma 4. *If G is a connected N_1^m-locally connected graph, then $\ell(G) \leq 2m+1$ and G is 2-connected.*

Proof. By Proposition 1, G is 2-connected. If there is an edge xy in G such that xy does not belong to a cycle of length at most $2m+1$, then x is not N_1^m-locally connected, implying that G is not N_1^m-locally connected, a contradiction. Therefore every edge of G belongs to a cycle of length at most $2m+1$. □

Proof of Theorem 7.
Let G be a connected N_1^m-locally connected graph. By Proposition 1, G is 2-connected. If G is N_2-locally connected, then by Theorem 8 G has a DCT. By Theorem C, $h(G) \leq 1$.

If G is N_1^m-locally connected for m a positive integer, then the statement of this Theorem follows from Theorem 6 and Lemma 4. $\square$

The conditions in Theorem 6 are sharp except possibly for the case $\ell(G) = 4$ and for the case $\ell(G) = 5$.

If $\ell(G) = 3$, then a sharpness example for Theorem 6 is the graph $K_{1,1,n}$ (see Fig. 3). Note that this graph is not hamiltonian for $n \geq 3$. For $\ell(G) \geq 6$ we set m to be the smallest positive integer such that $\ell(G) \leq 2m + 1$. Then the sharpness example is the following. Consider two vertices x, y and three vertex-disjoint x, y-paths, one of length m and two others of length m or $m+1$. It is easy to see that

- G is 2-connected,
- every edge of G belongs to a cycle of length at most $2m + 1$,
- $h_G = m - 1$.

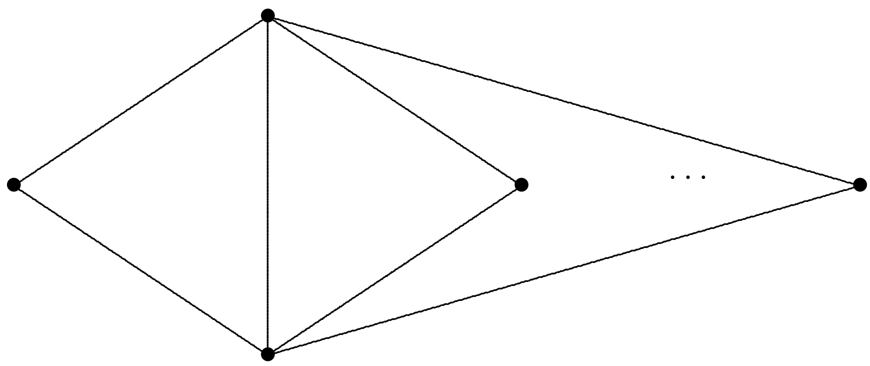

Fig. 3

The conditions in Theorem 7 are sharp except possibly for the case of N_1^2-local connectivity.

If $m = 1$, then a sharpness example is the graph $K_{1,1,n}$, $n \geq 3$. Note that the graph $K_{1,1,n}$ is locally connected but not hamiltonian. Clearly, this graph is also N_2-locally connected, thus $K_{1,1,n}$ is the sharpness example for the case N_2-local connectivity too. If $m \geq 3$, then a sharpness example is the following. We consider two vertices x, y and three vertex-disjoint x, y-paths, one of length m, and two others of length $m+1$. It is easy to see that G is N_1^m-locally connected and $h_G = m-1$.

For the cases of N_1^2-local connectivity and $\ell(G) = 5$ we are not able to find the sharpness example. We believe that Theorem 6 and Theorem 7 can be improved in these cases.

Conjecture 1. *Let G be a connected N_1^2-locally connected graph. Then G has a DCT.*

The following conjecture is an immediate consequence of Conjecture 1.

Conjecture 2. *Let G be a connected N_1^2-locally connected graph. Then $h(G) \leq 1$.*

Lai proved that any 2-connected graph G in which every edge belongs to a cycle of length at most 4 contains a DCT. The next conjecture could be a strengthening of the previous one and also of the theorem of Lai.

Conjecture 3. *Let G be a 2-connected graph such that every edge of G belongs to a cycle of length at most 5. Then $h(G) \leq 1$.*

The following theorems for claw-free graphs suggest that the previous conjectures could be true. Note that the line graph of a graph is claw-free.

Theorem 9. *Let G be a connected claw-free graph such that every edge of G belongs to a cycle of length at most 5. Then $h(G) \leq 1$.*

Proof. It is an immediate consequence of Theorem E and Theorem C. □

Theorem 10. *Let G be a connected claw-free graph, let m be a positive integer. If G is N_1^m-locally connected, then $h(G) \leq m-1$.*

Proof.

1) If G is a connected, locally connected claw-free graph, then G is hamiltonian by Theorem B.
2) If G is a connected, N_1^2-locally connected claw-free graph, then, by Lemma 4 and Theorem 9, $h(G) \leq 1$.
3) The case $m \geq 3$ is an immediate consequence of Theorem 7.

□

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