On a conjecture of Erdős and Turán for additive basis

Jaroslav Nešetřil Oriol Serra

Abstract

An old conjecture of Erdős and Turán states that the representation function of an additive basis of the positive integers can not be bounded. We survey some results related to this still wide open conjecture.

1 Introduction

Let A be a set of positive integers. The *representation function* $r_A(n)$ of A counts for each $n \in \mathbb{N}$ the number of pairs $a, a' \in A,\ a \leq a'$ with $a + a' = n$. The set A is an *asymptotic additive basis of order two* (*additive basis* for short in what follows) if $r_A(n) \geq 1$ for each large enough positive integer. Erdős and Turán [14] formulated in 1941 the following conjecture:

Conjecture 1.1 (ET Conjecture)**.** *The representation function of an additive basis can not be bounded.*

The purpose of this paper is to survey some results on this still wide open conjecture, some of them suggesting a positive answer and some in the opposite direction.

The book of Halberstam and Roth [17], the survey on representation functions of sets of integers by Sárközy and Sós [28] or the extensive account of the work of Erdős in Number Theory by Ruzsa [27] are excellent references for the problem. We will mostly concentrate on results not covered by these references.

A characteristic feature concerning boundedness of the representation function of an infinite set of integers is the order of magnitude of its counting function. The ET Conjecture is only meaningful for the so–called *thin* basis, the ones whose counting function is $O(n^{1/2})$. The ET Conjecture can be easily verified for the known constructions of thin basis. On the other hand, there have been efforts to construct dense infinite sets with bounded representation function. These constructions provide sets whose counting function has an asymptotic growth comparable in a sense to the one of thin basis. We discuss these questions in Section 2.

Section 3 deals with averages of the representation function. The celebrated Erdős-Fuchs Theorem [11] shows that the value of the representation function may not be well approximated by its average. More recently Ruzsa [25] constructed an additive base whose representation function is bounded in the square mean. These results give a better understanding of the difficulties involved in the ET Conjecture.

Among the few positive results for the ET Conjecture is its verification for the class of so–called d–bounded basis which is presented in Section 4.

In trying to get a wider picture of the problem, some semigroups other than the natural numbers with addition have been considered. There one may find positive results (for some semigroups) and negative ones (for groups) which are presented in Section 5. We finally discuss in Section 6 similar problems for linear functions of the form $ax + by$ with $a \neq b$, for which the analogous notion of basis may lead to bounded representation functions. We conclude with some final remarks and open problems, some of them probably easier than the original problem.

2 Thin basis and $B_h[g]$ sets

Let $A(n) = |A \cap [1, n]|$ denote the counting function of A. Note that, if $r_A(n)$ is bounded by a constant c, then $\binom{A(n)}{2} \leq \sum_{x \leq n} r_A(x) \leq cn$ which implies $A(n) \ll n^{1/2}$. On the other hand if A is an additive basis for the positive integers then we certainly have $A(n) \gg n^{1/2}$.

Erdős and Rényi [13] proved that, for each $\epsilon > 0$, there is a set A verifying $A(n) \gg n^{1/2-\epsilon}$ such that the representation function of A is bounded. Thus, in terms of density, the value $\alpha = 1/2$ is a threshold for the boundedness of the representation function of a set A verifying $A(n) \asymp n^{\alpha}$. This fact enhances the relevance of the ET Conjecture, and it shows that it is only relevant for *thin* basis, namely the ones for which $\limsup_{n\to\infty} n^{-1/2} A(n) < \infty$. In fact Erdős and Turán further strengthened the ET Conjecture by extending it to infinite sets A verifying only the condition $A(n) \gg n^{1/2}$. We will refer to this as the *strong* Erdős–Turán (s–ET) Conjecture.

Conjecture 2.1 (s-ET Conjecture). *The representation function of an infinite set of positive integers whose counting function verifies $A(n) \gg n^{1/2}$ can not be bounded.*

Constructions of thin additive basis are known since the work of Raikov [24], Stöhr [29] and Cassels [3], and some progress has been made in improving the value of the constant $c = \inf_{A \text{ basis}} \limsup_{n\to\infty} n^{-1/2}A(n)$. The constructions are based essentially on two different principles.

The first kind of thin basis is based on d-adic representations. Let $J \subset \mathbb{N}$ and denote by A_J the set of elements of the form $\sum_{j=1}^{k} x_j 4^j$ where $x_i \in \{0,1\}$ if $j \in J$ and $x_i \in \{0,2\}$ otherwise. Then $A = A_J \cup A_{\mathbb{N}\setminus J}$ is clearly an additive base. By choosing J to be the set of even numbers Hofmeister [18] gets $c \leq 2\sqrt{5}/3 \approx 2.581$, a bound recently improved by Blomer [1] to a value arbitrarily close to $\sqrt{3}/((\sqrt{2}-1)8^{1/4}) \approx 2.486$. On the other hand Stöhr proved that $c \geq \sqrt{8/\pi} \approx 1.595$.

The resulting basis has obviously an unbounded representation function: choose a subset $I \subset J$ (where we may assume that J is infinite) and consider $n = \sum_{i\in I} 4^i$; for each partition I_1, I_2 of I into two nonempty parts we have $x_k = \sum_{i\in I_k} 4^i \in A_J$, $k = 1,2$ so that $r_A(n) \geq 2^{|I|-1}$.

The second kind of construction is originally due to Cassels [3] whose aim was to construct an additive basis for which the $\lim n^{-1/2}A(n)$ do exists. Again for illustration we consider a special case based on the Fibonacci numbers $f_1, f_2, \ldots$. The constructed basis is the union of arithmetic progressions $A = \cup_{j\geq 1}\{a_j + rf_j : 0 \leq r < f_{j+3}\}$ where $a_j = \sum_{2\leq i\leq j} f_{i-1}f_{i+2}$. The existence of arbitrarily large arithmetic progressions in A, which is common to this kind of constructions, makes the representation function unbounded.

On the other hand, there have been efforts to construct dense sets with bounded representation function. The class of infinite sets with representation function bounded by g is denoted by $B_2[g]$. When $g = 1$ they are called Sidon sets. Erdős showed that an infinite Sidon set A verifies $\liminf_{n\to\infty} A(n)/(\sqrt{n/\log n}) < \infty$, in particular the s-ET Conjecture is true if the representation function is bounded by $g = 1$, although it is open for any $g \geq 2$. In the opposite direction Ruzsa [26] gives a construction of a Sidon set A with $A(n) \gg n^{\sqrt{2}-1+o(1)}$, a substantial improvement on the previous known results.

Kolountzakis [19] has shown that already for $g = 2$ there are infinite sets $B_2[2]$ with $\limsup_{n\to\infty} n^{-1/2}A(n) = 1$. Cilleruelo and Trujillo [5] obtained the best known result up to now for all g by constructing infinite sets in $B_2[g]$ with $\limsup_{n\to\infty} n^{-1/2}A(n) \geq \sqrt{(9/8)(g-1)}$. Thus there are sets with bounded representation function which are more dense (in the sense of the considered limits) than thin bases.

3 Averaging the representation function

The known constructions of thin basis show that, for A an additive basis, the average $\frac{1}{N}\sum_{x\le N} r_A(x)$ can be bounded. Erdős showed that if instead we choose an analogous function for the differences, $d_A(x) = \{(a,a') \in A :\ x = a - a'\}$ then, for every set A verifying $A(x) \gg \sqrt{x}$ we have

$$\frac{1}{N}\sum_{x\le N} d_A(x) \gg \log N. \tag{1}$$

In particular (1) shows that the function $d_A(x) = \{(a,a') \in A : , x = a - a'\}$ of an additive base A (which verifies $A(x) \gg \sqrt{x)}$ can not be bounded. Moreover A can not be a Sidon set, since $r_A(x) \le 1$ for all x is equivalent to $d_A(x) \le 1$ for all x, which would contradict (1). However even establishing that $\limsup_{n\to\infty} r_A(n) \ge c$ for a fixed constant $c \ge 3$ when A is an additive base, appears to be a nontrivial problem. By analyzing the traces of additive basis in finite intervals, Grekos et al. [15] showed that we can always take $c \ge 3$. More recently Borwein, Cho and Chu [2] improved it to $c \ge 4$.

By extending a classical result of Hardy and Landau for the squares, the celebrated theorem of Erdős and Fuchs [11] states that the representation function of any infinite set A can not have an eventually constant average. More precisely,

$$\sum_{x\le N} r_A(x) = cN + o(N^{1/4}(\log N)^{-1/2})$$

can not hold for any constant c. They also proved that, for any infinite set A with $A(n) \gg n^{1/2}$ and for each constant $c \ge 0$, we have

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{x\le N} (r_A(x) - c)^2 > 0.$$

These two results indicate an irregularity of the representation function which is compatible with the fact that r_A is not bounded.

Note that if there exists an additive basis A whose representation function is bounded by g then

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{x\le N} r_A^2(x) \le g \limsup_{N\to\infty} \frac{1}{N}\sum_{x\le N} r_A(x) < \infty,$$

where the last inequality follows from the fact that A should be a thin basis. Thus a proof that the mean of squares of the representation function is always unbounded for additive basis would have settled the ET Conjecture. However Ruzsa [25] gave a construction of an additive basis A for which $\limsup_{N\to\infty} \frac{1}{N}\sum_{x\le N} r_A^2(x) < \infty$. This construction is also relevant for the existence of basis of abelian groups with representation function bounded by an absolute constant, which we discuss in Section 5.

4 Basis with the ET property

An important problem in additive number theory is to investigate sets of integers containing *Hilbert cubes* which are sets of the form

$$H_k = \{h_0 + \sum_{i=1}^{k} \epsilon_i h_i,\ \epsilon_i \in \{0,1\}\}, \tag{2}$$

where $h_0, h_1, \ldots, h_k$ are integers and k is the dimension of the cube. It is clear that if a set A of positive integers contains arbitrary large Hilbert cubes then its representation function can not be bounded. Indeed, for any subset $J \subset [1,k]$, we can write

$$n = 2h_0 + \sum_{i=1}^{k} h_i = \left(h_0 + \sum_{i\in J} h_i\right) + \left(h_0 + \sum_{i\in[1,k]\setminus J} h_i\right). \tag{3}$$

Hence $r_A(n) \ge |H_k|/2$.

We next give a sufficient condition for a base to contain arbitrarily large Hilbert cubes. In particular the resulting basis verify the ET Conjecture. For a positive integer n let us denote by $S(n)$ the binary support of n, so that $n = \sum_{i\in S(n)} 2^i$.

Definition: An additive basis A is 2–*bounded* if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that, for each n, there are $x, y \in A$ verifying

$$n = x + y \quad \text{and} \quad |S(x) \cup S(y)| \le f(|S(n)|). \tag{4}$$

Theorem 4.1. *Any* 2*–bounded base contains arbitrarily large Hilbert cubes.*

Proof. For any given positive integer k we shall construct a Hilbert cube in A of cardinality at least $2^{k/2}$.

Since A is 2–bounded there is $f : \mathbb{N} \to \mathbb{N}$ for which (4) holds. Let $l = f(k)$ and let $r = r(k, 2k, k2^{l+1})$ be the Ramsey number for the partition of k–subsets into $k2^{l+1}$ classes having an homogeneous set of cardinality $2k$.

Choose an arithmetic progression P starting at l of length r and difference l. Color its k–subsets as follows.

Given a k–subset $U \subset P$, let $n = \sum_{i\in U} 2^i$. Since the basis is bounded, there are elements $x, y \in A$ with $n = x + y$ and $|S(x) \cup S(y)| \leq l$. Therefore $S(x) \cup S(y)$ is contained in the union of intervals

$$S(x) \cup S(y) \subset \cup_{u\in U}[u - l + 1, u].$$

Let $u_1 < u_2 < \cdots < u_k$ be the elements of U and denote by $x_i = (x_{i1}, \ldots, x_{il})$ the binary vector with $x_{ij} = 1$ if $u_i - l + j \in S(x)$ and $x_{ij} = 0$ otherwise. Let $x_U = (x_1, x_2, \ldots, x_k)$. Likewise define y_U. Since the elements of P are at mutual distance l, at least half of the l–tuples in one of x_U or y_U, say x_U, are different from the all–zero vector. Then U is colored by x_U.

This way we color all k–subsets of P with at most $k2^{l+1}$ colors. By Ramsey theorem there is a subset $Z \subset P$ of cardinality $2k$ with all its k–subsets colored by the same color

$$\alpha = ((\alpha_{1,1}, \ldots, \alpha_{1,l}), \ldots, (\alpha_{k,1}, \ldots, \alpha_{k,l})).$$

Let $z_1 < z_2 < \cdots < z_{2k-1} < z_{2k}$ be the elements of the homogeneous set Z. Denote by

$$a_i = \sum_{j=1}^{l} \alpha_{i',j} 2^{z_i - l + j},\ i' = \lceil i/2 \rceil,\ 1 \leq i \leq 2k,\ .$$

Note that, since the elements of Z are at mutual distance at least l, the supports $S(a_1), \ldots, S(a_{2k})$ are pairwise disjoint. Let

$$A' = \{a_1, a_2\} + \{a_3, a_4\} + \cdots + \{a_{2k-1}, a_{2k}\}.$$

Observe that, for each element $a = a_{i_1} + \cdots + a_{i_k}$ in A', the set $\{z_{i_1}, \ldots, z_{i_{2k}}\}$ is colored α, which means that for the integer n with support $S(n) = \{z_{i_1}, \ldots, z_{i_{2k}}\}$ the element a was chosen from the base. Therefore $A' \subset A$. Moreover, by the definition of the coloring, there are at least $k/2$ sets $\{a_{2i-1}, a_{2i}\}$ different from $\{0\}$. Therefore A' is a Hilbert cube (with $h_0 = a_1 + a_3 + \cdots + a_{2k-1}$ and $h_i = a_{2i} - a_{2i-1}$, $i > 0$) with cardinality $|A'| = 2^{k/2}$. This completes the proof. □

Corollary 4.2 ([22]). *A 2–bounded additive basis verifies the ET conjecture.*

In [22] a more general form of Corollary 4.2 is proved by essentially the same arguments. The formulation given here through Theorem 4.1 highlights the fact that a 2–bounded base contains arbitrarily large symmetric sets with some additional structure: they are Hilbert cubes.

The proof of Theorem 4.1 easily extends to d–adic expansions with $d \geq 2$. In this case the support $S_d(x)$ of an element $x = \sum_{i\geq 0} x_i d^i$ is the set of indices for which x_i is nonzero, and the ET conjecture holds d–bounded basis (verifying (4) with S_d in the place of S).

Even if the property of being d–bounded is a particular one, there are relatively large subsets of integers which only contain such basis.

Corollary 4.3. *For every $\epsilon > 0$ there is a set $X \subset \mathbb{N}$ with $X(n) > n^{1-\epsilon}$ such that every additive base $A \subset X$ is 2–bounded. In particular A verifies the ET–Conjecture.*

Proof. Let l be a positive integer and consider the set X of integers x whose binary expansion $x = \sum_{i\geq 0} x_i 2^i$ verifies $x_{lm} = 0$ for each $m \geq 1$. For every pair $x, x' \in X$ and each interval $I_m = [lm+1, l(m+1)]$, if $(S(x) \cap S(y)) \cap I_m$ is nonempty then $S(x+y) \cap I_m$ is nonempty as well. Therefore,

$$|S(x) \cup S(x')| \leq l|S(x+x')|,$$

an every base $A \subset X$ is d-bounded. By choosing $l > \lceil \frac{1}{\epsilon} \rceil$ we have $X(n) \geq 2^{x-x/l} > 2^{x(1-\epsilon)} = n^{1-\epsilon}$ where $x = \log_2 n$. ☐

5 The ET property in semigroups

The difficulty involved in the ET Conjecture motivated the study of analogous problems in semigroups other than the additive positive integers. Erdős himself proved [9] that the set of positive integers with multiplication verifies the ET Conjecture.

Theorem 5.1 (Erdős [9], Nešetřil and Rödl [21])**.** *If every positive integer can be written as a product of two elements in a set A then, for every $k \in \mathbb{N}$ there is an integer which can be written in more than k different ways as a product of two elements in A.*

Nešetřil and Rödl [21] gave a simple proof of the above result by using Ramsey theorem. Their argument inspired the analogous proof of Theorem 4.1 and similar positive results for other classes of semigroups described below.

Let $(G, *)$ be a commutative semigroup and $X \subset G$. A subset $A \subset X$ is a basis for X if $A * A \supset X$. We say that X has the $ET(k)$ property if the representation function of any basis of X is not bounded by k. If X has the $ET(k)$ property for each $k \in \mathbb{N}$ then we simply say that X has the ET property. We say that an infinite class $\mathcal{G}$ of semigroups has the ET property if for every $k \in \mathbb{N}$, all but a finite number of members of $\mathcal{G}$ have the $ET(k)$ property.

The next Theorem provides a wide class of examples of semigroups which verify the ET property as defined above. We say that a finite subset R in a semigroup G with a distinguished idempotent element e is *antisymmetric* if $e \in R$ and the equation $x * y = e$ holds in R if and only if $x = y = e$. We denote by G^N the direct product of N copies of G. We proved [23] the following

Theorem 5.2. *Let $(G, *)$ be a commutative semigroup and let $e \in G$ be an idempotent element. Let R be a finite antisymmetric set with $|R| > 1$ containing e. For each $k \in \mathbb{N}$ there is $N = N(k)$ such that R^N has the $ET(k)$ property.*

Theorem 5.2 allows us to identify several examples of semigroups with the ET–property. In particular it is shown in [23] that if $(P, \vee)$ is a finite semilattice with maximum and minimum elements then the class $\{(P^N, \vee), N \in \mathbb{N}\}$ and $(P^{\mathbb{N}}, \vee)$ have the ET property. In particular this applies to the family of finite (or cofinite) subsets of a countable set X with respect to union or intersection. A more interesting example in our present setting is the following.

Corollary 5.3. *The family $\{(\mathbb{N}^N, +), N \in \mathbb{N}\}$, where the sum is componentwise, has the ET–property. Similarly, the semigroup of infinite sequences of positive integers with finite support has the ET–property.*

When we move from semigroups to groups the situation is very different. It is not difficult to show that there is $A \subset \mathbb{Z}$ with $r_A(x) = 1$ for every $x \in \mathbb{Z}$. A greedy construction of such an A produces a very sparse set. The contrast with the case of positive integers is illustrated by a construction of Cilleruelo and Nathanson [4] giving a dense set of integers whose representation function matches an arbitrary function. More precisely, these authors prove that, given $\epsilon > 0$ there is a constant $c = c(\epsilon)$ such that, for every function $f : \mathbb{Z} \to \mathbb{N}$ verifying $\liminf_{|n| \to \infty} f(n) > c(\epsilon)$, there is a set A with $A(x) \gg x^{1/2-\epsilon}$ and $r_A(x) = f(x)$ for all x (here $A(x)$ denotes the number of elements in $A \cap [-x, x]$.)

For finite groups a similar situation arises. Let $\mathcal{P}$ be the set of primes p for which 2 is not a quadratic residue modulo p. Ruzsa [25] constructs, for every prime $p \in \mathcal{P}$, a base $A \subset \mathbb{Z}_p \times \mathbb{Z}_p$ with representation function r_A bounded from above by 18. This shows that the class $\{\mathbb{Z}_p \times \mathbb{Z}_p,\ p \in \mathcal{P}\}$ does not have the ET–property. The construction has been extended by Haddad and Helou [16] to groups of the form $G \times G$ and direct sums $G^{\mathbb{N}}$ where G is the additive group of a finite field, or an infinite algebraically closed field, of characteristic $\neq 2$. The same ideas have been used by Min and Cheng [30] to show that the class of cyclic groups has not the ET property. This is in contrast with the next Corollary to Theorem 5.2 which states that the $ET(k)$ property holds in direct products of groups if we restrict ourselves to an *antisymmetric* set.

Corollary 5.4. *Let G be a finite abelian group and let $R \subset G$ with $R \cap (-R) = \{0\}$. For each $k \in \mathbb{N}$ there is $N = N(k)$ such that R^N has the $ET(k)$ property. In particular, the group of infinite sequences of elements of R with finite support has the ET property.*

Corollary 5.4 shows that there are abelian groups which admit basis with representation function bounded by an absolute constant (for instance the p–elementary group $\mathbb{Z}_p^N$ which admits a basis whose representation function is bounded by 18) while containing antisymmetric sets with the ET property. The ET Conjecture says that this is the case for the group of integers.

6 Other linear equations

When moving from $A + A$ to $a \cdot A + b \cdot A$, where $t \cdot A = \{ta : \ a \in A\}$ the analogous problem of the resulting representation function may change drastically. Let us denote by $r_{A;a,b}(n)$ the number of representations of the form $n = ax + by$ with $x, y \in A$. We may assume without loss generality that $\gcd(a, b) = 1$.

Moser [20] showed that when $a = 1$ and $b > 1$ there exists a (unique) set A such that every positive integer can be uniquely written as $x + by$ with $x, y \in A$. For instance, the set $A = \{\sum_{i\geq 0} \epsilon_i 2^{2i}, \ \epsilon = 0 \text{ or } 1\}$ verifies this property when $b = 2$.

By solving a problem in [28, Problem 7.1] for the case of two–fold sums, Cilleruelo and Rué [6] recently proved that, when $a > b > 1$, the representation function $r_{A;a,b}$ can not be eventually constant. This extends an old result of Dirac and Newman who had proved the analogous result for the case $a = b = 1$, and shows that Moser's result is a quite particular one.

On the other hand, a set A such that $r_{A;a,b}(n) \leq 1$ for each n and $S = a \cdot A + b \cdot A$ has positive density is constructed in [7]. In particular, $A(n) \gg n^{1/2}$. This indicates that, if true, the strengthened version of the ET Conjecture which just requires $A(x) \gg x^{1/2}$, may only hold in the case $a = b = 1$.

7 Final remarks and problems

The validity of the s–ET Conjecture would imply that in any finite partition of an additive base there is a part which has unbounded representation function. This has been proved to be the case [22] for the class of d–bounded basis, and it is probably an easier problem to solve than the s–ET Conjecture itself. The following problem was already proposed in [22]:

Problem 7.1. *Let A be an additive basis of the positive integers. Assume that the representation function of A is not bounded. Is it true that in any finite partition $A = A_1 \cup \cdots \cup A_t$ there is a part A_i with unbounded representation function?*

The answer is strongly negative if we replace $\mathbb{N}$ by the set of all integers. It was shown in [22] that, for an arbitrary function $f : \mathbb{Z} \to \mathbb{N}$ there is a basis A with representation function f which can be split into two B_2 sets.

One can formulate a Ramsey version of the s–ET Conjecture. According to our discussion in Section 2, if a monochromatic set in a coloring of the positive integers has cardinality $\Omega(n^{1/2}\omega(n))$ for some increasing function $\omega : \mathbb{N} \to \mathbb{N}$, then the representation function of this set can not be bounded. This suggests the following problem.

Problem 7.2. *Is it true that, for every coloring $\chi : \mathbb{N} \to \mathbb{N}$ of the positive integers verifying $|\chi^{-1}([1, n])| \sim n^{1/2}$, there is a monochromatic set with unbounded representation function?*

One can formulate a finite version of the above problem. Let $f(k, n)$ be the minimum number of colors t such that every t–coloring of the interval $[1, n]$ contains a monochromatic set B such that $\max r_B(x) \geq k$.

Problem 7.3. *Is there a function $\omega : \mathbb{N} \to \mathbb{N}$ with $\limsup_{n\to\infty} \omega(n) = \infty$ such that $\limsup_{n\to\infty} n^{-1/2} f(\omega(n), n) > 0$?*

The analogous problem for abelian groups, where a symmetric set verifies $(B - x) = -(B - x)$ for some element x of the group, may have a negative answer in a strong sense. For instance, one can partition $G_p = \mathbb{Z}_p \times \mathbb{Z}_p$ into the p Sidon sets $A_i = \{(x, x^2 + i) : x \in \mathbb{Z}_p\}$, $0 \leq i \leq p - 1$ so that, for the class $\{G_p,\ p \text{ prime}\}$, we have $\lim_{p\to\infty} |G_p|^{-1/2} f(k, |G_p|) = 0$ for each $k > 2$.

The notion of 2–bounded basis discussed in Section 4 can be extended to general sets. Say that a set A of integers is 2–bounded if there is a function f such that $|S(x) \cup S(y)| \leq f(|S(x + y)|)$ for every pair $x, y \in A$. One may consider the s–ET Conjecture restricted to 2–bounded sets.

Problem 7.4. *Let A be a 2–bounded infinite set such that $A(x) \gg x^{1/2}$. Is it true that the representation function of A can not be bounded?*

As a final remark let us mention that we have only considered the ET Conjecture with basis of order 2. A set $A \subset \mathbb{N}$ is a basis of order $h \geq 2$ if every positive integer can be expressed as a sum of h elements in A and the ET Conjecture can be extended to these basis as well. When $h = 2k$ is an even number, then a basis A of order h gives rise to the basis kA of order two, and if the latter has unbounded representation function so does A. However, when h is odd there is no obvious way to reduce the problem to basis of order 2. Nevertheless, the positive results in sections 4 and 5 can be extended to basis of arbitrary order h, see [22, 23].

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Jaroslav Nešetřil
Department of Applied Mathematics and ITI,
Charles University,
Prague, Czech Republic
nesetril@kam.ms.mff.cuni.cz

Oriol Serra
Departament de Matemàtica Aplicada IV
Universidad Politècnica de Catalunya
08034 Barcelona, Spain
oserra@ma4.upc.edu