MINORS OF BOOLEAN FUNCTIONS WITH RESPECT TO CLIQUE FUNCTIONS AND HYPERGRAPH HOMOMORPHISMS

ERKKO LEHTONEN AND JAROSLAV NEŠETŘIL

Abstract. Each clone $\mathcal{C}$ on a fixed base set A determines a quasiorder on the set of all operations on A by the following rule: f is a $\mathcal{C}$-minor of g if f can be obtained by substituting operations from $\mathcal{C}$ for the variables of g. By making use of a representation of Boolean functions by hypergraphs and hypergraph homomorphisms, it is shown that a clone $\mathcal{C}$ on $\{0,1\}$ has the property that the corresponding $\mathcal{C}$-minor partial order is universal if and only if $\mathcal{C}$ is one of the countably many clones of clique functions or the clone of self-dual monotone functions. Furthermore, the $\mathcal{C}$-minor partial orders are dense when $\mathcal{C}$ is a clone of clique functions.

1. Introduction

This paper is a study of substitution instances of functions when the inner functions are taken from a given set of functions. Several variants of this idea have been used in the theory of Boolean functions. Harrison [5] studied the equivalence relation of Boolean functions where n-ary functions are considered equivalent if they are substitution instances of each other with respect to the general linear group $\mathrm{GL}(n, \mathbb{F}_2)$ or the affine general linear group $\mathrm{AGL}(n, \mathbb{F}_2)$ ($\mathbb{F}_2$ denotes the two-element field). Wang and Williams [27] defined a Boolean function f to be a *minor* of another Boolean function g, if f can be obtained from g by substituting for each variable of g a variable, a negated variable, or one of the constants 0 or 1. Wang [26] characterized classes of Boolean functions by forbidden minors. Feigelson and Hellerstein [3] and Zverovich [29] presented variants of minors and characterized classes of Boolean functions by forbidden minors. Further generalizations of the notion of minor to operations on arbitrary finite sets were presented by Pippenger [21].

A common framework for these results is provided by the notions of $\mathcal{C}$-minor and $\mathcal{C}$-equivalence, where $\mathcal{C}$ is an arbitrary clone. Let $\mathcal{C}$ be a fixed clone on a base set A, and let f and g be operations on A. We say that f is a $\mathcal{C}$-minor of g, if f can be obtained from g by substituting operations from $\mathcal{C}$ for the variables of g, i.e., $f = g(h_1, \ldots, h_n)$ for some $h_1, \ldots, h_n \in \mathcal{C}$. If f and g are $\mathcal{C}$-minors of each other, then we say that f and g are $\mathcal{C}$-equivalent. These general notions of $\mathcal{C}$-minor and $\mathcal{C}$-equivalence first appeared in print in [13], where the first author studied the $\mathcal{C}$-minor quasiorder for clones $\mathcal{C}$ of monotone and linear functions.

This work was supported by the Academy of Finland (grant no. 120307), by the Czech Ministry of Education (grant no. 1M0021620808), and by AEOLUS.

In this paper we study the $\mathcal{C}$-minor relations of Boolean functions when the parametrizing clone $\mathcal{C}$ is one of the countably many clones of clique functions or the clone of self-dual monotone functions. We represent Boolean functions by hypergraphs and we establish a connection between the $\mathcal{C}$-minor relations determined by the clones of clique functions and certain variants of hypergraph homomorphisms, called sup-homomorphisms. Our main result is that the $\mathcal{C}$-minor partial order of Boolean functions is universal in the sense that it admits an embedding of every countable poset if and only if $\mathcal{C}$ is one of the clones mentioned above. Furthermore, we show that the $\mathcal{C}$-minor partial orders determined by clones of clique functions are dense. Density and universality are perhaps the two most studied general properties of homomorphism order; see [8].

2. Preliminaries

We denote n-tuples by bold face letters and their components by italic letters, e.g., $\mathbf{a} = (a_1, \dots, a_n)$.

Let A be a fixed nonempty base set. An *operation* on A is a mapping $f \colon A^n \to A$ for some positive integer n, called the *arity* of f. Denote by $\mathcal{O}_A = \bigcup_{n \geq 1} A^{A^n}$ the set of all operations on A. Operations on $\mathbb{B} = \{0, 1\}$ are called *Boolean functions.*

For $1 \leq i \leq n$, the i-th n-ary *projection* is the operation $(a_1, \dots, a_n) \mapsto a_i$, and it is denoted by $\mathfrak{p}_i^{(n)}$. The *constant operation* $(a_1, \dots, a_n) \mapsto a$ for some $a \in A$ is denoted by $\mathfrak{c}_a^{(n)}$. We may omit the superscripts indicating the arity and write simply $\mathfrak{p}_i$ or $\mathfrak{c}_a$ when the arity is clear from the context.

If f is an n-ary operation and $g_1, \dots, g_n$ are all m-ary operations, then the *composition* of f with $g_1, \dots, g_n$, denoted $f(g_1, \dots, g_n)$, is the m-ary operation defined by

$$f(g_1, \dots, g_n)(\mathbf{a}) = f\big(g_1(\mathbf{a}), \dots, g_n(\mathbf{a})\big)$$

for all $\mathbf{a} \in A^m$.

A *class* is a subset $\mathcal{C} \subseteq \mathcal{O}_A$. A *clone* on A is a class $\mathcal{C}$ that contains all projections and is closed under composition, i.e., $f, g_1, \dots, g_n \in \mathcal{C}$ implies $f(g_1, \dots, g_n) \in \mathcal{C}$ whenever the composition is defined. The clones on A form an inclusion-ordered lattice. The clones of Boolean functions were completely described by Post [23], and they are called *Post classes.* The lattice of clones on $\{0, 1\}$ is referred to as the *Post lattice.* The clones on sets with more than two elements remain largely unknown and are an important topic of investigation in universal algebra. See Appendix for the nomenclature for Post classes. For general background on clones, see, e.g., [11, 22, 25].

Let $\mathcal{C}$ be a fixed class of operations on A. We say that f is a $\mathcal{C}$-*minor* of g, denoted $f \leq_{\mathcal{C}} g$, if $f = g(h_1, \dots, h_m)$ for some $h_1, \dots, h_m \in \mathcal{C}$. We say that f and g are $\mathcal{C}$-*equivalent,* denoted $f \equiv_{\mathcal{C}} g$, if they are $\mathcal{C}$-minors of each other.

The $\mathcal{C}$-minor relation $\leq_{\mathcal{C}}$ is a quasiorder (a reflexive and transitive relation) on $\mathcal{O}_A$ if and only if the parametrizing class $\mathcal{C}$ is a clone. If $\mathcal{C}$ is a clone, then $\equiv_{\mathcal{C}}$ is an equivalence relation on $\mathcal{O}_A$, and the $\equiv_{\mathcal{C}}$-class of f is denoted by $[f]_{\mathcal{C}}$.

As for quasiorders, the $\mathcal{C}$-minor relation $\leq_\mathcal{C}$ induces a partial order $\preccurlyeq_\mathcal{C}$ on the quotient $\mathcal{O}_A/{\equiv_\mathcal{C}}$: $[f]_\mathcal{C} \preccurlyeq_\mathcal{C} [g]_\mathcal{C}$ if and only if $f \leq_\mathcal{C} g$.

Let us make a few simple observations on $\mathcal{C}$-minors. It is clear from the definition that if $\mathcal{C}$ and $\mathcal{K}$ are clones such that $\mathcal{C} \subseteq \mathcal{K}$, then $\leq_\mathcal{C} \subseteq \leq_\mathcal{K}$ and $\equiv_\mathcal{C} \subseteq \equiv_\mathcal{K}$. Since $\mathfrak{c}_a^{(n)}(\phi_1, \dots, \phi_n) = \mathfrak{c}_a^{(m)}$ for any m-ary operations $\phi_1, \dots, \phi_n$, it is easy to see that $[\mathfrak{c}_a] = \{\mathfrak{c}_a^{(n)} : n \geq 1\}$ is a $\equiv_\mathcal{C}$-class for any clone $\mathcal{C}$ and it is a minimal element of the $\mathcal{C}$-minor partial order $\preccurlyeq_\mathcal{C}$.

In this paper we will mostly deal with Boolean functions, and we note the following basic fact which is specific to the clones of Boolean functions.

Lemma 1. *Let $\mathcal{C}$ be a clone of Boolean functions. Then $\mathcal{C} \setminus ([\mathfrak{c}_0] \cup [\mathfrak{c}_1])$ is a $\equiv_\mathcal{C}$-class.*

Proof. It is clear that $f \leq_\mathcal{C} \mathfrak{p}_1^{(1)}$ if and only if $f \in \mathcal{C}$. We show that $\mathfrak{p}_1^{(1)} \leq_\mathcal{C} f$ for every $f \in \mathcal{C} \setminus ([\mathfrak{c}_0] \cup [\mathfrak{c}_1])$, from which the statement follows by the transitivity of $\mathcal{C}$-equivalence.

Let $f \in \mathcal{C} \setminus ([\mathfrak{c}_0] \cup [\mathfrak{c}_1])$ and consider $f(\mathfrak{p}_1^{(1)}, \mathfrak{p}_1^{(1)}, \dots, \mathfrak{p}_1^{(1)})$, which will be a unary member of the clone $\mathcal{C}$ since $f, \mathfrak{p}_1^{(1)} \in \mathcal{C}$. There are four unary Boolean functions: $\mathfrak{p}_1$, $\bar{\mathfrak{p}}_1$, $\mathfrak{c}_0$, $\mathfrak{c}_1$, where $\bar{\mathfrak{p}}_1$ denotes the *negation map* $0 \mapsto 1$, $1 \mapsto 0$. If $f(\mathfrak{p}_1, \mathfrak{p}_1, \dots, \mathfrak{p}_1) = \mathfrak{p}_1$, then $\mathfrak{p}_1 \leq_\mathcal{C} f$ and we are done. If $f(\mathfrak{p}_1, \mathfrak{p}_1, \dots, \mathfrak{p}_1) = \bar{\mathfrak{p}}_1$, then $\bar{\mathfrak{p}}_1 \in \mathcal{C}$, and $f(\bar{\mathfrak{p}}_1, \bar{\mathfrak{p}}_1, \dots, \bar{\mathfrak{p}}_1) = f(\mathfrak{p}_1, \mathfrak{p}_1, \dots, \mathfrak{p}_1)(\bar{\mathfrak{p}}_1) = \bar{\mathfrak{p}}_1 \circ \bar{\mathfrak{p}}_1 = \mathfrak{p}_1$, and we are done.

Consider then the case that $f(\mathfrak{p}_1, \mathfrak{p}_1, \dots, \mathfrak{p}_1) = \mathfrak{c}_0$. Since f is assumed to be nonconstant, there is an n-tuple $\mathbf{a} \in \mathbb{B}^n$ such that $f(\mathbf{a}) = 1$. For $i = 1, \dots, n$, let $\phi_i = \mathfrak{c}_0$ if $a_i = 0$ and let $\phi_i = \mathfrak{p}_1$ if $a_i = 1$. Now $f(\phi_1, \phi_2, \dots, \phi_n) = \mathfrak{p}_1$, and since $\mathfrak{c}_0, \mathfrak{p}_1 \in \mathcal{C}$, we have that $\mathfrak{p}_1 \leq_\mathcal{C} f$.

The case that $f(\mathfrak{p}_1, \mathfrak{p}_1, \dots, \mathfrak{p}_1) = \mathfrak{c}_1$ is proved similarly. Now there is an n-tuple $\mathbf{a} \in \mathbb{B}^n$ such that $f(\mathbf{a}) = 0$, and for $i = 1, \dots, n$, we let $\phi_i = \mathfrak{c}_1$ if $a_i = 1$ and we let $\phi_i = \mathfrak{p}_1$ if $a_i = 0$. Then $f(\phi_1, \phi_2, \dots, \phi_n) = \mathfrak{p}_1$ and we are done. □

In our study of the $\mathcal{C}$-minor relations of Boolean functions determined by the clones of clique functions, we will make use of hypergraph homomorphisms, which will be introduced in the following section.

3. Hypergraphs and homomorphisms

The *power set* of a set A, i.e., the set of all subsets of A, is denoted by $\mathcal{P}(A)$. Denote by $\mathcal{P}'(A)$ the set of all nonempty subsets of A, i.e., $\mathcal{P}'(A) = \mathcal{P}(A) \setminus \{\emptyset\}$. For an integer $k \geq 1$, denote $\mathcal{P}'_k(A) = \{S \subseteq A : 0 < |S| \leq k\}$. The sets $\mathcal{P}(A)$, $\mathcal{P}'(A)$, $\mathcal{P}'_k(A)$ are partially ordered by subset inclusion $\subseteq$. A subset $B \subseteq P$ of a partially ordered set $(P, \leq)$ is *downward closed* if $S \in B$ and $S' \leq S$ together imply $S' \in B$ for all $S, S' \in P$. Dually, a subset $B \subseteq P$ is *upward closed* if $S \in B$ and $S \leq S'$ together imply $S' \in B$ for all $S, S' \in P$. The complement of any downward closed set is upward closed, and vice versa. For any subset B of a partially ordered set P, the smallest upward closed set containing B is denoted by $\uparrow B$, and the smallest downward closed set containing B is denoted by $\downarrow B$.

Let V be a finite set and let $E \subseteq \mathcal{P}'(V)$. The couple $G = (V, E)$ is called a *hypergraph* on V. The elements of V are called the *vertices* and the elements of E are called the *edges.* We may also denote the set of vertices and the set of edges of a hypergraph G by $V(G)$ and $E(G)$, respectively. An edge of order 1 is called a *loop,* and a hypergraph with no loops is called *loopless.* The couple $(\emptyset, \emptyset)$ is called the *empty hypergraph.* For general background on hypergraphs, see, e.g., [1].

The *rank* of a hypergraph $G = (V, E)$ is the number $\max_{S \in E} |S|$. Hypergraphs of rank at most 2 are ordinary undirected graphs with loops allowed but with no parallel edges. A hypergraph $G = (V, E)$ is said to be *k-uniform* if for every edge $S \in E$ we have that $|S| = k$.

The *complement* of a hypergraph $G = (V, E)$ is the hypergraph $\overline{G} = \bigl(V, \mathcal{P}'(V) \setminus E\bigr)$. The *$k$-complement* of $G = (V, E)$ is the hypergraph $\overline{G}^k = \bigl(V, \mathcal{P}'_k(V) \setminus E\bigr)$. It is straightforward to verify that $\overline{\overline{G}} = G$ for every hypergraph G, and that

$$\overline{\overline{G}^k}^k = G$$

for every hypergraph G of rank at most k.

Let $G = (V, E)$ be a hypergraph. We say that G is *upward closed* if E is an upward closed subset of $\mathcal{P}'(V)$. We say that G is *upward closed of rank k* if E is an upward closed subset of $\mathcal{P}'_k(V)$. The *upward closure* of G is the hypergraph ${\uparrow}G = (V, {\uparrow}E)$. The *upward closure of rank k* of G is the hypergraph ${\uparrow}^k G = (V, E')$, where $E' = {\uparrow}E \cap \mathcal{P}'_k(V)$.

Let $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$ be a finite family of n distinct nonempty sets. The *intersection hypergraph* of $\mathcal{F}$ is the hypergraph $\operatorname{Int} \mathcal{F} = (V, E)$ where $V = \{1, 2, \dots, n\}$ and a nonempty subset S of V is an edge if and only if

$$\bigcap_{i \in S} F_i \neq \emptyset.$$

Note that an intersection hypergraph has all possible loops as edges. Note also that $\operatorname{Int} \emptyset = (\emptyset, \emptyset)$.

Proposition 2. *A hypergraph $G = (V, E)$ is (isomorphic to) the intersection hypergraph of some finite family of sets if and only if the edge set E is downward closed and contains all singletons $\{x\}$, $x \in V$.*

Proof. Let $G = (V, E)$ be the intersection hypergraph of $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$. By definition, $\{x\} \in E$ for all $x \in V$. If $S \in E$, then $\bigcap_{i \in S} F_i \neq \emptyset$ and it is clear that for every $\emptyset \neq S' \subseteq S$, we also have that $\bigcap_{i \in S'} F_i \neq \emptyset$ and hence $S' \in E$.

For the converse implication, assume that E is downward closed and contains all singletons $\{x\}$, $x \in V$. We may assume without loss of generality that $V = \{1, 2, \dots, n\}$. We construct a finite family $\mathcal{F}$ of pairwise distinct sets whose intersection hypergraph is G. For each $i \in V$, let $F_i = \{S \in E : i \in S\}$. The sets $F_1, F_2, \dots, F_n$ are pairwise distinct since all singletons are edges by assumption and hence for each $1 \leq i \leq n$, $\{i\} \in F_i$ but $\{i\} \notin F_j$ for $i \neq j$. Let $S \in \mathcal{P}'(V)$. If $S \in E$, then $S \in F_i$ for all $i \in S$ and so $\bigcap_{i \in S} F_i \neq \emptyset$. Conversely, if $\bigcap_{i \in S} F_i \neq \emptyset$, then there is an $S' \in E$ such that $S' \in F_i$ for all

$i \in S$. By the definition of the F_i, $i \in S'$ for all $i \in S$, so $S \subseteq S'$, and hence, by the assumption that E is downward closed, we have that $S \in E$. We conclude that $G = \operatorname{Int} \mathcal{F}$. $\square$

Corollary 3. *A hypergraph $G = (V, E)$ is (isomorphic to) the complement of an intersection hypergraph if and only if G is loopless and the edge set E is upward closed.*

Proof. Follows immediately from Proposition 2. $\square$

Corollary 4. *A hypergraph $G = (V, E)$ is (isomorphic to) the k-complement of an intersection hypergraph if and only if G is loopless and the edge set E is an upward closed subset of $\mathcal{P}'_k(V)$.*

Proof. Assume first that $G = \overline{H}^k$ for an intersection hypergraph $H = (V, E')$. Then E' is downward closed and contains all singletons, and so $E = \overline{E'}^k = \overline{E'} \cap \mathcal{P}'_k(V)$ is an upward closed subset of $\mathcal{P}'_k(V)$ that contains no singletons.

Assume then that E is an upward closed subset of $\mathcal{P}'_k(V)$ with no singletons. Let $H = \overline{G}^k$. Then the edge set of H is downward closed and contains all singletons $\{x\}$, $x \in V$, and hence H is an intersection hypergraph by Proposition 2, and we have that $G = \overline{\overline{G}^k}^k = \overline{H}^k$. $\square$

Let $G = (V, E)$ and $H = (V', E')$ be hypergraphs. A mapping $h \colon V \to V'$ is called a *homomorphism* of G to H, denoted $h \colon G \to H$, if $h[S] \in E'$ for all $S \in E$. If there exists a homomorphism $h \colon G \to H$, we say that G is homomorphic to H and denote $G \leq H$. Every hypergraph is homomorphic to itself by the identity map on its vertex set. The composition of homomorphisms is again a homomorphism. Thus the relation $\leq$ is a quasiorder on the set of all hypergraphs. If $G \leq H$ and $H \leq G$, we say that G and H are *homomorphically equivalent* and denote $G \equiv H$. For general background on graphs and homomorphisms, see [6].

Proposition 5. *Let $G = (V, E)$ and $H = (V', E')$ be hypergraphs.*

(i) *If G and H have rank at most $k \geq 2$ and $G \leq H$, then $\uparrow^k G \leq \uparrow^k H$.*
(ii) *If G and H are loopless hypergraphs of rank 2 and $\uparrow^k G \leq \uparrow^k H$, then $G \leq H$.*
(iii) *If $G \leq H$, then $\uparrow G \leq \uparrow H$.*
(iv) *If G and H are loopless hypergraphs of rank 2 and $\uparrow G \leq \uparrow H$, then $G \leq H$.*

Proof. (i) Let $h \colon G \to H$ be a homomorphism. We show that h is also a homomorphism of $\uparrow^k G$ to $\uparrow^k H$. Let S be an edge of $\uparrow^k G$. Then there is an edge T of G such that $T \subseteq S$. Since h is a homomorphism, $h[T]$ is an edge of H. Since $T \subseteq S$, we have that $h[T] \subseteq h[S]$, and furthermore $|h[S]| \leq |S| \leq k$, so we have that $h[S]$ is an edge of $\uparrow^k H$.

(ii) Let $h \colon \uparrow^k G \to \uparrow^k H$ be a homomorphism. We show that h is also a homomorphism of G to H. Let S be an edge of G. Then S is also an edge of $\uparrow^k G$ and therefore $h[S]$ is an edge of $\uparrow^k H$. Then $2 \leq |h[S]| \leq |S| = 2$, so $h[S]$ is also an edge of H.

(iii) and (iv) are proved similarly. $\square$

By Proposition 5, the homomorphism order of graphs is embedded in the homomorphism order of the family of the upward closed loopless hypergraphs (of rank $k \geq 2$). It is a well-known fact that the homomorphism order of graphs is universal in the sense that every countable poset can be embedded in it (see [24]; for a simpler proof, see [8]). Hence, the homomorphism order of upward closed loopless hypergraphs (of rank $k \geq 2$) also has this universal property.

4. Minors of Boolean functions with respect to the clones of clique functions

We now discuss Boolean functions and call them simply functions. Let $\mathbb{B} = \{0, 1\}$. The set $\mathbb{B}^n$ is a Boolean (complemented distributive) lattice with respect to the coordinatewise order $0 < 1$. The smallest and the largest elements are the n-tuples $\mathbf{0} = (0, \dots, 0)$ and $\mathbf{1} = (1, \dots, 1)$, respectively. The *complement* of an n-tuple $\mathbf{a} \in \{0, 1\}^n$ is $\overline{\mathbf{a}} = (1 - a_1, \dots, 1 - a_n)$. The *set of true points* of a function $f \colon \mathbb{B}^n \to \mathbb{B}$ is $\mathfrak{T}_f = \{\mathbf{a} : f(\mathbf{a}) = 1\} = f^{-1}(1)$, and the elements of $\mathfrak{T}_f$ are called *true points* of f. We denote the set of minimal elements of $\mathfrak{T}_f$ by $\mathfrak{T}_f^{\min}$, and we call its members the *minimal true points* of f.

Let $a \in \mathbb{B}$. A set $S \subseteq \mathbb{B}^n$ is said to be *a-separating* if there is an i, $1 \leq i \leq n$, such that for every $(a_1, \dots, a_n) \in S$ we have $a_i = a$. A function f is said to be *a-separating* if $f^{-1}(a)$ is a-separating. A function f is said to be *a-separating of rank* $k \geq 2$ if every subset of $f^{-1}(a)$ of size at most k is a-separating. Sometimes these functions are also referred to as *clique functions.*

For $k \geq 2$, the class U_k of all 1-separating functions of rank k and the class W_k of all 0-separating functions of rank k are clones, and so are the classes U_∞ and W_∞ of all 1-separating functions and all 0-separating functions, respectively. We have that $U_\infty = \bigcap_{k \geq 2} U_k$ and $W_\infty = \bigcap_{k \geq 2} W_k$.

A Boolean function f is said to be 0-*preserving,* if $f(\mathbf{0}) = 0$. Similarly, f is said to be 1-*preserving,* if $f(\mathbf{1}) = 1$. If f is both 0-preserving and 1-preserving, we say that f is *constant-preserving.* Denote by T_0, T_1, and T_c the clones of 0-, 1-, and constant-preserving functions, respectively, i.e., $T_0 = \{f \in \mathcal{O}_\mathbb{B} : f(\mathbf{0}) = 0\}$, $T_1 = \{f \in \mathcal{O}_\mathbb{B} : f(\mathbf{1}) = 1\}$, $T_c = T_0 \cap T_1 = \{f \in \mathcal{O}_\mathbb{B} : f(\mathbf{0}) = 0, f(\mathbf{1}) = 1\}$, and denote by M the clone of monotone functions, i.e., $M = \{f \in \mathcal{O}_\mathbb{B} : \mathbf{a} \leq \mathbf{b} \Rightarrow f(\mathbf{a}) \leq f(\mathbf{b})\}$. For $k = 2, \dots, \infty$, denote $T_c U_k = T_c \cap U_k$ and $T_c W_k = T_c \cap W_k$; $MU_k = M \cap U_k$ and $MW_k = M \cap W_k$; $M_c U_k = MU_k \cap T_c$ and $M_c W_k = MW_k \cap T_c$.

In this section, we discuss the $\mathcal{C}$-minor relations determined by the various clones of 1-separating functions: U_k, $T_c U_k$, MU_k, $M_c U_k$ for $k = 2, 3, \dots, \infty$. Whatever results we obtain can be translated into similar results about the $\mathcal{C}$-minor relations determined by the clones of 0-separating functions using the following duality principle. Denoting by f^{d} the *dual* of a function f, defined by $f^{\mathrm{d}}(\mathbf{a}) = \overline{f(\overline{\mathbf{a}})}$, the *dual* of a class $\mathcal{C}$ is the class $\mathcal{C}^{\mathrm{d}} = \{f \in \mathcal{O}_\mathbb{B} : f^{\mathrm{d}} \in \mathcal{C}\}$. The dual of a clone is a clone, and in particular, U_k and W_k are duals of each other, and so are $T_c U_k$ and $T_c W_k$; MU_k and MW_k; and $M_c U_k$ and $M_c W_k$, for $k = 2, 3, \dots, \infty$.

Lemma 6. *Let $\mathcal{C}$ be a clone of Boolean functions. Then $f \leq_{\mathcal{C}} g$ if and only if $f^{\mathrm{d}} \leq_{\mathcal{C}^{\mathrm{d}}} g^{\mathrm{d}}$.*

Proof. If $f \leq_{\mathcal{C}} g$, then $f = g(h_1, \ldots, h_m)$ for some $h_1, \ldots, h_m \in \mathcal{C}$. Then for all $\mathbf{a} \in \mathbb{B}^n$,

$$\begin{aligned} f^{\mathrm{d}}(\mathbf{a}) &= \overline{f(\overline{\mathbf{a}})} = \overline{g(h_1, \ldots, h_m)(\overline{\mathbf{a}})} = \overline{g\big(h_1(\overline{\mathbf{a}}), \ldots, h_m(\overline{\mathbf{a}})\big)} \\ &= \overline{g\big(\overline{\overline{h_1(\overline{\mathbf{a}})}}, \ldots, \overline{\overline{h_m(\overline{\mathbf{a}})}}\big)} = g^{\mathrm{d}}\big(h_1^{\mathrm{d}}(\mathbf{a}), \ldots, h_m^{\mathrm{d}}(\mathbf{a})\big) = g^{\mathrm{d}}(h_1^{\mathrm{d}}, \ldots, h_m^{\mathrm{d}})(\mathbf{a}), \end{aligned}$$

so $f^{\mathrm{d}} = g^{\mathrm{d}}(h_1^{\mathrm{d}}, \ldots, h_m^{\mathrm{d}})$, and since $h_1^{\mathrm{d}}, \ldots, h_m^{\mathrm{d}} \in \mathcal{C}^{\mathrm{d}}$, we have that $f^{\mathrm{d}} \leq_{\mathcal{C}^{\mathrm{d}}} g^{\mathrm{d}}$. The converse implication is proved similarly. □

For the sake of simplicity, we assume in most of our proofs that k is finite. Proofs for the case $k = \infty$ can be obtained with obvious changes, i.e., instead of considering a set of at most k elements, consider instead a finite set with no upper bound on the number of its elements. We have presented in [12] a proof for the existence of an infinite descending chain in $\preccurlyeq_{U_\infty}$, $\preccurlyeq_{T_c U_\infty}$, $\preccurlyeq_{M U_\infty}$, $\preccurlyeq_{M_c U_\infty}$, using a construction based on threshold functions. We now apply substantially different methods and make use of hypergraph homomorphisms.

We first observe that whenever $\mathcal{C} \subseteq T_0$ and $f \leq_{\mathcal{C}} g$, we necessarily have that $f(\mathbf{0}) = g(\mathbf{0})$. It is also straightforward to verify that for any clone $\mathcal{C}$, $\overline{f} \leq_{\mathcal{C}} \overline{g}$ if and only if $f \leq_{\mathcal{C}} g$. Thus, in our analysis of the $\mathcal{C}$-minor relations when $\mathcal{C} \subseteq T_0$, we can confine ourselves to the restriction $\leq_{\mathcal{C}}|_{T_0}$ of $\leq_{\mathcal{C}}$ to the class of 0-preserving functions, because functions f and g with $f(\mathbf{0}) \neq g(\mathbf{0})$ cannot be related and the restriction of $\leq_{\mathcal{C}}$ to the set of functions f with $f(\mathbf{0}) = 1$ is an isomorphic copy of $\leq_{\mathcal{C}}|_{T_0}$.

For any 0-preserving function f, we define the *disjointness hypergraph of rank* k of f, denoted $G(f, k)$, as follows: $V\big(G(f, k)\big) = \mathfrak{T}_f$ and $S \in E\big(G(f, k)\big)$ if and only if $2 \leq |S| \leq k$ and $\bigwedge S = \mathbf{0}$. We alternatively call $G(f, 2)$ the *disjointness graph* of f. Similarly, we define the *disjointness hypergraph* of f, denoted $G(f, \infty)$, as follows: $V\big(G(f, \infty)\big) = \mathfrak{T}_f$ and $S \in E\big(G(f, \infty)\big)$ if and only if $2 \leq |S|$ and $\bigwedge S = \mathbf{0}$. These are in fact the various k-complements and the complement of the intersection hypergraph of the set of true points of f. Thus, by Corollary 3, a loopless hypergraph is the disjointness hypergraph of some Boolean function if and only if its edge set is upward closed. Similarly, by Corollary 4, for each $k \geq 2$, a loopless hypergraph is the disjointness hypergraph of rank k of some Boolean function if and only if its edge set is an upward closed subset of $\mathcal{P}'_k(V)$.

Proposition 7. *Let f and g be* 0*-preserving functions. Then $f \leq_{U_k} g$ if and only if $G(f, k)$ is homomorphic to $G(g, k)$.*

Proof. Let $f \colon \mathbb{B}^n \to \mathbb{B}$ and $g \colon \mathbb{B}^m \to \mathbb{B}$ be 0-preserving. Assume first that $f \leq_{U_k} g$. Then there exist funtions $h_1, \ldots, h_m \in U_k$ such that $f = g(h_1, \ldots, h_m)$. The map $h = (h_1, \ldots, h_m) \colon \mathbb{B}^n \to \mathbb{B}^m$ maps the true points of f to true points of g. We show that the restriction $h|_{\mathfrak{T}_f} \colon \mathfrak{T}_f \to \mathfrak{T}_g$ is in fact a homomorphism of $G(f, k)$ to $G(g, k)$. Let $S = \{\mathbf{a}_1, \ldots, \mathbf{a}_p\}$ ($\mathbf{a}_1, \ldots, \mathbf{a}_p \in \mathfrak{T}_f$ distinct, $2 \leq p \leq k$) be an edge of $G(f, k)$. In order to establish that

$h|_{\mathfrak{T}_f}[S] = h[S] = \{h(\mathbf{a}_1), \dots, h(\mathbf{a}_p)\}$ is an edge of $G(g, k)$, we first make use of the fact that the h_i are 1-separating of rank k. Since $\bigwedge S = \mathbf{0}$ by the definition of $G(f, k)$, we have that for $1 \leq i \leq m$, $h_i(\mathbf{a}_j) = 0$ for some $1 \leq j \leq p$, and so $\bigwedge h[S] = \mathbf{0}$. Furthermore, since $\mathbf{0} \notin \mathfrak{T}_g$, $h[S]$ cannot be a singleton, so we have that $2 \leq |h[S]| \leq p$, and thus $h|_{\mathfrak{T}_f}[S] = h[S]$ is an edge of $G(g, k)$.

Assume then that there is a homomorphism $h\colon G(f, k) \to G(g, k)$. Define the mapping $\gamma\colon \mathbb{B}^n \to \mathbb{B}^m$ by

$$\gamma(\mathbf{a}) = \begin{cases} h(\mathbf{a}), & \text{if } \mathbf{a} \in \mathfrak{T}_f, \\ \mathbf{0}, & \text{if } \mathbf{a} \notin \mathfrak{T}_f. \end{cases}$$

If $\mathbf{a} \notin \mathfrak{T}_f$, then $(g \circ \gamma)(\mathbf{a}) = g\bigl(\gamma(\mathbf{a})\bigr) = g(\mathbf{0}) = 0$. If $\mathbf{a} \in \mathfrak{T}_f$, then $(g \circ \gamma)(\mathbf{a}) = g\bigl(\gamma(\mathbf{a})\bigr) = g\bigl(h(\mathbf{a})\bigr) = 1$, because $h(\mathbf{a}) \in \mathfrak{T}_g$ for any $\mathbf{a} \in \mathfrak{T}_f$. Thus, $g \circ \gamma = f$. We still have to show that the components of $\gamma = (\gamma_1, \dots, \gamma_m)$ are members of U_k. Let $\mathbf{a}_1, \dots, \mathbf{a}_p$ $(1 \leq p \leq k)$ be true points of γ_i. Then $\mathbf{a}_1, \dots, \mathbf{a}_p \in \mathfrak{T}_f$ and $h(\mathbf{a}_1)(i) = \cdots = h(\mathbf{a}_p)(i) = 1$, which implies that $\bigwedge_{j=1}^{p} h(\mathbf{a}_j) \neq \mathbf{0}$, and therefore $\{h(\mathbf{a}_1), \dots, h(\mathbf{a}_p)\}$ is not an edge of $G(g, k)$. Since h is a homomorphism, $\{\mathbf{a}_1, \dots, \mathbf{a}_p\}$ is not an edge of $G(f, k)$, which implies $\bigwedge_{j=1}^{p} \mathbf{a}_j \neq \mathbf{0}$ or $\mathbf{a}_1 = \cdots = \mathbf{a}_k = \mathbf{0}$. The latter case is not possible since $\mathbf{0} \notin \mathfrak{T}_f$. Thus, $\gamma_i \in U_k$, and we conclude that $f \leq_{U_k} g$. □

Thus, the U_2-minor partial order $\preccurlyeq_{U_2}$, when restricted to the set of 0-separating functions, is isomorphic to the homomorphism partial order of graphs. Let us make a few simple observations based on this fact.

- $G(\mathfrak{c}_0^{(n)}, 2) = (\emptyset, \emptyset)$. This is the smallest element of the homomorphism order, and so is $[\mathfrak{c}_0]$ the smallest element of the restriction of $\preccurlyeq_{U_2}$ to the set T_0 of 0-preserving functions.
- If $\mathfrak{c}_0 \neq f \in U_2$, then by definition, $G(f, 2)$ is a nonempty graph with no edges. Such graphs are homomorphic to all nonempty graphs, and hence they are homomorphically equivalent to each other, but they are not equivalent to any other graphs. In other words, the nonempty graphs with no edges constitute an equivalence class of graphs. This is in agreement with Lemma 1, which asserts that $U_2 \setminus [\mathfrak{c}_0]$ is a $\equiv_{U_2}$-class.
- Bipartite graphs are homomorphically equivalent to K_2, the complete graph on two vertices. This is the disjointness graph of $x_1 + x_2$.
- It is known (see [6, 18, 28]) that the homomorphism order of graphs is dense (i.e., if $G < H$, then there is a K such that $G < K < H$) with two exceptions: $(\emptyset, \emptyset) < K_1 < K_2$ but there is no graph G such that $(\emptyset, \emptyset) < G < K_1$ or $K_1 < G < K_2$. This translates to a similar statement about the density of $\preccurlyeq_{U_2}$.

As discussed in Section 3, $\preccurlyeq_{U_2}$ is universal in the sense that every countable partial order can be embedded in it, and so are $\preccurlyeq_{U_k}$ for all $k = 2, 3, \dots, \infty$. We will show that the partial order $\preccurlyeq_{\mathcal{C}}$ is universal as well, whenever $\mathcal{C}$ is any one of the clones of 1-separating functions (i.e., U_k, $T_c U_k$, MU_k, $M_c U_k$, for $k = 2, 3, \dots, \infty$). For this end, we need some auxiliary results.

Recall that $\mathfrak{T}_f^{\min}$ denotes the set of minimal true points of f.

Proposition 8. (i) *If f and f' are 0-preserving functions of the same arity such that $\mathfrak{T}_f \subseteq \mathfrak{T}_{f'}$, then $f \leq_{U_k} f'$.*

(ii) *If f and f' are functions of the same arity such that f is 0-preserving and for each $\mathbf{a} \in \mathfrak{T}_{f'}$ there exists a $\mathbf{b} \in \mathfrak{T}_f$ with $\mathbf{b} \leq \mathbf{a}$, then $f' \leq_{U_k} f$.*

(iii) *If f and g are 0-preserving functions of the same arity such that $\mathfrak{T}_f^{\min} = \mathfrak{T}_g^{\min}$, then $f \equiv_{U_k} g$.*

Proof. (i) Since every edge of $G(f,k)$ is an edge of $G(f',k)$, the map $\mathbf{a} \mapsto \mathbf{a}$ is clearly a homomorphism of $G(f,k)$ to $G(f',k)$, so $f \leq_{U_k} f'$.

(ii) The function f' is necessarily 0-preserving, so we can prove the claim by finding a homomorphism $G(f',k) \to G(f,k)$. Define a mapping $h \colon \mathfrak{T}_{f'} \to \mathfrak{T}_f$ as follows. For each $\mathbf{a} \in \mathfrak{T}_{f'}$, let $h(\mathbf{a}) = \mathbf{a}^*$, where $\mathbf{a}^* \in \mathfrak{T}_f$ is such that $\mathbf{a}^* \leq \mathbf{a}$ (such a true point of f always exists by the assumption, and if there are several such points, choose any). We show that h is a homomorphism of $G(f',k)$ to $G(f,k)$. Let $S = \{\mathbf{a}_1, \dots, \mathbf{a}_p\}$ $(2 \leq p \leq k)$ be an edge of $G(f',k)$. Then $h[S] = \{h(\mathbf{a}_1), \dots, h(\mathbf{a}_p)\} = \{\mathbf{a}_1^*, \dots, \mathbf{a}_p^*\}$. Since $\mathbf{a}_i^* \leq \mathbf{a}_i$ for $1 \leq i \leq p$, we have that $\mathbf{0} \leq \bigwedge \mathbf{a}_i^* \leq \bigwedge \mathbf{a}_i = \mathbf{0}$. Furthermore, $2 \leq |h[S]| \leq k$; the latter inequality is obvious, and if it were the case that $|h[S]| = 1$, then $\mathbf{a}_1^* = \dots = \mathbf{a}_p^* \neq \mathbf{0}$, and so $\mathbf{a}_1^* \leq \mathbf{a}_i$ for all $1 \leq i \leq p$, and we would have $\mathbf{0} = \bigwedge \mathbf{a}_i \geq \mathbf{a}_1^* > \mathbf{0}$, which is impossible. Thus, $h[S]$ is an edge of $G(f,k)$, and h is indeed a homomorphism, and we conclude that $f' \leq_{U_k} f$.

(iii) Let f' be the function satisfying $\mathfrak{T}_{f'} = \mathfrak{T}_f^{\min}$. Then f and f' satisfy the conditions of parts (i) and (ii), so $f \equiv_{U_k} f'$. Similarly, f' and g satisfy the conditions and so $f' \equiv_{U_k} g$, and by the transitivity of $\equiv_{U_k}$, we have $f \equiv_{U_k} g$. □

The *monotone closure* of $f \colon \mathbb{B}^n \to \mathbb{B}$ is the function $f^{\mathrm{m}} \colon \mathbb{B}^n \to \mathbb{B}$ defined by $f^{\mathrm{m}}(\mathbf{a}) = 1$ if and only if $f(\mathbf{b}) = 1$ for some $\mathbf{b} \leq \mathbf{a}$. In other words, f^{m} is the function satisfying $\mathfrak{T}_{f^{\mathrm{m}}} = {\uparrow}\mathfrak{T}_f$.

Lemma 9. *If $f \in U_k$, then $f^{\mathrm{m}} \in MU_k$. Furthermore, if f is nonconstant, then $f^{\mathrm{m}} \in M_cU_k$.*

Proof. By definition, the monotone closure of any function is monotone, so $f^{\mathrm{m}} \in M$. Let $\mathbf{a}_1, \dots, \mathbf{a}_p$ $(1 \leq p \leq k)$ be true points of f^{m}. Then there exist true points $\mathbf{b}_1, \dots, \mathbf{b}_p$ of f such that $\mathbf{b}_i \leq \mathbf{a}_i$ $(1 \leq i \leq p)$. Since $f \in U_k$, we have that $\bigwedge \mathbf{b}_i \neq \mathbf{0}$, and thus $\bigwedge \mathbf{a}_i \geq \bigwedge \mathbf{b}_i > \mathbf{0}$, and so $f^{\mathrm{m}} \in U_k$. Thus, $f^{\mathrm{m}} \in M \cap U_k = MU_k$.

If f is not a constant function, then $f^{\mathrm{m}}(\mathbf{1}) = 1$. Since $f \in U_k$, $f(\mathbf{0}) = 0$ and so $f^{\mathrm{m}}(\mathbf{0}) = 0$, and we have that $f^{\mathrm{m}} \in T_c \cap M \cap U_k = M_cU_k$. □

Proposition 10. *Let f and g be monotone nonconstant functions. Then, for each $k = 2, 3, \dots, \infty$, the following statements are equivalent.*

(i) $f \leq_{M_cU_k} g$.
(ii) $f \leq_{MU_k} g$.
(iii) $f \leq_{T_cU_k} g$.
(iv) $f \leq_{U_k} g$.

Proof. The implications (i) ⇒ (ii) ⇒ (iv) and (i) ⇒ (iii) ⇒ (iv) are trivial by the subclone inclusions in the Post lattice. We only need to prove (iv) ⇒ (i).

Assume that $f \leq_{U_k} g$. Then $f = g(h_1, \dots, h_m)$ for some $h_1, \dots, h_m \in U_k$, and by the construction in the proof of Proposition 7, we can assume that $h(\mathbf{a}) = \mathbf{0}$ whenever $f(\mathbf{a}) = 0$. Denoting $h_i^{\mathrm{M}} = h_i^{\mathrm{m}}$ if $h_i \neq \mathfrak{c}_0$, and $h_i^{\mathrm{M}} = x_1 \wedge \dots \wedge x_n$ if $h_i = \mathfrak{c}_0$, Lemma 9 implies that $h_i^{\mathrm{M}} \in M_c U_k$. We show that $f = g(h_1^{\mathrm{M}}, \dots, h_m^{\mathrm{M}})$, establishing that $f \leq_{M_c U_k} g$.

Let $\mathbf{a} \in \mathbb{B}^n$. If $f(\mathbf{a}) = 0$, then $f(\mathbf{b}) = 0$ for all $\mathbf{b} \leq \mathbf{a}$ by the monotonicity of f and $\mathbf{a} \neq \mathbf{1}$ because f is assumed to be nonconstant. Hence $h(\mathbf{b}) = \mathbf{0}$ for all $\mathbf{b} \leq \mathbf{a}$ by the assumption we made on h, and thus $h^{\mathrm{M}}(\mathbf{a}) = (h_1^{\mathrm{M}}, \dots, h_m^{\mathrm{M}})(\mathbf{a}) = \mathbf{0}$. Therefore, $g(h_1^{\mathrm{M}}, \dots, h_m^{\mathrm{M}})(\mathbf{a}) = g\bigl(h^{\mathrm{M}}(\mathbf{a})\bigr) = g(\mathbf{0}) = 0$.

If $f(\mathbf{a}) = 1$, then $g\bigl(h(\mathbf{a})\bigr) = 1$, and $g(\mathbf{b}) = 1$ for all $\mathbf{b} \geq h(\mathbf{a})$ by the monotonicity of g. Since $h^{\mathrm{M}}(\mathbf{a}) \geq h(\mathbf{a})$, we have that $g(h_1^{\mathrm{M}}, \dots, h_m^{\mathrm{M}})(\mathbf{a}) = g\bigl(h^{\mathrm{M}}(\mathbf{a})\bigr) = 1$. $\square$

Theorem 11. *For any clone $\mathcal{C}$ such that $M_c U_\infty \subseteq \mathcal{C} \subseteq U_2$ or $M_c W_\infty \subseteq \mathcal{C} \subseteq W_2$, every countable poset can be embedded into the $\mathcal{C}$-minor partial order $\preccurlyeq_{\mathcal{C}}$.*

Proof. It follows from Corollaries 3 and 4, Propositions 5 and 7 and the universality of the homomorphicity order of graphs that $\preccurlyeq_{U_k}$ is also a universal partial order for each $k = 2, 3, \dots, \infty$.

By Proposition 8, every 0-preserving nonconstant function f is U_k-equivalent to its monotone closure f^{m}. By Proposition 10, the restrictions of the relations $\leq_{U_k}$, $\leq_{T_c U_k}$, $\leq_{MU_k}$, $\leq_{M_c U_k}$ to the class of monotone constant-preserving functions coincide, and therefore $\preccurlyeq_{T_c U_k}$, $\preccurlyeq_{MU_k}$, $\preccurlyeq_{M_c U_k}$ are also universal partial orders, the universality already achieved within the restriction of the corresponding $\mathcal{C}$-minor relations to the class of monotone functions.

The claim about the cases where $M_c W_\infty \subseteq \mathcal{C} \subseteq W_2$ follows by duality (Lemma 6). $\square$

5. Minors with respect to the clone of self-dual monotone functions

A Boolean function f is *self-dual* if $f = f^{\mathrm{d}}$, i.e, the equality $f(\mathbf{a}) = \overline{f(\overline{\mathbf{a}})}$ holds for all $\mathbf{a} \in \mathbb{B}^n$. The class of self-dual monotone functions is a clone and we denote it by SM. Using methods similar to the ones used in the previous section, we can show that the partial order $\preccurlyeq_{SM}$ is also universal.

Denote by $\mathcal{K}$ the class of functions $f \colon \mathbb{B}^n \to \mathbb{B}$ ($n \geq 2$) satisfying the following conditions:

- f is 0-preserving,
- $f(\mathbf{a}) = 0$ whenever $a_1 = 1$,
- the $(n-1)$-ary function f' defined by $f'(\mathbf{a}) = f(0, \mathbf{a})$ is monotone.

Every 0-preserving function $f \colon \mathbb{B}^n \to \mathbb{B}$ is U_2-equivalent to a member of $\mathcal{K}$. To see this, note that f is U_2-equivalent to its monotone closure f^{m} by Proposition 8. Define the function $f' \colon \mathbb{B}^{n+1} \to \mathbb{B}$ by $f'(1, \mathbf{a}) = 0$, $f'(0, \mathbf{a}) = f^{\mathrm{m}}(\mathbf{a})$ for all $\mathbf{a} \in \mathbb{B}^n$. Clearly $f' \in \mathcal{K}$ and it is easy to see that $G(f', 2)$ is isomorphic to $G(f^{\mathrm{m}}, 2)$, so $f' \equiv_{U_2} f^{\mathrm{m}} \equiv_{U_2} f$.

Proposition 12. *Let $f, g \in \mathcal{K}$. Then $f \leq_{SM} g$ if and only if $f \leq_{U_2} g$.*

Proof. Let f and g be n-ary and m-ary, respectively. If $f \leq_{SM} g$ then $f \leq_{U_2} g$ by the subclone inclusion $SM \subseteq U_2$ in the Post lattice. Assume then that $f \leq_{U_2} g$, and let f' and g' be the functions satisfying $\mathfrak{T}_{f'} = \mathfrak{T}_f^{\min}$, $\mathfrak{T}_{g'} = \mathfrak{T}_g^{\min}$. By Proposition 8, $f \equiv_{U_2} f'$ and $g \equiv_{U_2} g'$. Thus $f' \leq_{U_2} g'$, and so there exists a homomorphism $h\colon G(f', 2) \to G(g', 2)$. Define $\psi\colon \mathbb{B}^n \to \mathbb{B}^m$ by

$$\psi(\mathbf{a}) = \begin{cases} \mathbf{0} & \text{if } a_0 = 0 \text{ and } \mathbf{b} \not\leq \mathbf{a} \text{ for all } \mathbf{b} \in \mathfrak{T}_f^{\min}, \\ \displaystyle\bigvee_{\mathbf{a} \geq \mathbf{b} \in \mathfrak{T}_f^{\min}} h(\mathbf{b}) & \text{if } a_0 = 0 \text{ and } \mathbf{b} \leq \mathbf{a} \text{ for some } \mathbf{b} \in \mathfrak{T}_f^{\min}, \\ \overline{\psi(\overline{\mathbf{a}})} & \text{if } a_0 = 1. \end{cases}$$

By definition, ψ is self-dual. It is also easy to see that $f = g \circ \psi$, since h maps minimal true points of f to minimal true points of g and the join of true points of g is again a true point of g.

In order to complete the proof that $f \leq_{SM} g$, we still have to show that ψ is monotone. Let $\mathbf{u}, \mathbf{v} \in \mathbb{B}^n$ such that $\mathbf{u} < \mathbf{v}$. We need to consider several cases.

Case 1. If $u_0 = 0$ and $\mathbf{b} \not\leq \mathbf{u}$ for all $\mathbf{b} \in \mathfrak{T}_f^{\min}$, then $\psi(\mathbf{u}) = \mathbf{0} \leq \psi(\mathbf{v})$. Similarly, if $v_0 = 1$ and $\mathbf{b} \not\leq \overline{\mathbf{v}}$ for all $\mathbf{b} \in \mathfrak{T}_f^{\min}$, then $\psi(\mathbf{v}) = \mathbf{1} \geq \psi(\mathbf{u})$.

Case 2. If $u_0 = 0$ and $\mathbf{b} \leq \mathbf{u}$ for some $\mathbf{b} \in \mathfrak{T}_f^{\min}$ and $v_0 = 0$, then

$$\psi(\mathbf{u}) = \bigvee_{\mathbf{u} \geq \mathbf{b} \in \mathfrak{T}_f^{\min}} h(\mathbf{b}) \leq \bigvee_{\mathbf{v} \geq \mathbf{b} \in \mathfrak{T}_f^{\min}} h(\mathbf{b}) = \psi(\mathbf{v}).$$

Case 3. If $u_0 = 0$ and $\mathbf{b} \leq \mathbf{u}$ for some $\mathbf{b} \in \mathfrak{T}_f^{\min}$ and $v_0 = 1$, then $\mathbf{u} \wedge \overline{\mathbf{v}} = \mathbf{0}$. Assuming that $\mathbf{v}$ is such that it is not already taken care of by Case 1, we have that $\mathbf{c} \leq \overline{\mathbf{v}}$ for some $\mathbf{c} \in \mathfrak{T}_f^{\min}$. In fact, for any $\mathbf{b}, \mathbf{c} \in \mathfrak{T}_f^{\min}$ with $\mathbf{b} \leq \mathbf{u}$, $\mathbf{c} \leq \mathbf{v}$, we have that $\mathbf{b} \wedge \mathbf{c} = \mathbf{0}$, so $\{\mathbf{b}, \mathbf{c}\}$ is an edge of $G(f', 2)$. Since h is a homomorphism, $\{h(\mathbf{b}), h(\mathbf{c})\}$ is an edge of $G(g', 2)$, so $h(\mathbf{b}) \wedge h(\mathbf{c}) = \mathbf{0}$ and thus

$$\Bigl(\bigvee_{\mathbf{u} \geq \mathbf{b} \in \mathfrak{T}_f^{\min}} h(\mathbf{b}) \Bigr) \wedge \Bigl(\bigvee_{\mathbf{v} \geq \mathbf{c} \in \mathfrak{T}_f^{\min}} h(\mathbf{c}) \Bigr) = \bigvee_{\substack{\mathbf{u} \geq \mathbf{b} \in \mathfrak{T}_f^{\min} \\ \mathbf{v} \geq \mathbf{c} \in \mathfrak{T}_f^{\min}}} \bigl(h(\mathbf{b}) \wedge h(\mathbf{c}) \bigr) = \mathbf{0}$$

and so

$$\psi(\mathbf{u}) = \bigvee_{\mathbf{u} \geq \mathbf{b} \in \mathfrak{T}_f^{\min}} h(\mathbf{b}) \leq \overline{\bigvee_{\overline{\mathbf{v}} \geq \mathbf{c} \in \mathfrak{T}_f^{\min}} h(\mathbf{c})} = \psi(\mathbf{v}).$$

Case 4. If $u_0 = 1$ then also $v_0 = 1$ and we have that $\overline{\mathbf{u}} > \overline{\mathbf{v}}$. Then $\psi(\mathbf{u}) = \overline{\psi(\overline{\mathbf{u}})} \leq \overline{\psi(\overline{\mathbf{v}})} = \psi(\mathbf{v})$ by the previous cases considered. □

Theorem 13. *Every countable poset can be embedded into the SM-minor partial order $\preccurlyeq_{SM}$.*

Proof. The theorem follows from the universality of $\preccurlyeq_{U_2}$ (Theorem 11), from the fact that each $\equiv_{U_2}$-class of 0-preserving functions has a representative in $\mathcal{K}$, and from Proposition 12. □

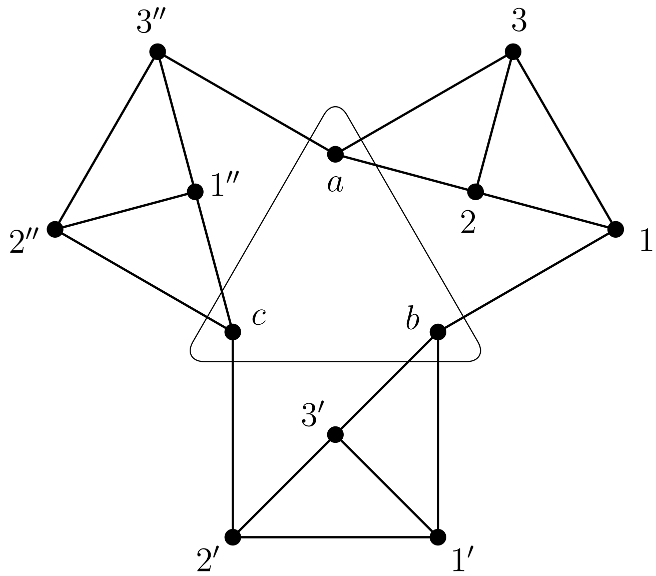

FIGURE 1. A hypergraph that is a homomorphism core but not a sup-homomorphism core.

6. SUP-HOMOMORPHISMS OF HYPERGRAPHS

We are yet to analyze the density of the universal partial orders $\preccurlyeq_{U_k}$, $k = 2, \ldots, \infty$. Homomorphisms of upward closed hypergraphs suggest another interpretation. Let $G = (V, E)$ and $H = (V', E')$ be hypergraphs. A *sup-homomorphism* of G to H is a mapping $f \colon V \to V'$ such that for all $S \in E$ there exists a $T \in E'$ such that $f[S] \supseteq T$. This will be denoted by $f \colon (V, E) \overset{\mathrm{SH}}{\to} (V', E')$. We write $G \overset{\mathrm{SH}}{\to} H$ to denote the fact that there exists a sup-homomorphism of G to H, and we write $G \overset{\mathrm{SH}}{\not\to} H$ if there does not exist any sup-homomorphism of G to H.

Clearly any homomorphism $f \colon (V, E) \to (V', E')$ is also a sup-homomorphism. The converse is not true as can be easily seen, and an example of a core hypergraph H which fails to be a sup-homomorphism core is given in Fig. 1 (note that here $\{a, b, c\} \in E(H)$ while all other edges of H are 2-element sets). H is obviously a homomorphism core but H can be mapped by a sup-homomorphism to any triangle. (Recall that a core is a hypergraph H for which any homomorphism $f \colon H \to H$ is an automorphism. Every hypergraph is homomorphically equivalent to a core.)

However, $f \colon (V, E) \to (V', E')$ is a sup-homomorphism if and only if $f \colon (V, E) \to (V', {\uparrow}E')$ is a homomorphism. From this it follows that if (V, E) and (V', E') are both k-uniform hypergraphs ($k \in \mathbb{N}$), then $f \colon (V, E) \to (V', E')$ is a sup-homomorphism if and only if f is a homomorphism. It follows that, when restricted to k-uniform hypergraphs, many results about homomorphisms also hold for sup-homomorphisms. Thus for example the ordering $\preccurlyeq_{U_2}$ of Boolean functions is dense (as was proved for undirected graphs in [28]; see, e.g., [6]). Here we refine the proof given in [18] to prove the density of the orders $\preccurlyeq_{U_k}$ ($k = 2, 3, \ldots, \infty$) and, by duality, of $\preccurlyeq_{W_k}$ ($k = 2, 3, \ldots, \infty$).

For $1 \leq n \leq m$, denote by K_n^m the hypergraph (V, E) with $V = \{1, \dots, m\}$ and whose edges are all n-element subsets of $\{1, \dots, m\}$. Also, denote by K_0^m the hypergraph $(\{1, \dots, m\}, \emptyset)$. Thus, $K_m^m = \big(\{1, \dots, m\}, \{\{1, \dots, m\}\}\big)$ for $m \geq 1$.

Theorem 14. *Let G_1 and G_2 be sup-homomorphism core hypergraphs of rank at most $k \geq 1$. Assume that $G_1 \overset{\text{SH}}{\to} G_2 \overset{\text{SH}}{\not\to} G_1$. Then there exists a hypergraph G of rank at most k such that $G_1 \overset{\text{SH}}{\to} G \overset{\text{SH}}{\to} G_2$ while $G_2 \overset{\text{SH}}{\not\to} G \overset{\text{SH}}{\not\to} G_1$ if and only if $(G_1, G_2) \neq (\emptyset, K_0^1)$ or $(G_1, G_2) \neq (K_0^1, K_k^k)$.*

Proof. If $(G_1, G_2) = (\emptyset, K_0^1)$ or $(G_1, G_2) = (K_0^1, K_k^k)$, then it is straightforward to verify that there is no hypergraph G of rank at most k such that $G_1 \overset{\text{SH}}{\to} G \overset{\text{SH}}{\to} G_2$ while $G_2 \overset{\text{SH}}{\not\to} G \overset{\text{SH}}{\not\to} G_1$.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be sup-homomorphism cores, and assume that $(G_1, G_2) \neq (\emptyset, K_0^1)$ and $(G_1, G_2) \neq (K_0^1, K_k^k)$. Consider first the case that G_2 is not a single edge, i.e., $G_2 \neq K_n^n$ for all $2 \leq n \leq k-1$. Assume that G_2 is connected. Let $H = (V', E')$ be a hypergraph satisfying the following properties:

(1) H is d-uniform where $d = \sum_{S \in G_2} |S|$. As H is $\sum_{S \in G_2} |S|$-uniform, we partition every edge $T \in E'$ into subsets T_S, $S \in G_2$, $|T_S| = |S|$. (Thus, explicitly, $\bigcup_{S \in G_2} T_S = T$ and $T_S \cap T_{S'} = \emptyset$ for $S \neq S'$.)
(2) For every colouring $c \colon V' \to |G_1|^{|G_2|}$ there exists a $T \in E'$ such that $c|_T$ is constant.
(3) (V', E') does not contain cycles of length $\leq |G_2|$.

The existence of such a hypergraph $H = (V', E')$ is well known [2, 17]; see [20] for a simple construction. (Remark that our proof is simple but the existence of H is the nontrivial part of the proof.)

We define G as follows: $G = G_1 + (G_2 \times H)$, where $+$ denotes the disjoint union and $\times$ is the following version of categorical product which suffices for our purposes: Let $(V_2, \leq)$, $(V', \leq)$ be arbitrary orderings. Then the set of vertices of $G_2 \times H$ is $V_2 \times V'$ and the edges of $G_2 \times H$ are all sets of the form $\{(y_1, z_1), \dots, (y_t, z_t)\}$ where $y_1 < y_2 < \dots < y_t$ in $(G_2, \leq)$ and $\{y_1, \dots, y_t\} = S \in E_2$ and $z_1 < z_2 < \dots < z_t$, $\{z_1, \dots, z_t\} = T_S \subseteq T \in E'$.

We prove that G has all the desired properties. It is easy to see that $G_1 \overset{\text{SH}}{\to} G \overset{\text{SH}}{\to} G_2$ (for $G \overset{\text{SH}}{\to} G_2$ we can take the projection $G_2 \times H \to G_2$ together with the sup-homomorphism $G_1 \overset{\text{SH}}{\to} G_2$). We have that $G_2 \overset{\text{SH}}{\not\to} G$. For, assume on the contrary that $G_2 \overset{\text{SH}}{\to} G$. Then, by the connectedness, $G_2 \overset{\text{SH}}{\to} G_2 \times H$ and thus also $f \colon G_2 \overset{\text{SH}}{\to} H$. However, then $f[G_2]$ is contained in $G_2 \times H'$ where H' has at most $|V_2|$ edges and thus H' is a (hypergraph) tree. But then $G_2 \times H' \overset{\text{SH}}{\to} G_2$, because in this case $G_2 \times H'$ can be mapped to a single edge hypergraph with $\min_{S \in E_2} |S|$ vertices and thus G_2 fails to be a core, a contradiction.

Finally we prove that $G \overset{\text{SH}}{\not\to} G_1$. Suppose, on the contrary, that $f \colon G \overset{\text{SH}}{\to} G_1$ is a sup-homomorphism. This mapping induces a mapping $f \colon G_2 \times H \to G_1$ (which we denote again by f). For each $v \in V'$, denote by f_v the restriction (a fibre) of f to the set $\{(u, v) : u \in V_2\}$. The colouring of V' given by

$v \mapsto f_v$ uses at most $|V_1|^{|V_2|}$ colours and thus there exists a $T \in E'$ such that $f_v = f_{v'}$ for all $v, v' \in T$. However, then the mapping $g\colon V_2 \to V_1$ defined by $g(u) = f_v(u) = f(u, v)$ (for any $v \in T$) is a sup-homomorphism $G_2 \to G_1$, a contradiction.

If G_2 is disconnected, say $G_2 = H_1 + H_2 + \cdots + H_t$ is a partition of G_2 into connected parts, then there is a H_i such that $H_i \stackrel{\mathrm{SH}}{\not\to} G_1$ and we can repeat the above proof for H_i in place of G_2.

Consider the the case that G_2 is a single edge, i.e., $G_2 = K_n^n$ for some $2 \leq n \leq k-1$. We will proceed as in the previous case. We choose $H = (V', E')$ to be $(n+1)$-uniform and satisfying properties (2) and (3) above, and we let $G = G_1 + (G_2 \otimes H)$, where $\otimes$ denotes the following modified categorical product: again the set of vertices of $G_2 \otimes H$ is $V_2 \times V'$ but now the edges are all sets of the form $\{(y_1, z_1), (y_2, z_2), \ldots, (y_n, z_n), (y_1, z_n)\}$ where $y_1 < y_2 < \cdots < y_n$ in $(V_2, \leq)$ and $z_1 < z_2 < \cdots < z_n$, $\{z_1, \ldots, z_n\} \subseteq T \in E'$.

It is again easy to see that $G_1 \stackrel{\mathrm{SH}}{\to} G \stackrel{\mathrm{SH}}{\to} G_2$. In this case it is obvious that $G_2 \stackrel{\mathrm{SH}}{\not\to} G$ because the edges of G have $n+1$ elements. To prove $G \stackrel{\mathrm{SH}}{\not\to} G_1$, suppose, on the contrary, that $f\colon G \stackrel{\mathrm{SH}}{\to} G_1$ is a sup-homomorphism. This mapping induces a mapping $f\colon G_2 \times H \to G_1$ (which we denote again by f). For each $v \in V'$, denote by f_v the restriction of f to the set $\{(u, v) : u \in V_2\}$. The colouring of V' given by $v \mapsto f_v$ uses at most $|V_1|^{|V_2|}$ colours and thus there exists an edge $\{(y_1, z_1), (y_2, z_2), \ldots, (y_n, z_n), (y_1, z_n)\} = T \in E'$ such that $f_v = f_{v'}$ for all $v, v' \in T$. Then $f(y_1, z_1) = f(y_1, z_n)$, and thus $|f[T]| \leq n$, implying that there is an edge $f[T] \supset S \in E_1$. But then any surjective map from V_2 onto S would be a sup-homomorphism of G_2 to G_1, and we have reached a contradiction. □

7. Concluding remarks

We have shown that the $\mathcal{C}$-minor partial order $\preccurlyeq_{\mathcal{C}}$ is universal if $\mathcal{C}$ is one of the following clones: SM, U_k, T_cU_k, MU_k, M_cU_k, W_k, T_cW_k, MW_k, M_cW_k, for $k = 2, 3, \ldots, \infty$. In fact, these are the only clones of Boolean functions that have this universal property, which follows from certain previously known results for the other Post classes, which we will summarize below.

It was shown in [16] that if a clone $\mathcal{C}$ on A contains the *discriminator function*

$$t(x, y, z) = \begin{cases} z, & \text{if } x = y, \\ x, & \text{otherwise} \end{cases} \qquad (x, y, z \in A),$$

then there are only a finite number of $\equiv_{\mathcal{C}}$-classes, and the smallest clone containing the discriminator function is minimal with respect to this finiteness property. For Boolean functions, the clones containing the discriminator function are precisely the clones that have this finiteness property, i.e., there are only a finite number of $\equiv_{\mathcal{C}}$-classes if and only if $S_c \subseteq \mathcal{C}$.

As regards the clones M, M_0, M_1, M_c of monotone functions, it was shown in [13] that $\preccurlyeq_M$ is isomorphic to the homomorphicity order of nonempty 2-lattices (see [14] for definitions and terminology). Kosub and Wagner [10] pointed out that every 2-lattice is homomorphically equivalent to its longest alternating chain. An alternating 2-chain is completely determined by its length and the label of its smallest element. Denoting by $C(n, b)$ the alternating 2-chain of length n with its smallest element labeled by b we have that $C(n, b)$ is homomorphic to $C(n', b')$ if and only if either $n = n'$ and $b = b'$ or $n < n'$. Thus, this partial order has width 2 and has no infinite descending chains. The partial orders $\preccurlyeq_{M_0}$, $\preccurlyeq_{M_1}$ and $\preccurlyeq_{M_c}$ are minor variants of $\preccurlyeq_M$, and they also have width 2 and are well-founded.

It was shown in [13] that if $\mathcal{C}$ is the clone L of linear functions or the clone L_0 of 0-preserving linear functions, then $\preccurlyeq_{\mathcal{C}}$ is well-founded and it has an infinite antichain. By duality, this is also true for the clone L_1, and the proof can be straightforwardly modified to show that these properties also hold for the remaining two clones of linear functions: LS and L_c.

Is is shown in [15] that if $\mathcal{C}$ is one of the clones of semilattice operations, i.e., Λ_c, Λ_0, Λ_1, Λ, V_c, V_0, V_1, V, then $\preccurlyeq_{\mathcal{C}}$ is well-founded. In this case the partial order $\preccurlyeq_{\mathcal{C}}$ clearly contains an infinite antichain, because $\mathcal{C}$ is a subclone of a clone having the universality property.

If $\mathcal{C}$ is a clone that contains only essentially at most unary functions (i.e., $I_c \subseteq \mathcal{C} \subseteq \Omega(1)$), then it is clear that $\mathcal{C}$-equivalent functions have the same number of essential variables, and if f is a proper $\mathcal{C}$-minor of g, then the number of essential variables of f is smaller that that of g. Thus, there cannot be infinite descending chains of $\mathcal{C}$-minors. Since $\mathcal{C}$ is a subclone of a clone $\mathcal{C}'$ for which $\preccurlyeq_{\mathcal{C}'}$ is known to contain an infinite antichain, $\preccurlyeq_{\mathcal{C}}$ also contains an infinite antichain.

Thus, we have established the following theorem. For a partially ordered set $(P, \leq)$, and for subsets S, T of P, we denote by $[S, T]$ the interval

$$[S, T] = \{x \in P : a \leq x \leq b \text{ for some } a \in S,\ b \in T\}.$$

Theorem 15. *Partition the Post lattice in the following intervals:*

$$\begin{aligned}
I_1 &= [\{S_c\}, \{\Omega\}],\\
I_2 &= [\{M_c\}, \{M\}],\\
I_3 &= [\{SM, M_c U_\infty, M_c W_\infty\}, \{U_2, W_2\}],\\
I_4 &= [\{I_c\}, \{L, \Lambda, V\}].
\end{aligned}$$

Let $\mathcal{C}$ be a clone of Boolean functions.

(i) $\preccurlyeq_{\mathcal{C}}$ *is a finite partial order if and only if* $\mathcal{C} \in I_1$.
(ii) $\preccurlyeq_{\mathcal{C}}$ *is a countably infinite well-founded partial order without infinite antichains if and only if* $\mathcal{C} \in I_2$.
(iii) $\preccurlyeq_{\mathcal{C}}$ *is a universal partial order if and only if* $\mathcal{C} \in I_3$.
(iv) $\preccurlyeq_{\mathcal{C}}$ *is well-founded and it contains an infinite antichain if and only if* $\mathcal{C} \in I_4$.

This is perhaps a surpsising result. There are only a handful of results which give a full characterization of universality. One such example is the universality of minor closed subclasses of oriented paths; see [7, 19].

Little is known about the $\mathcal{C}$-minors determined by the clones on larger base sets. We would like to mention here that it was shown in [13, 14] that if A

is a finite set with at least three elements and $\leq$ is a partial order on A with comparable elements and $M_\leq$ is the clone of monotone functions with respect to $\leq$, then $\preccurlyeq_{M_\leq}$ is universal.

Appendix. Post classes

We make use of notations and terminology appearing in [4] and [9].

- Ω denotes the clone of all Boolean functions;
- T_0 and T_1 denote the clones of 0- and 1-preserving functions, respectively, i.e.,

$$T_0 = \{f \in \Omega : f(0, \ldots, 0) = 0\}, \quad T_1 = \{f \in \Omega : f(1, \ldots, 1) = 1\};$$

- T_c denotes the clone of constant-preserving functions, i.e., $T_c = T_0 \cap T_1$.
- M denotes the clone of all monotone functions, i.e.,

$$M = \{f \in \Omega : f(\mathbf{a}) \leq f(\mathbf{b}) \text{ whenever } \mathbf{a} \leq \mathbf{b}\};$$

- $M_0 = M \cap T_0$, $M_1 = M \cap T_1$, $M_c = M \cap T_c$;
- S denotes the clone of all self-dual functions, i.e., $S = \{f \in \Omega : f^{\mathrm{d}} = f\}$;
- $S_c = S \cap T_c$, $SM = S \cap M$;
- L denotes the clone of all linear functions, i.e.,

$$L = \{f \in \Omega : f = c_0 + c_1x_1 + \cdots + c_nx_n \text{ for some } n \text{ and } c_0, \ldots, c_n \in \mathbb{B}\};$$

- $L_0 = L \cap T_0$, $L_1 = L \cap T_1$, $LS = L \cap S$, $L_c = L \cap T_c$;

Let $a \in \{0, 1\}$. A set $A \subseteq \{0, 1\}^n$ is said to be *a-separating* if there is i, $1 \leq i \leq n$, such that for every $(a_1, \ldots, a_n) \in A$ we have $a_i = a$. A function f is said to be *a-separating* if $f^{-1}(a)$ is a-separating. The function f is said to be *a-separating of rank $k \geq 2$* if every subset $A \subseteq f^{-1}(a)$ of size at most k is a-separating.

- For $m \geq 2$, U_m and W_m denote the clones of all 1- and 0-separating functions of rank m, respectively;
- U_∞ and W_∞ denote the clones of all 1- and 0-separating functions, respectively, i.e., $U_\infty = \bigcap_{k \geq 2} U_k$ and $W_\infty = \bigcap_{k \geq 2} W_k$;
- $T_cU_m = T_c \cap U_m$ and $T_cW_m = T_c \cap W_m$, for $m = 2, \ldots, \infty$;
- $MU_m = M \cap U_m$ and $MW_m = M \cap W_m$, for $m = 2, \ldots, \infty$;
- $M_cU_m = M_c \cap U_m$ and $M_cW_m = M_c \cap W_m$, for $m = 2, \ldots, \infty$;
- Λ denotes the clone of all conjunctions and constants, i.e.,

$$\Lambda = \{f \in \Omega : f = x_{i_1} \wedge \cdots \wedge x_{i_n} \text{ for some } n \geq 1 \text{ and } i_j\text{'s}\} \cup [\mathfrak{c}_0] \cup [\mathfrak{c}_1];$$

- $\Lambda_0 = \Lambda \cap T_0$, $\Lambda_1 = \Lambda \cap T_1$, $\Lambda_c = \Lambda \cap T_c$;
- V denotes the clone of all disjunctions and constants, i.e.,

$$V = \{f \in \Omega : f = x_{i_1} \vee \cdots \vee x_{i_n} \text{ for some } n \geq 1 \text{ and } i_j\text{'s}\} \cup [\mathfrak{c}_0] \cup [\mathfrak{c}_1];$$

- $V_0 = V \cap T_0$, $V_1 = V \cap T_1$, $V_c = V \cap T_c$;
- $\Omega(1)$ denotes the clone of all projections, negations, and constants;
- I^* denotes the clone of all projections and negations;
- I denotes the clone of all projections and constants;
- $I_0 = I \cap T_0$, $I_1 = I \cap T_1$;
- I_c denotes the smallest clone containing only projections, i.e., $I_c = I \cap T_c$.

Acknowledgements

This joint research project was initiated during the second author's visit to the University of Turku. We would like to thank professor Juhani Karhumäki for introducing us to each other. Part of the work was carried out when the first author was visiting Charles University in Prague.

References

[1] C. Berge, *Graphs and Hypergraphs,* North-Holland, Amsterdam, London, 1973.

[2] P. Erdős, A. Hajnal, On chromatic numbers of graphs and set systems, *Acta Math. Acad. Sci. Hungar.* **17** (1966) 61–99.

[3] A. Feigelson, L. Hellerstein, The forbidden projections of unate functions, *Discrete Appl. Math.* **77** (1997) 221–236.

[4] S. Foldes, G. R. Pogosyan, Post classes characterized by functional terms, *Discrete Appl. Math.* **142** (2004) 35–51.

[5] M. A. Harrison, On the classification of Boolean functions by the general linear and affine groups, *J. Soc. Indust. Appl. Math.* **12**(2) (1964) 285–299.

[6] P. Hell, J. Nešetřil, *Graphs and Homomorphisms,* Oxford Lecture Series in Mathematics and Its Applications **28**, Oxford University Press, Oxford, New York, 2004.

[7] J. Hubička, J. Nešetřil, Finite paths are universal, *Order* **21** (2004) 181–200.

[8] J. Hubička, J. Nešetřil, Universal partial order represented by means of oriented trees and other simple graphs, *European J. Combin.* **26** (2005) 765–778.

[9] S. W. Jablonski, G. P. Gawrilow, W. B. Kudrjawzew, *Boolesche Funktionen und Postsche Klassen,* Vieweg, Braunschweig, 1970.

[10] S. Kosub, K. W. Wagner, The Boolean hierarchy of NP-partitions, in H. Reichel, S. Tison (eds.), *STACS 2000, 17th Annual Symposium on Theoretical Aspects of Computer Science,* Lecture Notes in Comput. Sci. **1770**, Springer-Verlag, Berlin, 2000, pp. 157–168. An expanded version is available as Technical Report TUM-I0209, Institut für Informatik, Technische Universität München, München, 2002.

[11] D. Lau, *Function Algebras on Finite Sets,* Springer-Verlag, Berlin, Heidelberg, 2006.

[12] E. Lehtonen, An infinite descending chain of Boolean subfunctions consisting of threshold functions, *Contributions to General Algebra* **17**, Proceedings of the Vienna Conference 2005 (AAA70), Verlag Johannes Heyn, Klagenfurt, 2006, pp. 145–148.

[13] E. Lehtonen, Descending chains and antichains of the unary, linear, and monotone subfunction relations, *Order* **23** (2006) 129–142.

[14] E. Lehtonen, Labeled posets are universal, *European J. Combin.* **29** (2008) 493–506.

[15] E. Lehtonen, A note on minors determined by clones of semilattices, arXiv:0809.3234v1.

[16] E. Lehtonen, Á. Szendrei, Equivalence of operations with respect to discriminator clones, *Discrete Math.* (2008), doi:10.1016/j.disc.2008.01.003.

[17] L. Lovász, On chromatic number of finite set systems, *Acta Math. Acad. Sci. Hungar.* **19** (1968) 59–67.

[18] J. Nešetřil, The homomorphism structure of classes of graphs, *Combin. Probab. Comput.* **8** (1999) 177–184.

[19] J. Nešetřil, Y. Nigussie, Density of universal classes of series-parallel graphs, *J. Graph Theory* **54** (2007) 13–23.

[20] J. Nešetřil, V. Rödl, A short proof of the existence of highly chromatic hypergraphs without short cycles, *J. Combin. Theory Ser. B* **27** (1979) 225–227.

[21] N. Pippenger, Galois theory for minors of finite functions, *Discrete Math.* **254** (2002) 405–419.

[22] R. Pöschel, L. A. Kalužnin, *Funktionen und Relationenalgebren: Ein Kapitel der diskreten Mathematik,* Birkhäuser, Basel, Stuttgart, 1979.

[23] E. L. Post, *The Two-Valued Iterative Systems of Mathematical Logic,* Annals of Mathematical Studies **5**, Princeton University Press, Princeton, 1941.
[24] A. Pultr, V. Trnková, *Combinatorial, Algebraic and Topological Representations of Groups, Semigroups and Categories,* North-Holland, Amsterdam, 1980.
[25] Á. Szendrei, *Clones in Universal Algebra,* Séminaire de mathématiques supérieures **99**, Les Presses de l'Université de Montréal, Montreal, 1986.
[26] C. Wang, Boolean minors, *Discrete Math.* **141** (1991) 237–258.
[27] C. Wang, A. C. Williams, The threshold order of a Boolean function, *Discrete Appl. Math.* **31** (1991) 51–69.
[28] E. Welzl, Color-families are dense, *Theoret. Comput. Sci.* **17** (1982) 29–41.
[29] I. E. Zverovich, Characterizations of closed classes of Boolean functions in terms of forbidden subfunctions and Post classes, *Discrete Appl. Math.* **149** (2005) 200–218.

(E. Lehtonen) Department of Mathematics, Tampere University of Technology, P.O. Box 553, FI–33101 Tampere, Finland

Department of Combinatorics and Optimization, University of Waterloo, 200 University Avenue West, Waterloo, Ontario, N2L 3G1, Canada
E-mail address: erkko.lehtonen@tut.fi

(J. Nešetřil) Department of Applied Mathematics and Institute of Theoretical Computer Sciences, Charles University, Malostranské nám. 25, 11800 Praha, Czech Republic
E-mail address: nesetril@kam.mff.cuni.cz