Collapse closed classes of graphs and graph parameters

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Abstract

We characterize classes of multi-graphs that by a transformation of Schrijver give so called invariant ring in tensor algebra. This property enables us subsequently to show, that whenever we have an edge reflection positive and multiplicative graph parameter defined on this class such that its value on vertexless loop is a natural number, it can be represented in edge coloring model.

1 Introduction

The area of graph parameters has received close attention in the recent years. One of the central places of this area is counting of graph homomorphisms. A very insightful survey on this subject is by Borgs, Chayes, Lovász, Sós, Szegedy and Vesztergombi [BCL+06]. One direction springing out of the counting the homomorphisms is the theory of a graph limit object and its application to the graph parameters and property testing as well as the extremal graph theory.

Another direction (that further intersects with the first one though) that we can follow is to study graph parameters coming out of the physical statistical models, so called partition functions. (Formally, a graph parameter is a function $f : \mathcal{G} \to \mathbb{R}$ where $\mathcal{G}$ is the class of all graphs and isomorphic graphs are considered to be the same.) For instance it can be a partition function of icing models. For more examples and explanation see [dlHJ93]. Although the physical statistical models have a physical interpretation, they include many purely combinatorial parameters.

The most general model and the one relevant to this paper is *edge coloring model.* What are the combinatorial examples of graph[1] parameters that are in the edge coloring model? For instance the number of perfect matchings, the number of proper edge colorings or the permanent of the adjacency matrix of

[1] from now on we will always consider graphs with multiple edges and every edge can have 2 endpoints, 1 endpoint (a loop) or even no endpoint (a vertexless loop).

a given graph. For more examples of the parameters we refer reader to [Sze07]. And what do these parameters have in common? For each of them (and for each parameter f in the edge coloring model) there is a finite set of colors, here it will always be denoted as $[n] := \{1, \ldots, n\}$ and a function $b : A_n \to \mathbb{R}$ where A_n is a set of all multisets of colors such that the following holds for each graph G:

$$f(G) = f_b(G) := \sum_{\phi: EG \to [n]} \prod_{v \in VG} b(\phi[\delta(v)]) \tag{1}$$

where $\delta(v)$ is a multisubset of edges of G incident with v (loops will appear twice) and $\phi[\delta(v)]$ is a corresponding multiset of colors given by edge coloring ϕ. Note that for each graph G, H and their disjoint union then holds

$$f_b(G) f_b(H) = f_b(G \uplus H).$$

Graph parameters satisfying the previous are called *multiplicative.*

Now the main question is, given a graph parameter f, can we decide whether there is a natural number n and a function $b : A_n \to \mathbb{R}$ such that $f_b = f$? This question can be solved by providing some suitable characterization of parameters in the edge coloring model. Such one was first given by Szegedy [Sze07] and later Schrijver [Sch08b] proved another one, that is very similar to the Szegedy's but in some sense it is more compact.

We will not go into details but we just mention that the sufficient and necessary conditions for a graph parameter to be in the edge coloring model are the multiplicativeness and positive semidefiniteness of certain matrices defined by the values of the parameter (the so called *edge reflection positivity*).

The topic of our paper starts with the following question: Can we decide whether certain parameter is in the edge coloring model even in situation when the parameter is defined only on a certain class of graphs (let us call it a **partial parameter**)? For which classes of graphs can we do that?

For the simplest instance, take as the partial parameter the number of perfect matchings defined on graphs with even number of vertices. To decide whether this parameter is in the edge coloring model using the theorem of Szegedy (or Schrijver) we would have to guess its value on graphs with odd number of vertices. Should it be zero or the number of matchings with maximal cardinality possible (i.e. number of matchings omitting exactly one vertex)?

In addition, this guessing could be done for us since if we manage to recognize that a partial parameter is in the edge coloring model, i.e. if we find a number n and a proper function $b : A_n \to \mathbb{R}$ such that $f = f_b$ on the class of graphs, then f_b is an extension of f since by the formula 1 it is defined on all graphs (the extension does not necessarily need to be unique).

If we compared the edge coloring model parameters to the continuous functions and the class of all graphs to a Hausdorff topological space then our sought classes of graphs would be analogues to the closed sets and the version of Szegedy's theorem for the partial parameters would be analogue to the classical Tietze-Urysohn extension theorem.

Our approach will be closest to the one described by Schrijver in [Sch06] where the scope of Szegedy's method is widened by applying Schrijver's previous result [Sch08a, Theorem 1] providing characterization of certain tensor subalgebras.

An important step in the technique of [Sch06] is a convenient transformation of graphs into tensors. Such a transformation of all graphs with bounded degree (i.e., all graphs that do not have a vertex of degree higher than some Δ) is shown to generate a tensor algebra of form $T(R^m)^G$ (for some m and for some subgroup G of orthogonal group $O(\mathbb{R}^m)$), the so called *invariant ring*. The notation $T(\mathbb{R}^m)^G$ means the set of all tensors invariant under the action of the group G where the action is uniquely determined by the action of the group G on $\mathbb{R}^m$. We will not further define this notion more precisely and we refer the interested reader to [Sch08a, Theorem 1] where certain characterization of invariant rings was found and that characterization we will state as definition and use in our paper.

In a hope that such classes of graphs, that transformed into tensors turn out to generate some invariant ring, might be good candidates for the "closed sets" of the "space" of all graphs, we ask the first question of this paper: how do they look like?

By lifting the conditions of Schrijver's characterization [Sch08a, Theorem 1] into the language of graphs, we get that sufficient conditions for any class of graphs $\mathcal{G}$ are:

1. Whenever graphs G and H are contained in $\mathcal{G}$, then their disjoint union $G \cup H$ is contained in $\mathcal{G}$.

2. For any graph $G \in \mathcal{G}$ and vertices $u \neq v \in VG$ (with $\deg(u) = \deg(v)$) and any bijection of edges incident with u to edges incident with v, a graph obtained by deleting u and v and gluing the edges incident with u to their image edges incident with v is also contained in $\mathcal{G}$.

3. Whenever $G \cup L \in \mathcal{G}$, then $G \in \mathcal{G}$ where L is vertexless loop.

A class $\mathcal{G}$ satisfying conditions 1., 2. and 3. will be referred to as *collapse closed*.

Sufficient (and necessary) are actually only the first two conditions, but as we will see, deleting the vertexless loop does not change the image of the transformation (it means only multiplication by a real number).

Additionally, when a parameter is defined on a graph $G \cup L$ then we can expect that we also know the value on G, hence it does not make much sense to consider classes that do not satisfy the third condition.

Can we really obtain the extended version of the Szegedy's theorem for the collapse closed classes? The answer is almost yes. If a class $\mathcal{G}$ satisfy a certain condition (that holds for all interesting cases of the collapse closed classes) then a Szegedy-type statement holds for the partial parameters defined on $\mathcal{G}$. To give the statement of the resulting theorem we would again need the notion of the edge reflection positivity (on a class $\mathcal{G}$). We leave the precise statement to the fifth section.

Our paper is organized as follows: In the following section we introduce preliminary notions required for our work.

The third section explains the main notion of the paper – the collapse closed class of graphs together with its algebraic meaning and states their description in a "description theorem". The fourth section is fully devoted to the proof of the description theorem. Both the statement and the proof of the description theorem involves only combinatorics, thus reader, who is not interested in the following application, can skip all algebraic parts of the second and the third section.

In the fifth section we investigate the relation of collapse closed classes to the edge coloring model which is a place where the algebraic meaning of the collapse closed classes of graphs comes into play. Here we prove an extended version of Szegedy's characterization of graph parameters in the edge coloring model. Concluding section contains final remarks and some open questions.

2 Background and Notation

In the paper by a **graph** G we will denote a triple (VG, EG, δ) where VG is a finite set of vertices, EG is a finite set of edges and δ is a function that assigns to each vertex $v \in VG$ a multiset of edges $\delta(v)$ (that we call **incident** with v) such that each edge appears in $\delta[VG]$ twice or never. If an edge e does not appear in $\delta[VG]$ we call it a **vertexless loop**, if it appears twice in some $\delta(v)$ we call it a **loop**. If an edge appears twice in $\delta[VG]$ with distinct vertices v, w, it is an edge between v and w. By a **degree** of a vertex $v \in VG$ we denote the cardinality $|\delta(v)| =: \deg(v)$.

By an **isomorphism** $f : G \to H$ we mean a pair of bijections $f_V : VG \to VH$ and $f_E : EG \to EH$ such that for each vertex $v \in VG$ it holds that $\delta(f_V(v)) = f_E[\delta(v)]$. We will only write v' instead of $f(v)$ and e' instead of $f(e)$ for G and G' where we construct G' as a disjoint copy of G so there is an implicit isomorphism f between them.

Let G be a graph and let $u, v \in VG$ be two of its distinct vertices and let $\pi : \delta(u) \leftrightarrow \delta(v)$ be a multiset bijection from $\delta(u)$ to $\delta(v)$. Such a bijection will give us a relation $\overset{\pi}{\sim}$ on EG defined by $e \overset{\pi}{\sim} f$ whenever an end of e is mapped by π to an end of f.

Later on, we will slightly abuse the notation by writing $\pi(e) = f$ for some $e \in \delta(u)$ and $f \in \delta(v)$ when it is clear that e and f are not loops or with additional specification which ends of edges do we mean. If we were to be precise and formal, we would represent $\delta(u)$ and $\delta(v)$ as sets of ends of edges and π a bijection between them (thus there is $|\delta(u)|!$ different choices for π) but we believe this is more understandable way of notation.

By **collapsing** in G vertex u with v with respect to π we get a graph denoted as $G_{u,v,\pi}$ and obtained from G by deleting u and v and gluing the end of each edge e incident with u to the end of an edge $\pi(e)$ incident with v. Formally, $G_{u,v,\pi} := (VG \setminus \{u,v\}, E', \delta')$ where E' is the set of classes of the unique minimal equivalence $\sim_\pi$ containing $\overset{\pi}{\sim}$ and where δ' is obtained from δ by replacing each edge by its class of equivalence: $\delta'(v) := \{[e]_{\sim_\pi} \mid e \in \delta(v)\}$.

Alternatively, we can describe the same operation using ordering of the incident edges. Let for each $u \in VG$ be $\Delta_u : [\deg(u)] \to \delta(u)$ a map such that $\Delta_u[[\deg(u)]] = \delta(u)$. We will call such Δ as a **local ordering**. Then by collapsing in G vertices u and v of the same degree with respect to ordering Δ we get a graph $G_{u,v,\Delta}$ defined as $(VG \setminus \{u,v\}, E', \delta')$ where E' is set of classes of the smallest equivalence $\sim$ such that $\Delta_u(j) \sim \Delta_v(j)$ for each $j \in [\deg(u)]$ and where δ' is obtained from δ by replacing each edge by its class of equivalence: $\delta'(v) := \{[e]_\sim \mid e \in \delta(v)\}$.

An example of collapsing a vertex u with a vertex v in a graph G is depicted on the Figure 1. Here and further on we will always depict the graph in a way that the planar cyclic ordering around collapsed vertices is compatible with the ordering Δ and collapsed vertices are connected by a red stripe narrowed in the middle.

In this example $\Delta_u = (f_1, l_1, l_1, e_1, g)$ and $\Delta_v = (f_2, e_2, l_2, l_2, g)$.

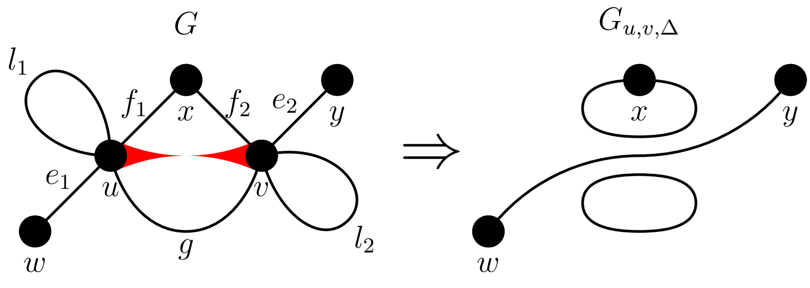


Figure 1: Illustration of a collapse. Note that the edges e_1, l_1, e_2 and l_2 turn into one single edge.

Since after the collapse, the local ordering Δ can be naturally replaced by Δ' defined by $\Delta'_u(i) := [\Delta_u(i)]_\sim$ for each $u \in VG$ and $j \in [\deg(u)]$, for any $\{u_1, \ldots, u_k\} \subseteq VG$ and $\{v_1, \ldots, v_k\} \subseteq VG$ we can define series of collapses of u_1 and v_1, ... and u_k and v_k with respect to the local ordering Δ whenever $\deg(u_1) = \deg(v_1), \ldots, \deg(u_k) = \deg(v_k)$. Formally, it means to do each collapse in the series with respect to an local ordering $\Delta^i, i \in [k]$ where $\Delta^1 := \Delta$ and $\Delta^i := \Delta^{i-1\prime}$. The result is independent of reordering the vertices since it is determined by the smallest equivalence relation $\sim$ on EG such that $\Delta_{u_i}(j) \sim \Delta_{v_i}(j)$ for each $i \in [k]$ and $j \in [\deg(u_i)]$.

Let S be a real inner product vector space. The **tensor algebra** $T(S)$ is equal to

$$T(S) := \bigoplus_{n\in\mathbb{N}} S^{\otimes n}.$$

It is an $\mathbb{R}$-algebra, with a product $\otimes$. A subalgebra $\mathcal{A}$ of $T(S)$ is called **graded** if $\mathcal{A} = \bigoplus_{n\in\mathbb{N}}(\mathcal{A} \cap S^{\otimes n})$. For distinct $i, j \in \mathbb{N}$, the **contraction operator** $C_{i,j}$ is the linear function $T(S) \to T(S)$ such that $C_{i,j}(z) = 0$ where $z \in S^{\otimes n}$ with $i > n$ or $j > n$, and

$$C_{i,j}(x_1 \otimes \cdots \otimes x_n) = \langle x_i, x_j\rangle x_1 \otimes \cdots \hat{x}_i, \hat{x}_j \cdots \otimes x_n$$

if $i, j \le n$ and $x_1, \ldots, x_n \in S$.

A subalgebra $\mathcal{A}$ is called **contraction closed** if $C_{i,j}\mathcal{A} \subseteq \mathcal{A}$ for all i, j.

For a given class $\mathcal{G}$ we define the class of graphs with an ordered set of vertices (called **ordered graphs**) $\mathcal{OG}$ to contain all graphs from $\mathcal{G}$ with all possible orderings of their vertex sets.

In this paper we will suppose that each ordered graph in $\mathcal{OG}$ has the vertex set VG equal to $\{1, \ldots, |VG|\}$.

Now, for any n (where the interpretation of n is the number of colors) we define a transformation of ordered graphs into tensors over a real inner product space S_n with orthonormal basis $\{e_\alpha | \alpha \in A_n\}$. (Again, A_n is the set of all multisets on elements $1, \ldots, n$, so S_n is an infinitely dimensional vector space.)

The transformation $p_n \colon \mathcal{OG} \to T(S_n)$ is defined in the following way:

$$p_n(G) := \sum_{\phi:EG\to[n]} \bigotimes_{v=1}^{|VG|} c_{\phi[\delta(v)]} e_{\phi[\delta(v)]}$$

where the coefficient c_α is defined as

$$c_\alpha := \sqrt{\prod_{i=1}^{n} \mu_i(\alpha)!},$$

where $\mu_i(\alpha)$ denotes the multiplicity of i in α.

Finally, for the main part we will need the following notions.

Let G be a graph with its vertex set labelled by the set of numbers $[|VG|] := \{1, 2, \ldots, |VG|\}$. Then the adjacency function $\mathcal{F}(G) : [|VG|]^{\le 2} \to \mathbb{N}$ of a graph G is defined by number of edges connecting the given set of 0,1 or 2 vertices.

Let G be a graph. Then we define $\mathrm{score}(G)$ as a sequence $(\mathrm{score}(G)_k; k \in \mathbb{N}) := (|\{v \in VG; \deg(v) = k\}| \mid k \in \mathbb{N})$ that is nonzero at finitely many places. By $[\mathrm{score}(G)]$ we denote the same sequence where we replace each entry by its class of congruence modulo 2.

3 Collapse closed classes of graphs and their algebraic meaning

By L, further on, we denote a graph consisting of a single vertexless loop, by K_0 a graph with no vertices and no edges. By the union of graphs G and H denoted by $G \cup H$ we always mean disjoint union. Formally $G \cup H := (VG \uplus VH, EG \uplus EH, \delta_G \uplus \delta_H)$. In case of ordered graphs G, H we define (instead of union) their tensor product $G \otimes H$ to be a disjoint union of G with a copy of H where every vertex $i \in VH$ is replaced by $i + |VG|$.

Definition 1 *We call a nonempty class of graphs $\mathcal{G}$* **collapse closed** *if and only if the following conditions hold:*

1. *If $G, H \in \mathcal{G}$ then $G \cup H \in \mathcal{G}$.*
2. *If G is element of $\mathcal{G}$, u, v are vertices of G and π is a bijection of $\delta(u)$ to $\delta(v)$ then a graph obtained by collapsing u with v with respect to π is an element of $\mathcal{G}$ as well.*
3. *If $G \cup L \in \mathcal{G}$ then $G \in \mathcal{G}$.*

We can ask the following simple question about a collapse closed class of graph $\mathcal{G}$: Does it contain the vertexless loop L (and as a consequence of the third condition also the empty graph K_0?) The answer is positive whenever $\mathcal{G}$ contains some G with at least one edge. Then it also contains $G \cup G'$ where G' is a copy of G. By collapsing every vertex $v \in VG$ with $v' \in VG'$ (with respect to arbitrary bijections) we get a graph with no vertex and at least one edge that has to form a vertexless loop.

What is the linear subspace $\langle p_n(\mathcal{O}\mathcal{G})\rangle \subseteq T(S_n)$ generated by the image of a collapse closed class $\mathcal{G}$?

First, note that for any ordered graph G holds $p_n(G \otimes L) = np_n(G)$ so the third condition does not influence the answer to this question.

As for the other two operations we have the following compatibility relations:

$$p_n(G \otimes H) = p_n(G) \otimes p_n(H)$$

for any two ordered graphs G, H and

$$C_{i,j}(p_n(G)) = \sum_{\pi:\delta(i)\leftrightarrow\delta(j)} p_n(G_{i,j,\pi})$$

for any ordered graph G and two of its vertices i, j where $\pi : \delta(i) \leftrightarrow \delta(j)$ is a bijection of $\delta(i)$ and $\delta(j)$.

This is true since

$$\begin{aligned}
\sum_{\pi:\delta(i)\leftrightarrow\delta(j)} p_n(G_{i,j,\pi}) &= \sum_{\pi:\delta(i)\leftrightarrow\delta(j)} \sum_{\substack{\phi:EG\to[n] \\ \phi(e)=\phi(f) \text{ when } e\sim_\pi f}} \bigotimes_{v\in VG\setminus\{i,j\}} c_{\phi[\delta(v)]} e_{\phi[\delta(v)]} \\
&= \sum_{\phi:EG\to[n]} \sum_{\substack{\pi:\delta(i)\leftrightarrow\delta(j) \\ \phi(e)=\phi(f) \text{ when } e\sim_\pi f}} \bigotimes_{v\in VG\setminus\{i,j\}} c_{\phi[\delta(v)]} e_{\phi[\delta(v)]} \\
&= \sum_{\alpha\in A_n^{|\delta(i)|}} \sum_{\substack{\phi:EG\to[n] \\ \phi[\delta(i)]=\phi[\delta(j)]=\alpha}} \sum_{\substack{\pi:\delta(i)\leftrightarrow\delta(j) \\ \phi(e)=\phi(f) \text{ when } e\sim_\pi f}} \bigotimes_{v\in VG\setminus\{i,j\}} c_{\phi[\delta(v)]} e_{\phi[\delta(v)]} \\
&= \sum_{\alpha\in A_n^{|\delta(i)|}} \sum_{\substack{\phi:EG\to[n] \\ \phi[\delta(i)]=\phi[\delta(j)]=\alpha}} c_\alpha^2 \bigotimes_{v\in VG\setminus\{i,j\}} c_{\phi[\delta(v)]} e_{\phi[\delta(v)]} \\
&= C_{i,j} p_n(G)
\end{aligned}$$

where $A_n^{|\delta(i)|}$ denotes the set of multisets of colors of cardinality $|\delta(i)|$. The coefficient c_α^2 denotes by definition the number of color preserving bijections from $\delta(i)$ to $\delta(j)$ where coloring of both gives the multiset α.

Thus we get that $\langle p_n(\mathcal{OG})\rangle$ is an algebra (it is closed under the tensor product) which is, moreover, contraction closed. Since $\langle p_n(\mathcal{OG})\rangle$ is generated by tensors contained in one grade of the tensor algebra $T(S_n)$ (namely the grade of $p_n(G)$ is equal to the number of vertices of the ordered graph G), the algebra $\langle p_n(\mathcal{OG})\rangle$ has to be graded. At last, $\langle p_n(\mathcal{OG})\rangle$ is symmetric: this is due to the fact that with each ordered graph G all graphs differing only in the vertex ordering are also contained in $\mathcal{OG}$.

This gives all required conditions for [Sch06, Theorem 2] which gives us a possibility of extending an algebra homomorphism $\hat{f} : \langle p_n(\mathcal{OG})\rangle \to \mathbb{R}$ to an algebra homomorphism $\tilde{f} : T(S_n) \to \mathbb{R}$. This will be the final step of the proof in Section 5 by which we obtain an edge coloring model for a partial graph parameter.[2]

Since we are already algebraically motivated for studying the collapse closed classes, let us specify which of them we will particularly study and try to pre-

[2]We note that when a tensor algebra $\mathcal{A}$ satisfies the same conditions as $\langle p_n(\mathcal{OG})\rangle$ does, namely that it is contraction closed and graded, and when we set S to be the minimal vector space such that $\mathcal{A} \subset T(S)$ then by [Sch08a, Corollary 1e] there is a subgroup H of the orthogonal group $\mathcal{O}(S)$ such that $\mathcal{A}$ is exactly the set of all tensors of $T(S)$ that are invariant under the action of the group H.

cisely describe and after that we will mention some examples that we can figure out without any deep understanding.

Definition 2 *We additionally define a collapse closed class $\mathcal{G}$ as* **nondegenerate** *if it satisfies the following:*

Whenever a graph G with a vertex of degree $k \in \mathbb{N}$ is contained in $\mathcal{G}$ then there is a graph H in $\mathcal{G}$ having a vertex of degree k without loops and the vertex belongs to a component of at least 3 vertices.

Since now on, we will be mainly interested in nondegenerate collapse closed classes because of the following reasons:

1. For this classes of graphs we can easily provide the extension of the Szegedy's [Sze07, Teorem 2.2].

2. The most important classes are nondegenerate. As we are going to explain in the end of this section, the degenerate classes do not provide any important enrichment, yet their description (and the proof of the description as well) would become inadequately complicated.

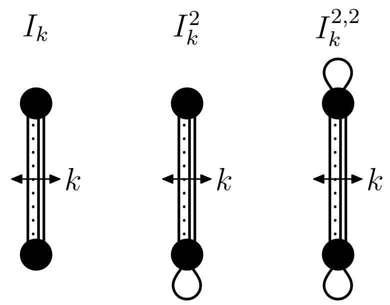
Examples

By first three examples we would like to support our claim that it is reasonable to omit the case of degenerate collapse closed classes from our attention:

1. The class containing graphs consisting only of isolated vertices (with some loops possibly) and vertexless loops.

2. The class containing only disjoint unions of copies of vertexless loop L and copies of two vertex components where only degrees k and $k+2$ appear.

   For example the three following graphs are contained in the class:



The examples of nondegenerate collapse closed classes follow:

1. The class of all graphs with vertices of degrees 2,4 or 6.

2. The class of graphs such that the number of their vertices of degree 2,4 or 6 is even.

3. The class of such graphs that any of their components of connectivity has either only vertices with degree 2 and 4 or only with degree 6.

As we are going to prove in our main description theorem, these three examples actually demonstrate the only three types of restrictions that any nondegenerate collapse closed class can represent. First let us formalize these types of restrictions.

Definition 3 *Let $\mathcal{G}$ be a collapse closed class of graphs.*

1. *By $D_{\mathcal{G}} \subset \mathbb{N}$ we denote the set of degrees of vertices of graphs in $\mathcal{G}$.*

2. *We define relation $\sim_{\mathcal{G}}$ on $D_{\mathcal{G}}$ by $a \sim_{\mathcal{G}} b$ if and only if there exists $G \in \mathcal{G}$ with two vertices of degrees a and b in the same component. Because $\mathcal{G}$ is closed on collapses, it is easily seen that $\sim_{\mathcal{G}}$ is an equivalence relation.*

3. *By $V_{\mathcal{G}}$ we denote the set $\{[\text{score}(G)] \mid G \in \mathcal{G}\}$ which is a subset of the $\mathbb{Z}_2$-vector space of zero-one sequences $\bigoplus_{d \in D} \mathbb{Z}_2$. The set $V_{\mathcal{G}}$ is actually a linear subspace since $[\text{score}(G)] + [\text{score}(H)] = [\text{score}(G \cup H)]$ and the zero vector is in $V_{\mathcal{G}}$ since $K_0 \in \mathcal{G}$.*

Note that collapsing any two vertices in a graph G does not change the sequence $[\text{score}(G)]$ since two vertices of the same degree disappear. Also note that when $\mathcal{G}$ is nondegenerate and $1 \in D_{\mathcal{G}}$ then there has to be $k \in D_{\mathcal{G}}$ such that $1 \sim_{\mathcal{G}} k$ because there is no 1-regular component of connectivity of at least 3 vertices.

Definition 4 *Let D be a subset of $\mathbb{N}$, let $\sim$ be an equivalence relation on D such that there is $k \in D$ such that $1 \sim k$ and let $V \leq \mathbb{Z}_2^D$ be a vector space of $\mathbb{Z}_2$ sequences nonzero on finitely many places. Then we call a triple $P := (D, \sim, V)$ a* **graph class restriction***.*

By $\mathcal{G}(P)$ we will denote the class of all graphs G such that all vertices of G have degree in D, $[\text{score}(G)] \in V$ and for each two connected vertices $u, v \in VG$ holds that $\deg(u) \sim \deg(v)$.

Observation 1 *For any graph class restriction $P = (D, \sim, V)$ the class $\mathcal{G}(P)$ is collapse closed that is, moreover, nondegenerate.*

Proof. It is easily seen that deleting a vertexless loop from any graph changes neither its score nor the connectivity of any of its two vertices. It is also easily seen that none of the three operations of collapse closed classes can create a vertex of a new degree.

Since $[\mathrm{score}(G)] + [\mathrm{score}(H)] \in V$ whenever $G, H \in \mathcal{G}(P)$ and since any two vertices connected by a path in $G \cup H$ are already connected by a path either in G or in H, also $G \cup H \in \mathcal{G}(P)$ whenever $G, H \in \mathcal{G}(P)$.

For any G, $u, v \in VG$ and any bijection $\pi : \delta(u) \to \delta(v)$ it holds that $[\mathrm{score}(G_{u,v,\pi})] = [\mathrm{score}(G)]$ and that any two vertices $w, w' \in VG_{u,v,\pi}$ connected by a path in $G_{u,v,\pi}$ are either connected by the same path in G or each of them is connected by a path with either u or v, the vertices of the same degree. Hence $\deg(w) \sim \deg(w')$ and thus $G_{u,v,\pi} \in \mathcal{G}(P)$ if $G \in \mathcal{G}(P)$.

By constructing simple examples of graphs, we get that $\mathcal{G}$ is nondegenerate. □

By the Observation 1, picking an arbitrary graph class restriction is a way of constructing a nondegenerate collapse closed class of graphs. Our main result, the description of nondegenerate collapse closed classes claims that in this way we get all of them.

Theorem 1 (The description theorem) *For any nondegenerate collapse closed class of graphs $\mathcal{G}$ the following holds:*

$$\mathcal{G} = \mathcal{G}(P)$$

where $P := (D_{\mathcal{G}}, \sim_{\mathcal{G}}, V_{\mathcal{G}})$.

To give an rough idea about the description of all collapse closed classes we provide the following unproven and informal statement:

Each collapse closed class of graphs can be obtained from some nondegenerate class $\mathcal{G}$ by attaching to each vertex of degree d some fixed number of loops $f(d)$ for each $d \in D_{\mathcal{G}}$ and then adding components of cardinality at most 2 with degrees in $D' \supseteq D_{\mathcal{G}}$ to graphs in $\mathcal{G}$.

We do not consider this to be a true enrichment whereas the description (mainly the description of 2 vertex components) would become unnecessarily complicated and, however, the proof could be obtained by extending of our proof, it would be inadequately technical.

4 Proof of the description theorem

Proof of the Theorem 1. The inclusion $\mathcal{G} \subseteq \mathcal{G}(P)$ is trivial. For the opposite inclusion $\mathcal{G}(P) \subseteq \mathcal{G}$ we want to prove that when we take any graph $G \in \mathcal{G}(P)$ then also $G \in \mathcal{G}$. We will do that in three steps:

1. **Constructing the vertex set.**

   Claim 1 *The class $\mathcal{G}$ contains a graph G_0 with score equal to* $\mathrm{score}(G)$.

We will make use of the following simple lemma that will easily follow from the second step (that is independent on the first step).

Lemma 1 *For each degree $k \in D_{\mathcal{G}}$ the graph I_k consisting of two vertices connected by k edges is contained in $\mathcal{G}$.*

From the definition of $V_{\mathcal{G}}$ we know that there is a graph $H \in \mathcal{G}$ such that $[\text{score}(H)] = [\text{score}(G)]$. To get a graph such that it has the number of vertices of degree k equal to $\text{score}(G)_k$ we just need to make a union of H with the appropriate number of copies of I_k or apply to H the appropriate number of collapses of vertices of degree k. After we do this for each degree k appearing in the graph G, we get the desired graph. We will further denote it as G_0.

2. **Swapping gadget construction.** In this technical step we prepare for every two degrees $k \sim_{\mathcal{G}} l \in D_{\mathcal{G}}$ an auxiliary four-vertex graph $S_{k,l}$. These auxiliary graphs will be used for modification of G_0 into G.

   We first have to do some preparations, namely getting rid of the loops from any pair of vertices of some degrees k and l and at the same time making their distance greater than one. The following technical but not difficult lemma will do that for us.

   Lemma 2 *For each $k \sim_G l$ there is a graph $F_{k,l}$ contained in $\mathcal{G}$ such that it has two connected vertices of degrees k and l without loops with distance greater than one.*

   So let us take for any $k \sim_{\mathcal{G}} l$ the graph $F := F_{k,l} \in \mathcal{G}$ with connected vertices u and v of degrees k and l without loops such that their distance is greater than 1. Then there is a sequence of vertices $v_0 = u, v_1, \ldots, v_d = v$ and such that each subsequent pair v_i, v_{i+1} is connected by an edge e_i. We make a union $F \cup F'$ where F' is a copy of F. The following procedure is captured in the Figure 2.

   We fix an arbitrary local ordering Δ on $F \cup F'$ that is the same on F' as on F except that Δ on F' swaps the edges e'_{i-1} and e'_i at each v'_i (as the Figure 2 suggest: we want the edges of the same color to get glued together). Formally, Δ is such an local ordering that $\Delta_{v_i}(1) = e_{i-1}$ and $\Delta_{v_i}(2) = e_i$, $\Delta_{v'_i}(1) = e'_i$ and $\Delta_{v'_i}(2) = e'_{i-1}$ for $i = 1, \ldots, d-1$ and, finally, $\Delta_{v'}(j) = \Delta_v(j)'$ in all other cases.

   On the graph $F \cup F'$ we make a series of collapses of form v_i with v'_i, for $i = 1, \ldots, d-1, d+1, \ldots, |VF|$ with respect to the local ordering Δ where

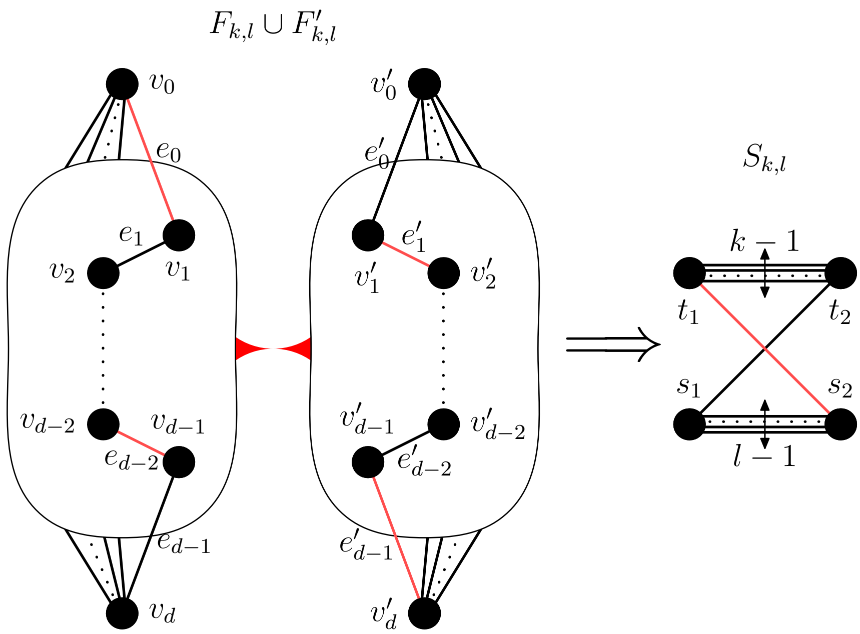


Figure 2: The swapping gadget construction for the case when distance of u and v is even.

$v_{d+1}, \ldots, v_{|VF|}$ is an arbitrary numbering of the remaining vertices of F. In the end we obtain a graph on vertices v_0, v_d, v'_0 and v'_d determined by such an equivalence $\sim$ on $EF \cup EF'$ that $e_0 \sim e'_1 \sim e_2 \sim e'_3 \sim \ldots$ and $e'_0 \sim e_1 \sim e'_2 \sim e_3 \sim \ldots$ and $e \sim e'$ for all non-indexed edges e. Thus we get that the edge $[e_0]_\sim$ connects v_0 with v_d or v'_d and the edge $[e'_0]_\sim$ connects v'_0 with v'_d or v_d depending on whether d is odd or even. For all $k-1$ of non-indexed edges e incident with v_0 the resulting edge $[e]_\sim$ connects v_0 with v'_0 and for all $l-1$ of not indexed edges e incident with v_d the resulting edge $[e]_\sim$ connects v_d with v'_d. All remaining edges e form a loop together with e'.

When we delete the vertexless loops from the result and denote the remaining four vertices as s_1, s_2, t_1, t_2 in such a way that t_2 is connected by a single edge to s_1 and t_1 is connected by a single edge to s_2, we get the desired graph $S_{k,l}$ as depicted on the right part of the Figure 2.

Thus we get:

Claim 2 *For each $k, l \in D_\mathcal{G}$ such that $k \sim_\mathcal{G} l$ it holds that $\mathcal{G}$ contains the graph $S_{k,l}$.*

Proof of Lemma 1. Since $S_{k,k} \in \mathcal{G}$ then also $(S_{k,k})_{s_1,s_2,\pi} \in \mathcal{G}$ for any π. But this is the graph I_k possibly with some extra vertexless loops. □

3. **Edge swapping.**

Claim 3 *For any labelling of VG by $\{1, 2, \ldots, |VG|\}$ and for each $0 \leq a \leq 2|EG|$ there is a graph $G_a \in \mathcal{G}$ with the vertex set labelled by $\{1, 2, \ldots, |VG|\}$ such that $|\mathcal{F}(G_a) - \mathcal{F}(G)|_1$ is at most $2|EG| - a$.*

Note that this directly implies our goal.

As the statement of the Claim 3 suggest, we prove it by induction, where the first step for $a = 0$ follows from the Claim 1 (Constructing the vertex set) where we use any labelling of VG_0 such that equally labelled vertices in VG and in VG_0 have the same degree. For the rest, let us suppose that $\mathcal{G}$ contains G_a, where $a < |EG|$ such that the hypothesis holds. Then we know that there are vertices u_1, u_2 and v_2 such that u_1 and u_2 are connected in G_a by more edges than in G (let us denote the superfluous edge as e) and u_1 and v_2 are connected in G_a by fewer edges than in G. Additionally, it follows that there is a vertex v_1 such that v_2 and v_1 are in G_a connected by more edges than in G (let us denote the superfluous edge as f). The situation is captured in the left part of the Figure 3.

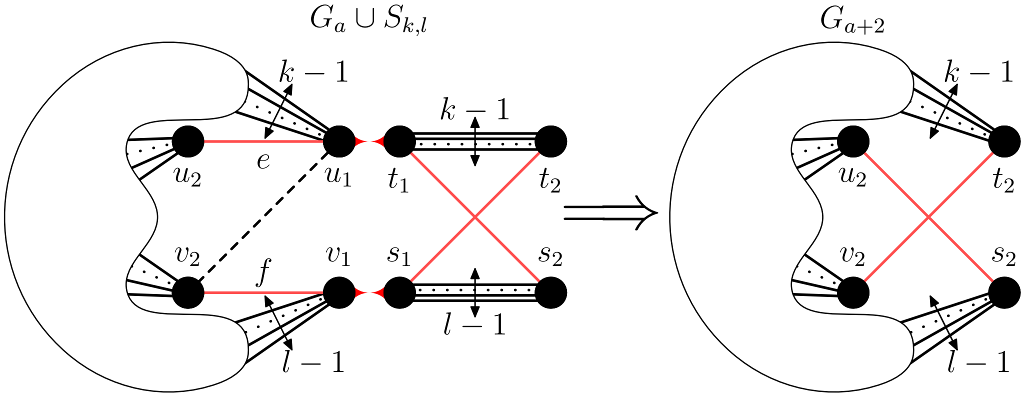


Figure 3: Swapping of a superfluous edge for the case when all vertices u_1, u_2, v_1 and v_2 are distinct.

Note that the vertices u_1, u_2, v_1 and v_2 do not necessarily need to be all distinct. (Also it can happen that $e = f$ when $u_1 = v_2$ and $u_2 = v_1$.) Our situation only guarantees that $u_2 \neq v_2$ and $u_1 \neq v_1$ since the number of edges connecting two vertices cannot be at the same time bigger and smaller than another number. Although the Figure 3 only captures the case, when all the vertices are distinct, the following procedure works for all cases.

Let us denote the degree of u_1 as k and the degree of v_1 as l. Since both G_a and G are in $\mathcal{G}$, we have that $k \sim_{\mathcal{G}} l$ and that is why by Claim 2 the graph $S_{k,l}$ is contained in $\mathcal{G}$. We make a union G_a with one copy of $S_{k,l}$ and perform a collapse of u_1 with s_1 with respect to any bijection that maps e to the only single edge of s_1 and, similarly, we collapse v_1 with t_1

with respect to any bijection that maps f to the only single edge of t_1. In the case that e or f is a loop, only one end is mapped as described, the other end is mapped (as all the other edges) arbitrarily. In the resulting graph (let us denote it as G_{a+2}) s_2 has the same neighbors as u_1 had except that one edge is now going to v_2 instead of u_2. And, similarly, t_2 has the same neighbors as v_1 had except that one edge is now going to u_2 instead of v_2.

When we label the vertex s_2 by the label of u_1 and t_2 by the label of v_1, we have that $\mathcal{G}$ contains G_{a+2} such that the hypothesis holds since the superfluous edge e has been deleted and the missing edge between u_1 and v_2 has been added. Beside that we have deleted other superfluous edge f but we have added an edge between u_1 and v_2 that might be again superfluous. □

Proof of the Lemma 2. We proceed in two steps. In the first one, we prove the lemma in the case when $k = l$. Then, because $\mathcal{G}$ is nondegenerate, there is a graph $H \in \mathcal{G}$ with a vertex u of degree k and without loops, and with vertices v and w, all of them being in one component. We fix an arbitrary local ordering Δ on H. We make a union $H \cup H' \cup H''$ where H' and H'' are copies of H with a local ordering $\Delta \cup \Delta' \cup \Delta''$ where Δ' and Δ'' are copies of Δ. It remains to set $F_{k,k}$ as a result of collapsing v with v' and w' with w'' in $H \cup H' \cup H''$ with respect to the local ordering $\Delta \cup \Delta' \cup \Delta''$. Indeed, in $F_{k,k}$ the vertices u and u'' are without loops, have degree k and all neighbors of u are in $VH \cup \{u'\}$ since any edge e connecting u with v gets glued with the edge e' connecting v' with u'.

In the second step we prove the lemma in the case that $k \neq l$. We know that there is a graph H in $\mathcal{G}$ with vertices x and y of degrees k and l connected by a path. We make a union $F_{k,k} \cup H$ and collapse u'' with x with respect to a bijection mapping the first edge on the path from u'' to u to the first edge on the path between x and y. If we do the same with $F_{l,l}$ and the vertex y, we get the desired graph $F_{k,l}$. □

5 Extending the Szegedy's characterization to collapse closed classes

As we promised earlier, now we will introduce the characterization of partial graph parameters in the edge coloring model. For this let $\mathcal{G}$ again be a collapse closed class of graphs. Let $f : \mathcal{G} \to \mathbb{R}$ be a real parameter on $\mathcal{G}$. We say that f is **multiplicative** on $\mathcal{G}$ whenever for any $G, H \in \mathcal{G}$ we have that $f(G \cup H) = f(G)f(H)$.

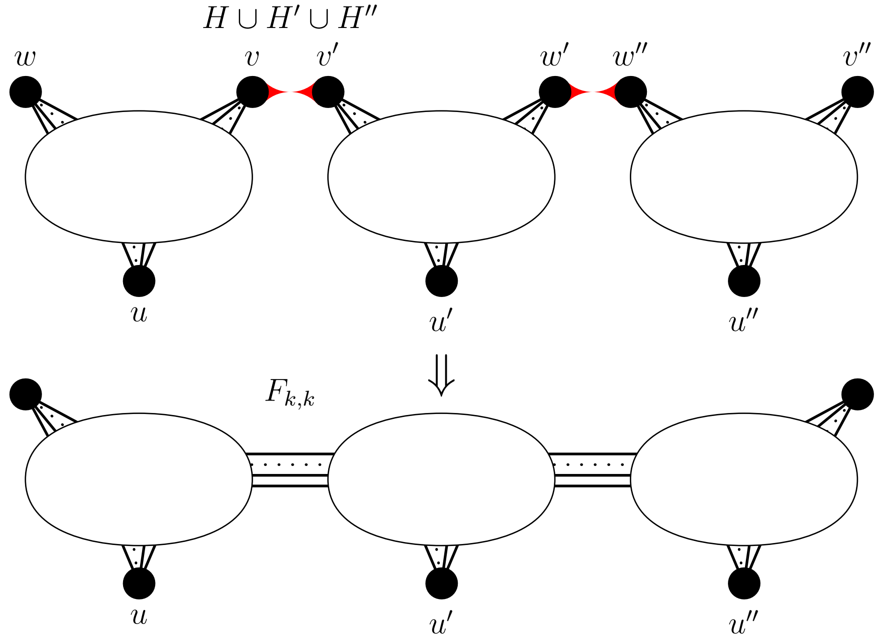


Figure 4: Construction of the graph $F_{k,k}$.

For further statements we will need the following. For a class of graphs $\mathcal{G}$ we denote $\mathcal{QG}$ a collection of all formal linear combinations of graphs $\sum_G \gamma_G G$ with at most finitely many γ_G nonzero. These are called **quantum graphs**. By taking the linear extension of the disjoint union $G \cup H$ as multiplication, $\mathcal{QG}$ becomes a commutative algebra. The partial parameter f can be extended linearly to $\mathcal{QG}$.

By $(G)_{u_1,v_1,\pi_1,\ldots,u_k,v_k,\pi_k}$ we will denote $(\ldots(G)_{u_1,v_1,\pi_1}\ldots)_{u_n,v_n,\pi_n}$. For $G, H \in \mathcal{G}$ and $k \in \mathbb{N}$, define the quantum graph $\lambda_k(G,H)$ by

$$\lambda_k(G,H) := \sum_{u_1,v_1,\pi_1,\ldots,u_k,v_k,\pi_k} (G \cup H)_{u_1,v_1,\pi_1,\ldots,u_k,v_k,\pi_k}, \tag{2}$$

where the sum is taken over all distinct $u_1,\ldots,u_k \in VG$, distinct $v_1,\ldots,v_k \in VH$ and all bijections $\pi_i : \delta(u_i) \to \delta(v_i)$, for $i = 1,\ldots,k$. We can linearly extend λ_k to the whole $\mathcal{QG}$.

In this paper we state only a symmetric version of edge reflection positivity for which it was shown by Schrijver [Sch08b] that it can replace edge reflection positivity of Szegedy.

For $k \in \mathbb{N}$ let us define $\mathcal{G} \times \mathcal{G}$ matrix $M_{f,k}$ by expressing the entry indexed by $G, H \in \mathcal{G}$:

$$(M_{f,k})_{(G,H)} := f(\lambda_k(G,H))$$

We define a partial parameter f to be edge reflection positive on class $\mathcal{G}$ if and only if the matrix $M_{f,k}$ is positive semidefinite for each k. Positive semidefiniteness of $M_{f,k}$ is equivalent to the fact that $f(\lambda_k(\gamma,\gamma)) \geq 0$ for each $\gamma \in \mathcal{QG}$.

In [Sch08b] it is proven:

Proposition 1 *Be $\mathcal{G}$ the class of all graphs and let $f : \mathcal{G} \to \mathbb{R}$. Then $f = f_b$ for some $n \in \mathbb{N}$ and for some $b : A_n \to \mathbb{R}$ if and only if f is multiplicative and edge reflection positive on $\mathcal{G}$.*

What we want to show is that actually the class of all graphs can be replaced by any nondegenerate collapse closed class of graphs. First note that we need to have unions and collapses included in the class if only for the condition of multiplicativity and edge reflection positivity. So we need to have $\mathcal{G}$ collapse closed unless we would completely change the statement.

Why cannot $\mathcal{G}$ be degenerate collapse closed classes? In our proof of the Theorem 2 (namely in the proof of Claim 4) we need that for each $d \in D_{\mathcal{G}}$ the graph I_d is an element of $\mathcal{G}$. Though this is a weaker condition than nondegenerateness we leave the statement in the following form since we consider the case of nondegenerate classes to be far the most important:

Theorem 2 *Let $\mathcal{G}$ be a collapse closed class of graphs that is nondegenerate and let $f : \mathcal{G} \to \mathbb{R}$. Then $f = f_b$ for some $n \in \mathbb{N}$ and for some $b : A_n \to \mathbb{R}$ if f is multiplicative, edge reflection positive on $\mathcal{G}$ and $f(L) \in \mathbb{N}$.*

To validate that Theorem 2 is indeed an extension of Proposition 1 we need to refer to [Sch08b] where the condition $f(L) \in \mathbb{N}$, in the case of $\mathcal{G}$ equals to the class of all graphs, is derived from the multiplicativity and the edge reflection positivity.

In our case we were not able to derive the same condition again. For more explanation see the last section.

And also note that the inverse implication follows from the Proposition 1 and from the fact that $f_b(L) = n$ for any $b : A_n \to \mathbb{R}$.

Proof. In the first part of the proof we will show an existence of an algebra homomorphism $\tilde{f} : \langle p_n(\mathcal{OG}) \rangle \to \mathbb{R}$ such that $f(G) = \tilde{f}(p_n(G))$. We will almost exactly follow the steps of the Schrijver's proof in [Sch08b].

Claim 4 *Let γ be a quantum graph consisting of k-vertex graphs. If $f(\lambda_k(\gamma,\gamma)) = 0$ then $f(\gamma) = 0$.*

Proof of the Claim 4. We prove the claim by induction on k. So assume that the claim holds for all quantum graphs made of graphs with less than k vertices.

We can assume that all graphs occurring in γ with nonzero coefficient have the same degree sequence $d_1, \ldots, d_k$, since if we would write $\gamma = \gamma_1 + \gamma_2$, where all graphs in γ_1 have degree sequence different from those in γ_2, then $\lambda_k(\gamma_1, \gamma_2) = 0$, whence $f(\lambda_k(\gamma_i, \gamma_i)) = 0$ for $i = 1, 2$.

Now $f(\lambda_k(\gamma, \gamma)) = 0$ implies, by the positive semidefiniteness of $M_{f,k}$:

$$f(\lambda_k(\gamma, H)) = 0 \text{ for each graph } H \in \mathcal{G}. \tag{3}$$

Let P be the graph $P := I_{d_1} \cup \ldots \cup I_{d_k}$. Since $d_1, \ldots, d_k \in D_{\mathcal{G}}$ and $[\text{score}(P)] = 0$ the graph P is in $\mathcal{G}$.

If $d_1, \ldots, d_k$ are all distinct, we are done, since then γ is a multiple of $\lambda_k(\gamma, P)$, implying with 3 that $f(\gamma) = 0$ — but generally there can be vertices of equal degrees.

The sum in 2 for λ_k can be decomposed according to the set I of those components of P with both vertices chosen among $v_1, \ldots, v_k$ and to the set J of those components of P with no vertices chosen among $v_1, \ldots, v_k$ (necessarily $|I| = |J|$). Let K denote the set of components of P, and for $J \subseteq K$, let P_J be the union of the components in J. Then

$$\lambda_k(\gamma, P) = \sum_{I,J \subseteq K, I \cap J = \emptyset, |I|=|J|} \alpha_{I,J} \gamma_I \cup P_J, \tag{4}$$

where $\alpha_{I,J} \in \mathbb{N}$ with $\alpha_{\emptyset,\emptyset} \neq 0$, and where

$$\gamma_I := \lambda_{2|I|}(\gamma, P_I).$$

Now for each $I \subseteq K$, we have $\lambda_{k-2|I|}(\gamma_I, \gamma_I) = \lambda_k(\gamma, \gamma_I \cup P_I)$. Hence

$$f(\lambda_{k-2|I|}(\gamma_I, \gamma_I)) = f(\lambda_k(\gamma, \gamma_I \cup P_I)) = 0,$$

by 3. So by induction, if $I \neq \emptyset$ then $f(\gamma_I) = 0$. Therefore, by 4, since $f(\lambda_k(\gamma, P)) = 0$ and $\alpha_{\emptyset,\emptyset} \neq 0$, $f(\gamma) = f(\gamma_\emptyset \cup P_\emptyset) = 0$. □

Since $f(L) \in \mathbb{N}$ we can set $n := f(L)$.

Then Claim 4 implies:

$$\text{there is a linear function } \hat{f} : \langle p_n(\mathcal{OG}) \rangle \to \mathbb{R} \text{ such that } f = \hat{f} \circ p_n. \tag{5}$$

Otherwise, there is a quantum graph γ with $p_n(\gamma) = 0$ and $f(\gamma) \neq 0$. We can assume that γ is homogenous, that is, all graphs in γ have the same number of vertices, k say, since $p_n(\gamma_k + \gamma_l) = 0$ where γ_k and γ_l consists of graphs on k and l vertices, respectively, implies $p_n(\gamma_k) = 0$ and $p_n(\gamma_l) = 0$.

Since $\lambda_k(\gamma, \gamma)$ is a polynomial in L, and since $f(L) = n = p_n(L)$,

$$\begin{aligned} f(\lambda_k(\gamma, \gamma)) &= p_n(\lambda_k(\gamma, \gamma)) \\ &= p_n \left(\sum_{u_1,v_1,\pi_1,\ldots,u_k,v_k,\pi_k} (\gamma \otimes \gamma)_{u_1,v_1,\pi_1,\ldots,u_k,v_k,\pi_k} \right) \\ &= \sum_{u_1,v_1,\ldots,u_k,v_k} C_{u_1,v_1} \circ \ldots \circ C_{u_k,v_k}(p_n(\gamma \otimes \gamma)) \\ &= 0 \end{aligned}$$

where we use the compatibility of contraction and collapse. So by Claim 4, $f(\gamma) = 0$. This proves 5.

In fact, $\hat{f}$ is an algebra homomorphism, since for all $G, H \in \mathcal{OG}$:

$$\hat{f}(p_n(G)p_n(H)) = \hat{f}(p_n(G \otimes H)) = f(GH) = f(G)f(H) = \hat{f}(p_n(G))\hat{f}(p_n(H)).$$

Because f does not depend on ordering of vertices, also $\hat{f}$ is symmetric. We define $(.,.) : \langle p_n(\mathcal{OG})\rangle \times \langle p_n(\mathcal{OG})\rangle \to \langle p_n(\mathcal{OG})\rangle$ by

$$(p_n(G), p_n(H)) := \sum_{i=1}^{k}\sum_{j=1}^{l} C_{i,k+j}(p_n(G) \otimes p_n(H)),$$

for k vertex ordered graph G and l vertex ordered graph H. But then $(p_n(G), p_n(H)) = p_n(\lambda_1(G, H))$ since by compatibility of contraction and collapse

$$\sum_{i=1}^{k}\sum_{j=1}^{l} C_{i,k+j}(p_n(G) \otimes p_n(H)) = \sum_{i=1}^{k}\sum_{j=1}^{l}\sum_{\pi:\delta(i)\leftrightarrow\delta(j)} p_n((G \otimes H)_{i,k+j,\pi}).$$

Then by positive semidefiniteness of $M_{f,1}$ for any quantum graph γ holds

$$\hat{f}((p_n(\gamma), p_n(\gamma))) = f(\lambda_1(\gamma, \gamma)) \geq 0.$$

So we can use [Sch06, Theorem 1] and we get that there is an algebra homomorphism $\tilde{f} : T(S_n) \to \mathbb{R}$ such that $f(G) = \tilde{f}(p_n(G))$. Now we only need to set $b(\alpha) := c_\alpha \tilde{f}(e_\alpha)$ since then $f_b(G) = \tilde{f}(p_n(G)) = f(G)$.

6 Further questions and problems

The similar question of description of the collapse closed classes could be raised for embedded graphs where a graph embedding is determined by a cyclic ordering of ends of edges around each vertex. Here we would require that the bijection in the collapse operation would respect the cyclic ordering of the edges around respective vertices. When the ordering is given, such a bijection is determined only by specifying a “first” end of edge at each of two collapsed vertices. In this way we get an embedded graph that can be realized by adding a handle connecting two appropriate areas incident with the two collapsed vertices and using the handle to connect the corresponding edges.

Can we then characterize all classes of embedded graphs closed for disjoint union, deleting loops and collapse? Again after a convenient transformation each such a class would give us a contraction closed, graded and symmetric tensor subalgebra.

Besides, it would be nice to state the Theorem 2 without the additional condition $f(L) \in \mathbb{N}$. This condition is fundamental in the proof of the Proposition 1 in [Sch08b] but there it is derived from the edge reflection positivity and the fact that the parameter is defined on every 2-regular graph on k vertices for some big enough k of a convenient parity. Whereas in the proof of the Theorem 2 we can only suppose that we have arbitrary d-regular graphs for some $d \in \mathbb{N}$ on even number of vertices. It is unclear to us whether the condition that $f \in \mathbb{N}$ follows from this or there is an example of a multiplicative and edge reflection positive parameter that violates this condition.

At last we conjecture the validity of the Theorem 2 even for all collapse closed classes of graphs. However, we expect technical difficulties in the proof and we did not consider it to be such an important improvement worth of the effort.

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