Long paths and cycles in hypercubes with faulty vertices

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Abstract

A fault-free path in the n-dimensional hypercube Q_n with f faulty vertices is said to be *long* if it has length at least $2^n - 2f - 2$. Similarly, a fault-free cycle in Q_n is long if it has length at least $2^n - 2f$. If all faulty vertices are from the same bipartite class of Q_n, such length is the best possible. We show that for every set of at most $2n - 4$ faulty vertices in Q_n and every two fault-free vertices u and v satisfying a simple necessary condition on neighbors of u and v, there exists a long fault-free path between u and v. This number of faulty vertices is tight and improves the previously known results. Furthermore, we show for every set of at most $n^2/10 + n/2 + 1$ faulty vertices in Q_n where $n \geq 15$

*Supported by the Czech Science Foundation Grants 201/05/H014.
†Supported by Czech Ministry of Education project MSM 0021620834.

that Q_n has a long fault-free cycle. This is a first quadratic bound, which is known to be asymptotically optimal.

1 Introduction

The n-dimensional hypercube Q_n is a (bipartite) graph with all binary vectors of length n as vertices and with edges joining every two vertices that differ in exactly one coordinate. The application of hypercubes as interconnection networks inspired many questions related to their fault-tolerance. In particular, in this paper we consider a problem of long fault-free cycles and long fault-free paths between two given vertices in hypercubes in which some vertices are faulty.

This problem is sometimes considered in a more general setting also with faulty edges, not only vertices. Assume that we have f_v faulty vertices and f_e faulty edges in Q_n. A path or a cycle in Q_n is said to be *fault-free* if it contains no faulty vertex and no faulty edge. Furthermore, a cycle in Q_n is *long* if it has length at least $2^n - 2f_v$. Similarly, a path in Q_n is *long* if it has length at least $2^n - 2f_v - 2$. Note that every long path between vertices u and v has length at least $2^n - 2f_v - 1$ if $d(u, v)$ is odd, where $d(u, v)$ is the distance between u and v. Furthermore, if all faulty vertices belong to the same bipartite class of Q_n, then every long fault-free cycle and long fault-free path is the longest possible. In this view, the problem of long fault-free cycles and paths is a relaxation of a substantially more difficult problem of Hamiltonian cycles and paths in hypercubes with balanced faulty vertices in the sense that in the former problem we are allowed to choose another up to f_v vertices that will be avoided (see e.g. [4] for some references on the latter problem).

As far as we know, the problem of long fault-free cycles in hypercubes was first studied by Tseng [14] who showed that such cycle in Q_n exists if $f_v + f_e \leq n-1$, $f_v \leq n-1$, and $f_e \leq n-4$. This bound was slightly improved by Sengupta [12] to $f_v + f_e \leq n - 1$, and $f_v > 0$ or $f_e \leq n - 2$. Then it was substantially strengthened by Fu [5] to $f_v \leq 2n - 4$ (and $f_e = 0$), and further naturally generalized by Hsieh [7] to $f_v + f_e \leq 2n - 4$ and $f_e \leq n - 2$. The latest improvement due to Castañeda and Gotchev [2] is for $f_v \leq 3n - 7$ ($f_e = 0$) and $n \geq 5$. Note that all these bounds are linear in the dimension n. We provide a first quadratic bound on f_v (and $f_e = 0$), which is known to be asymptotically optimal.

The similar problem for paths was first studied by Fu [6] who showed that there is a long fault-free path in Q_n between every two fault-free vertices if $f_v \leq n - 2$ (and $f_e = 0$). Hung, Chang, and Sun [8] showed that even a little longer path exists under similar conditions. More precisely, there is a fault-free path in Q_n of length at least $2^n - 2f_v$ between every two fault-free vertices if $f_v \leq n-2$ ($f_e = 0$) and at least one vertex from each bipartite class is faulty.

Recently, the bound of Fu was improved by Kueng, Liang, Hsu, and Tan [10] to $f_v \leq 2n-5$ (and $f_e = 0$), but with an additional (strong) condition that every vertex has at least two fault-free neighbors. We show for $f_v \leq 2n-4$ (and $f_e = 0$) that a much weaker condition is both necessary and sufficient (up to one exception in Q_4). We also show that our bound is tight.

Let us also mention related results on bipancyclicity and bipanconnectivity. Tsai [13] showed that every fault-free edge and every fault-free vertex of Q_n lies on a fault-free cycle of every even length from 4 to $2^n - 2f_v$ if $f_v \leq n-2$ (and $f_e = 0$). Ma, Liu, and Pan [11] showed that if $f_v + f_e \leq n-2$, then Q_n contains a fault-free path of length l between every two fault-free vertices u and v for every l from $d(u,v)+2$ to $2^n - 2f_v - 1$ such that $l - d(u,v)$ is even. There are also many results on long fault-free cycles and paths in various modifications of hypercubes, which we do not list here.

2 Main results

A long fault-free path between u and v in Q_n with a set F of faulty vertices is shortly called an *(F,u,v)-path*. An edge $uv \in E(Q_n)$ is *fault-free* if both vertices u and v are fault-free. Note that for $n \geq 2$, every long path has length at least 2 if $|F| \leq 2n-4$. A vertex u is *surrounded* by F if F contains all neighbors of u. Furthermore, a triple (F,u,v) is *blocked* in Q_n if

$$u \text{ is surrounded by } F \cup \{v\} \text{ in } Q_n \text{ or } v \text{ is surrounded by } F \cup \{u\} \text{ in } Q_n; \tag{1}$$

otherwise (F,u,v) is *free* in Q_n. The reference to the underlying graph Q_n may be omitted if it is clear from the context. Clearly, if (F,u,v) is *blocked*, there is no fault-free path between u and v of length more than 1. Thus, the triple (F,u,v) must be free for the existence of an (F,u,v)-path if $|F| \leq 2n-4$. The following theorem shows that this necessary condition is also sufficient, up to one exception in Q_4.

Theorem 2.1. *Let F be a set of at most $2n-4$ faulty vertices of Q_n where $n \geq 2$. For every two fault-free vertices u and v, there exists a long fault-free path between u and v in Q_n if and only if both (1) and (2) does not hold.*

On Figure 1 we have the following configuration for $n = 4$ and $|F| =$

$2n-4$:

$$\text{there are two vertices } a \text{ and } b \text{ with } d(a,b)=4 \text{ in } Q_4 \text{ such that} \\ F\cup\{u,v,a,b\} \text{ are the all 8 vertices of one bipartite class of } Q_4. \tag{2}$$

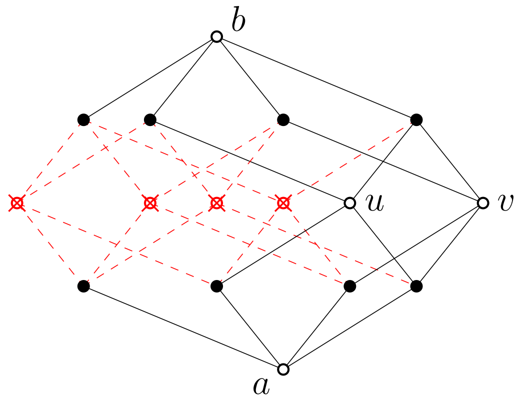

Figure 1: The exceptional configuration (2) in Q_4. The crossed points represent the faulty vertices and u, v are the prescribed endvertices for a requested long fault-free path.

Observe in this configuration that every fault-free path between u and v has length at most 4 because the graph $Q_4 \setminus (F \cup \{u,v\})$ has two components and no fault-free path between u and v can visit both components. Hence, there is no (F,u,v)-path although $|F| \leq 2n-4$ and (F,u,v) is free. Note that there are two non-isomorphic exceptional configurations since $d(u,v)$ can be 2 or 4.

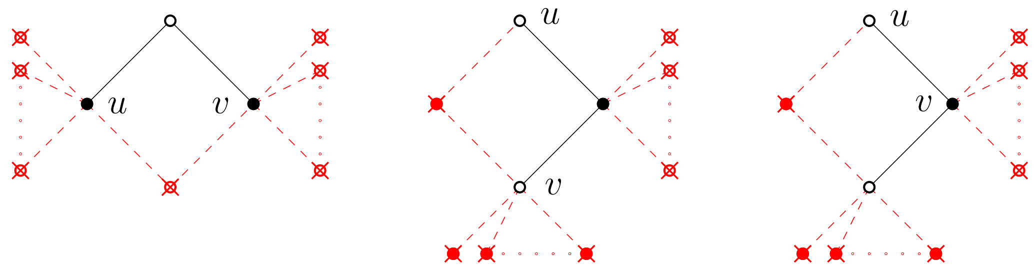

Figure 2: $|F|=2n-3$, $n\geq 4$, and (F,u,v) is free, but there is no (F,u,v)-path.

Moreover, observe that the inequality $|F| \leq 2n-4$ in Theorem 2.1 is tight for every $n \geq 4$. On Figure 2 we can see three configurations of $2n-3$ faulty vertices and two fault-free vertices u and v in Q_n such that (F,u,v) is free. Clearly, in all these configurations there is only one fault-free path between u and v of length 1 or 2, which is not long.

We prove Theorem 2.1 by induction on the dimension n. In Section 4 we prove the base of induction by a tedious case analysis for $n \leq 4$. In Section 5 we prove the induction step. In Section 6 we prove that for a sufficiently large n, we can find a long fault-free cycle in Q_n with a quadratic number of faulty vertices.

Theorem 2.2. *Let F be a set of at most $\frac{n^2}{10} + \frac{n}{2} + 1$ faulty vertices of Q_n where $n \geq 15$. Then Q_n contains a long fault-free cycle.*

On the other hand, Koubek [9] and independently Castañeda and Gotchev [3] noticed that for $n \geq 4$ there is a set F of $\binom{n}{2} - 1$ faulty vertices such that Q_n has no long fault-free cycle, so Theorem 2.2 is asymptotically optimal. Such a set F can be, for example, a set consisting of all but one vertex at distance 2 from the vertex $(0, \ldots, 0)$.

It remains an open question whether the bound given by Theorem 2.2 can be improved to meet the upper bound of $\binom{n}{2} - 2$ vertices, as Castañeda and Gotchev [3] conjectured.

3 Preliminaries

The main obstacle in the proof of Theorem 2.1 are vertices surrounded by faulty vertices. In the following auxiliary propositions we mainly show that there are only few such obstacles.

Proposition 3.1. *Let F be a set of at most $2n - 3$ faulty vertices in Q_n where $n \geq 2$. Then, at most one vertex of Q_n is surrounded by F.*

Proof. Suppose on the contrary that two vertices u and v of Q_n are surrounded by F. Since each of them has n faulty neighbors, and they have at most 2 faulty neighbors in common, it follows that $|F| \geq 2n - 2$, a contradiction. □

In the following proposition we show that at most one triple (F, u, v) is blocked when $|F| \leq 2n - 4$ and the vertex u is fixed and not surrounded by F itself.

Proposition 3.2. *Let F be a set of at most $2n - 4$ faulty vertices in Q_n where $n \geq 2$, and let $u \in V(Q_n)$ be not surrounded by F. Then, (F, u, v) is blocked for at most one vertex $v \in V(Q_n)$.*

Proof. First, assume that u has exactly one fault-free neighbor v. Thus, u is surrounded by $F \cup \{v\}$ and not surrounded by $F \cup \{w\}$ for any other vertex w. By Proposition 3.1, no other vertex than u is surrounded by $F \cup \{v\}$. It follows that no vertex is surrounded by $F \cup \{u\}$, so v is the only vertex such that (F, u, v) is blocked.

Now assume that u has at least 2 fault-free neighbors. Thus, u is not surrounded by $F \cup \{w\}$ for any vertex w. By Proposition 3.1, at most one vertex v is surrounded by $F \cup \{u\}$. Therefore, (F, u, v) is blocked for at most one vertex v. □

Next, we show that at most one triple (F, u, v) is blocked when $|F| \leq 2n - 5$ and uv is required to be a fault-free edge.

Proposition 3.3. *Let F be a set of at most $2n-5$ faulty vertices in Q_n where $n \geq 3$. Then, (F, u, v) is blocked for at most one fault-free edge $uv \in E(Q_n)$.*

Proof. Suppose on the contrary that triples (F, u, v) and (F, u', v') are blocked for two fault-free edges $uv, u'v' \in E(Q_n)$. Assume that u is surrounded by $F \cup \{v\}$, and u' is surrounded by $F \cup \{v'\}$. Observe that $u \neq u'$ since v and v' are fault-free. But then, both u and u' are surrounded by $F \cup \{v, v'\}$, which contradicts Proposition 3.1. □

The following proposition is useful in situations when we have a long fault-free path P in Q_L and we need to find an edge $a_L b_L$ on P such that there is a long fault-free path between a and b in Q_R.

Proposition 3.4. *Let F be a set of at most $2n - 4$ faulty vertices in Q_n where $n \geq 2$. For every path P in Q_n, if P contains at least three fault-free edges uv such that (F, u, v) is blocked, then it contains a fault-free edge ab such that (F, a, b) is free.*

Proof. Let uv be a fault-free edge of P such that (F, u, v) is blocked, and both u and v are inner vertices of P. Such edge exists since only two edges of P can contain an endvertex. Assume that u is surrounded by $F \cup \{v\}$, and let w be the other neighbor of v on P. Furthermore, assume that u' is surrounded by $F \cup \{v'\}$ for some other fault-free edge $u'v'$ of P. We show that the edge vw of P is fault-free and (F, v, w) is free.

Since both u and u' have exactly $n-1$ faulty neighbors and $|F| \leq 2n-4$, they must have two faulty neighbors in common. Thus $d(u, u') = 2$ and all faulty vertices together with v (and v') belong to the same bipartite class of Q_n. Hence w is fault-free and moreover, v is not surrounded by $F \cup \{w\}$. Since u is surrounded by $F \cup \{v\}$, it follows from Proposition 3.1 that w is not surrounded by $F \cup \{v\}$. Therefore, (F, v, w) is free for a fault-free edge vw of P. □

In order to apply induction, we need to split the hypercube Q_n with up to $2n-4$ faulty vertices into two $(n-1)$-dimensional subcubes Q_L and Q_R so that both Q_L and Q_R contain at most $2n-6$ faulty vertices. This is obtained by fixing some coordinate $i \in [n]$ where $[n] = \{1, \dots, n\}$. Formally, we define the subcube Q_L^i as the subgraph of Q_n induced by vertices that have 0 on the i-th coordinate. Similarly, the subcube Q_R^i is the subgraph of Q_n induced by vertices that have 1 on the i-th coordinate. The index i in Q_L^i and Q_R^i is omitted when it is clear or irrelevant. For $x \in V(Q_L)$, let x_R be the (only) neighbor of x in Q_R. Similarly for $x \in V(Q_R)$, let x_L be the (only) neighbor of x in Q_L.

Proposition 3.5. *Let F be a set of at most $2n-4$ vertices in Q_n where $n \geq 5$. Then Q_n can be split into Q_L and Q_R such that both subcubes contain at most $2n-6$ faulty vertices, unless $n = 5$, $|F| = 6$, and F consists of some vertex $w \in V(Q_n)$ and all his neighbors.*

Proof. If $|F| \leq 1$, we may split Q_n arbitrarily. If $2 \leq |F| \leq 2n-5$, we choose two faulty vertices and split Q_n so that they are in different subcubes. Clearly, in both these cases both Q_L and Q_R contain at most $2n-6$ faulty vertices. Now we assume that $|F| = 2n-4$.

Let A be the binary $|F| \times n$ matrix with faulty vertices in its rows. Assume that Q_n cannot be split into Q_L and Q_R such that both subcubes contain at most $2n-6$ faulty vertices. That is, each column of A contains at most one 1, or at most one 0. Without loss of generality we may assume that each column contains at most one 1. Thus A contains at most n 1's. Hence A has at most $n+1$ rows as all rows are different. Since $n+1 < 2n-4$ for $n \geq 6$, it follows that $n = 5$ and F consists of the vertex $(0, 0, \dots, 0)$ and all his neighbors. $\square$

Let us recall that a path between u and v is long if it has length at least $2^n - 2|F| - 2$. We represent paths by sequences of vertices, i.e. $(u_1, u_2, \dots, u_k)$ is a path P between u_1 and u_k of length $|E(P)| = k-1$ if all vertices $u_1, \dots, u_k$ are distinct and $u_i u_{i+1}$ is an edge for every $i \in [k-1]$. This allows us to define concatenation of paths as concatenation of their sequences. For example, if P_1 is a path between u_1 and v_1 and P_2 is a path between u_2 and v_2 such that P_1 and P_2 are vertex-disjoint and $v_1 u_2$ is an edge, then (P_1, P_2) is a path between u_1 and v_2 of length $|E(P_1)| + |E(P_2)| + 1$.

4 Long fault-free paths - small dimension

In this section we present the base of induction for Theorem 2.1. The case $n = 2$ is obvious since $|F| \leq 2n-4 = 0$. For $n = 3$ we even prove a stronger statement with one additional faulty vertex than in Theorem 2.1. Namely, for $|F| \le 2n-3 = 3$ and every two fault-vertices u and v there exists an (F,u,v)-path if (F,u,v) is free. Note that the opposite implication does not hold since the edge uv itself (if it exists) is an (F,u,v)-path when $|F|=3$.

Lemma 4.1. *Let F be a set of at most 3 vertices of Q_3, and let u and v be two fault-free vertices. If (F,u,v) is free, then there exists an (F,u,v)-path.*

Proof. *Case 1: $|F|=3$.*

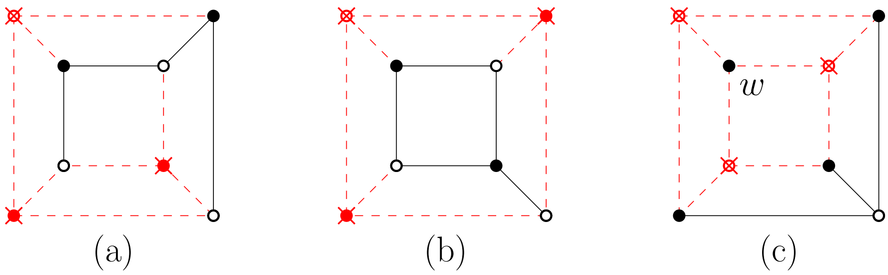

Figure 3: All configurations (up to isomorphism) of 3 faulty vertices in Q_3.

We want to find a path of length at least $2^3 - 3\cdot 2 - 2 = 0$, so it suffices to show that u and v belong to the same component of $Q_3 \setminus F$ if (F,u,v) is free. There are tree configurations (up to isomorphism) of F with $|F|=3$; see Figure 3. Observe that $Q_3 \setminus F$ on Figure 3(a,b) is connected. Also $Q_3 \setminus (F \cup \{w\})$ on Figure 3(c) is connected and w is surrounded by F. Hence the statement holds.

Case 2: $|F|=2$.

The graph $Q_3 \setminus F$ is connected because Q_3 is 3-connected, so there exists a path P between u and v in $Q_3 \setminus F$. We want to find a fault-free path between u and v of length at least $2^3 - 2\cdot 2 - 2 = 2$. If $d(u,v) \ge 2$, then P has this length.

Now assume that $d(u,v)=1$. There exist two disjoint edges x_iy_i such that ux_i and y_iv are edges of Q_3 for $i \in \{1,2\}$. If $x_i, y_i \notin F$ for some $i \in \{1,2\}$, then (u,x_i,y_i,v) is a requested path. If $x_1,x_2 \in F$ or $y_1,y_2 \in F$, then (F,u,v) is blocked. It remains to find an (F,u,v)-path for the case where $F=\{x_1,y_2\}$ (or isomorphically $F=\{x_2,y_1\}$). See Figure 4 for such path.

Case 3: $|F| \le 1$.

This case follows from the previous result by Fu [6] for at most $n-2$ faulty vertices. ☐

Assume that Q_n is split into Q_L and Q_R. The sets of faulty vertices in Q_L and Q_R are denoted by F_L and F_R, respectively.

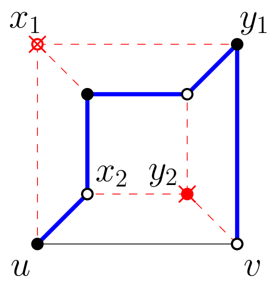

Figure 4: The (F, u, v)-path in Case 2 of Lemma 4.1.

In Q_4 we are often in a situation when Q_4 is split into Q_L and Q_R so that $u \in V(Q_L)$ and $v \in v(Q_R)$. We would like to find a vertex x in Q_L such that there exist an (F_L, u, x)-path P_L and an (F_R, x_R, v)-path P_R and their concatenation $P = (P_L, P_R)$ is an (F, u, v)-path. Now, we present sufficient conditions on the vertex x to apply such construction.

Lemma 4.2. *Let Q_4 be split into Q_L and Q_R so that $u \in V(Q_L)$, $v \in V(Q_R)$, $|F_L| \leq 3$, $|F_R| \leq 3$ and there exists a fault-free vertex x in Q_L such that $x_R \notin F_R$, (F_R, v, x_R) is free in Q_R and at least one of the following conditions holds.*

(a) (F_L, u, x) is free in Q_L, and $d(u, x)$ or $d(v, x_R)$ is odd.

(b) There exists a fault-free path P_L between u and x in Q_L of length at least $2^3 - 2|F_L| - 1$.

(c) $d(u, v)$ is even, $|F_L| = 3$, and $x = u$.

Then there exists an (F, u, v)-path in Q_4.

Proof. There exists an (F_R, x_R, v)-path P_R in Q_R by Lemma 4.1. In the first case, there exists an (F_L, u, x)-path P_L by Lemma 4.1. In the third case, let P_L be the trivial path between u and x. We show that the path $P = (P_L, P_R)$ has sufficient length in all three cases.

(a) Without lost of generality we assume that $d(u, x)$ is odd. Then the length of P is $|E(P)| = |E(P_L)| + 1 + |E(P_R)| \geq 2^3 - 2|F_L| - 1 + 1 + 2^3 - 2|F_R| - 2 = 2^4 - 2|F| - 2$.

(b) $|E(P)| = |E(P_L)| + 1 + |E(P_R)| \geq 2^3 - 2|F_L| - 1 + 1 + 2^3 - 2|F_R| - 2 = 2^4 - 2|F| - 2$.

(c) Since $d(x_R, v)$ is odd we have $|E(P)| \geq 1 + 2^3 - 2|F_R| - 1 \geq 2^4 - 2|F| - 2$.

□

Note that if $d(u, v)$ is even, then one of $d(u, x)$ and $d(v, x_R)$ is odd for every vertex x in Q_L. Let $N(u)$, $N_L(u)$ and $N_R(u)$ be the sets of neighbors of u in Q_n, Q_L and Q_R, respectively. We conclude this section with the following lemma that serves as the basis for induction in the proof of Theorem 2.1 for $n = 4$.

Lemma 4.3. *Let F be a set of at most 4 faulty vertices in Q_4. For every two fault-free vertices u and v, there is an (F, u, v)-path if and only if (F, u, v) is free and (2) does not hold.*

Proof. The necessity was discussed in Section 2.

Case 1: We can split Q_4 so that $|F_L| = 4$ or $|F_R| = 4$.

Assume that $|F_L| = 4$. Let $u' = u$ if $u \in V(Q_R)$, otherwise $u' = u_R$. Similarly, let $v' = v$ if $v \in V(Q_R)$, otherwise $v' = v_R$. Clearly, there is an (F_R, u', v')-path in Q_R which is a long path in Q_4. We prolong this path by the edge uu_R if $u \in V(Q_L)$ and vv_R if $v \in V(Q_L)$ and we obtain an (F, u, v)-path in Q_4.

For the rest of the proof, we assume that $|F_L| \le 3$ and $|F_R| \le 3$ for every splitting of Q_4 into Q_L and Q_R, which is one of the conditions of Lemma 4.2. Furthermore, we assume that $u \in V(Q_L)$ for every splitting of Q_4, otherwise we exchange the roles of Q_L and Q_R. We distinguish the following cases.

Case 2: We can split Q_4 so that $v \in V(Q_R)$, $|F_L| = 3$ or $|F_R| = 3$, and moreover, if $d(u, v)$ is odd, then u is not surrounded by F_L in Q_L and v is not surrounded by F_R in Q_R.

Without lost of generality we assume that $|F_L| = 3$. Since $|F_R| \le 1$, (F_R, z, v) is free in Q_R for every vertex z in Q_R. If u is surrounded by F_L in Q_L, then $d(u, v)$ is even and $u_R \notin F_R$. This configuration satisfies conditions of Lemma 4.2(c) for $x = u$. So we assume that u is not surrounded by F_L in Q_L.

Observe on Figure 3 that there are at least 3 vertices different from u in the component of $Q_L \setminus F_L$ containing u. Since $|F_R \cup \{v\}| \le 2$, there is a vertex $x \in V(Q_L)$ satisfying the requirements of Lemma 4.2(b).

Case 3: We can split Q_4 so that $u, v \in V(Q_L)$, $|F_L| = 0$ and $|F_R| \le 3$.

Observe that for every edge ab in Q_L such that $\{a, b\} \ne \{u, v\}$ there exists an (F_L, u, v)-path containing ab. Assume that $|F_R| = 3$. There exists fault-free edge ab in Q_R such that $\{a, b\} \ne \{u_R, v_R\}$ because Q_3 has 12 edges and one faulty vertex makes only 3 edges faulty. Let P_L be an (F_L, u, v)-path in Q_L containing the edge $a_L b_L$. We obtain an (F, u, v)-path from P_L by replacing the edge $a_L b_L$ with the path (a_L, a, b, b_L).

Now assume that $|F_R| \le 2$. There exist at least 5 fault-free edges in Q_R different from $u_R v_R$ because Q_3 has 12 edges and one faulty vertex makes only

3 edges faulty. If (F_R, x, y) is blocked in Q_R for some fault-free edge xy in Q_R, then there are 2 faulty vertices in Q_R in distance 2 and there is only another one fault-free edge $x'y'$ such that (F_R, x', y') is blocked in Q_R. Hence, there exists a fault-free edge ab in Q_R different from $u_R v_R$ such that (F_R, a, b) is free in Q_R. Let P_R be an (F_R, a, b)-path in Q_R and P_L be an (F_L, u, v)-path in Q_L containing $a_L b_L$. Let P be obtained from P_L by replacing the edge $a_L b_L$ with the path P_R. Since the length of P is $|E(P_L)|-1+2+|E(P_R)| \geq 2^4-2|F|-1$, it follows that P is an (F, u, v)-path.

Case 4: $d(u, v)$ is even.

We split Q_4 so that $u \in V(Q_L)$ and $v \in V(Q_R)$. If there exists splitting such that moreover $u_R \in F$ or $v_L \in F$, then we apply it. If $|F_R| = 3$ or $|F_L| = 3$, then this configuration satisfies the requirements of Case 2. So, we assume that $|F_R| \leq 2$ and $|F_L| \leq 2$.

By Proposition 3.2, there exists at most one vertex l in Q_L such that (F_L, l, u) is blocked in Q_L and at most one vertex r of Q_R such that (F_R, r, v) is blocked in Q_R. If there exists a vertex $x \in V(Q_L)$ such that $x, x_R \notin F \cup \{u, v, l, r\}$, then there exists an (F, u, v)-path by Lemma 4.2(a). When there is no such vertex x?

Note that $|F \cup \{u, v, r, l\}| \leq 8$ and Q_L has 8 vertices. There is no requested vertex x if and only if

$$\text{for every vertex } y \text{ of } Q_L \text{ exactly one of } y \text{ and } y_R \text{ belongs to } F \cup \{u, v, l, r\}. \tag{3}$$

Our aim is to show that we have the exceptional configuration (2) if (3) holds. So we assume for the rest of this case that (3) holds. Hence $|F_L| = |F_R| = 2$ and vertices l and r exist.

We know that u is surrounded by $F_L \cup \{l\}$ in Q_L or l is surrounded by $F_L \cup \{u\}$ in Q_L. Now, we show that u is not surrounded by $F_L \cup \{l\}$ in Q_L. Suppose on the contrary that u is surrounded by $F_L \cup \{l\}$ in Q_L. If $d(u, v) = 2$, then $v_L \in N_L(u) = F_L \cup \{l\}$ which contradicts (3). Now, $d(u, v) = 4$. Let f be some faulty neighbor of u. It follows from (3) that $u_R \notin F$ and $v_L \notin F$ which contradicts our requirements on splitting because it is possible to split Q_4 by the dimension in which f and u differ. Similarly, r is not surrounded by $F_R \cup \{v\}$.

Since l is surrounded by $F_L \cup \{u\}$, vertices of $F_L \cup \{u\}$ belong to the same bipartite class A of Q_4 and l belongs to the other bipartite class B of Q_4. Let a be the only vertex of Q_L in A that does not belong to $F_L \cup \{u\}$. Similarly, the three vertices of $F_R \cup \{v\}$ belong to the same bipartite class and let b be the fourth vertex of that bipartite class in Q_R. Since u and v are in the same bipartite class A, the vertices of $F \cup \{u, v, a, b\}$ form the bipartite class A. It follows from (3) that $a_R = r$ and $b_L = l$. See Figure 5 for an illustration.

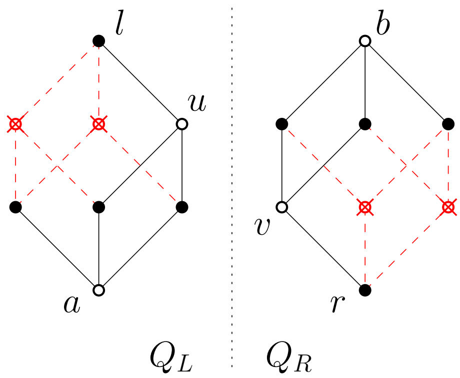

Figure 5: Case 4 in Lemma 4.3: the exceptional configuration (2).

We have $d(a,b) \geq 3$ because $a \in V(Q_L)$, $b \in V(Q_R)$, $a_R = r$, $N_R(r) = F_R \cup \{v\}$ and $b \notin F_R \cup \{v\}$. Since a and b belong to the same bipartite class, it follows that $d(a,b) = 4$. Hence, we conclude that if (3) holds, then we have the exceptional configuration (2).

Case 5: $d(u,v)$ is odd.

First, we show that we can split Q_4 so that $u \in V(Q_L)$, $v \in V(Q_R)$, u is not surrounded by F_L in Q_L, v is not surrounded by F_R in Q_R and $u_R \in F_R \cup \{v\}$.

If $d(u,v) = 1$ then we split Q_4 by the dimension in which u and v differs. Then, $u_R = v$ and the vertex u is not surrounded by F_L in Q_L and v is not surrounded by F_R in Q_R, otherwise (F,u,v) would be blocked.

Now, we assume that $d(u,v) = 3$. Let Q_A be the smallest subcube of Q_4 containing u and v. Since $d(u,v) = 3$, the dimension of Q_A is 3 and let Q_B be the complementary subcube. If there is no faulty vertex in Q_A, then we have the configuration of Case 3. If there exists a faulty vertex f in Q_A, then f is a neighbor of u or v, say u, so we split Q_4 by the dimension in which f and u differs so $u \in V(Q_L)$ and $v \in V(Q_R)$. Furthermore, u is not surrounded by F_L in Q_L, because (F,u,v) is free and $u_R = f$. If v is surrounded by F_R in Q_R, then $u_R = f$ is in $N_R(v) = F_R$ as $|F_R| \leq 3$ which contradicts the assumption that $d(u,v) = 3$.

Now, Q_4 is split so that $u \in V(Q_L)$, $v \in V(Q_R)$, u is not surrounded by F_L in Q_L, v is not surrounded by F_R in Q_R and $u_R \in F_R \cup \{v\}$. If $|F_R| = 3$ or $|F_L| = 3$, then we have Case 2. So we assume that $|F_R| \leq 2$ and $|F_L| \leq 2$.

First, we assume that u has only one fault-free neighbor u' in Q_L. The triple (F,u',v) is free and all neighbors of u are in $F \cup \{u',v\}$. Observe on Figure 2 that in the exceptional configuration (2) there is no vertex surrounded by faulty vertices and end-vertices. Hence, the triple (F,u',v) does not form the exceptional configuration (2). There exists an (F,u',v)-path by Case 4 which we prolong by the edge uu' to obtain an (F,u,v)-path.

Next, we assume that v has only one fault-free neighbor in Q_R. Observe that $d(u, v) = 1$, otherwise $u_R \notin F_R \cup \{v\}$. Thus, $v_L = u$ and by exchanging the roles of Q_L and Q_R and the roles of u and v, we may proceed as in the previous paragraph. Now, both u and v have at least two fault-free neighbors in their subcubes.

Note that there is at most one faulty vertex in $N_L(u)$ and at most one faulty vertex in $N_R(u_R)$ because $u_R \in F \cup \{v\}$. By Proposition 3.2, there exists at most one vertex l in Q_L such that (F_L, u, l) is blocked in Q_L. If a vertex l exists, then there is no faulty vertex in $N_L(u)$. Hence, there is at most one vertex x in $N_L(u)$ such that $x \in F$ or (F_L, u, x) is blocked. Similarly, there is at most one vertex x in $N_L(u)$ such that $x_R \in F$ or (F_R, v, x_R) is blocked. Therefore, there exists a vertex x in $N_L(u)$ satisfying the condition of Lemma 4.2(a). □

5 Long fault-free paths - general dimension

In this section we present the proof of our main result on long fault-free paths.

Theorem 2.1. *Let F be a set of at most $2n - 4$ faulty vertices of Q_n where $n \geq 2$. For every two fault-free vertices u and v, there exists an (F, u, v)-path in Q_n if and only if (F, u, v) is free and we do not have the exceptional configuration (2).*

Proof. The necessity was discussed in Section 2. We proceed by induction on n. The statement holds for $n \leq 4$ by the previous section. Now we assume that $n \geq 5$ and we have two fault-free vertices u and v in Q_n such that (F, u, v) is free.

First, we consider the case when u or v has exactly one neighbor uncovered by $F \cup \{u, v\}$. Assume that u has the only neighbor u' uncovered by $F \cup \{v\}$. Clearly, the vertex v is not surrounded by $F \cup \{u'\}$. Let v' be the vertex v if v has at least two neighbors uncovered by $F \cup \{u'\}$, otherwise let v' be the only neighbor of v uncovered by $F \cup \{u'\}$. Since $|F| \leq 2n - 4$, the vertex u' has at least two neighbors uncovered by $F \cup \{v\}$. Moreover, if u' has exactly two such neighbors, then all faulty vertices and the vertex v are neighbors of u or u', so v has at most 3 vertices covered by $F \cup \{u'\}$, and thus $v' = v$. Hence, u' and v' have at least two neighbors uncovered by $F \cup \{u', v'\}$. Furthermore, every (F, u', v')-path avoids u (and v if $v' \neq v$), so it can be prolonged to an (F, u, v)-path. Therefore, in the following we assume that both u and v have at least two neighbors uncovered by $F \cup \{u, v\}$.

Our aim is to split Q_n into Q_L and Q_R such that $|F_L| \leq 2n-6$ and $|F_R| \leq 2n-6$ where $F_L = F \cap V(Q_L)$ and $F_R = F \cap V(Q_R)$. By Proposition 3.5, this can be done with the only exception when $n = 5$, $|F| = 6$, and F consists of some vertex w and all his neighbors. But when this exception happens, we may remove the vertex w from F since it cannot be visited by any path that is fault-free with respect to $F \setminus \{w\}$, so we may assume that the requested split exists.

In what follows, note that whenever we apply induction for a free triple (F', a, b) in Q_L or in Q_R, the configuration (2) cannot occur since $d(a, b)$ is odd or $|F'| < 2n - 6$. We assume that $u \in V(Q_L)$ and we distinguish the following cases.

Case 1: $v \in V(Q_R)$.

We may assume that $|F_L| \geq |F_R|$. Thus $|F_R| \leq n - 2$. See Figure 6 for an illustration.

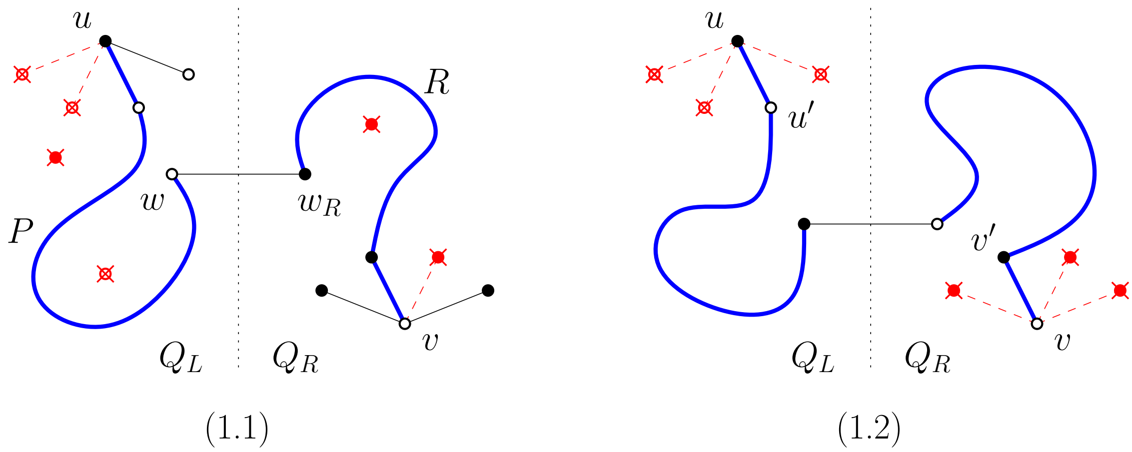

Figure 6: The construction of an (F, u, v)-path in Case 1 of Theorem 2.1.

Subcase 1.1: Both vertices u and v have at least 2 fault-free neighbors in their subcubes.

It follows for every $w \in V(Q_L)$ that if (F_L, u, w) is blocked in Q_L, then w is surrounded by $F_L \cup \{u\}$ in Q_L. Similarly for every $w_R \in V(Q_R)$, if (F_R, v, w_R) is blocked in Q_R, then w_R is surrounded by $F_R \cup \{v\}$ in Q_R.

We claim that there is a vertex $w \in V(Q_L)$ such that $d(u, w)$ is odd, $w_R \neq v$, both w and w_R are fault-free, (F_L, u, w) is free in Q_L, and (F_R, v, w_R) is free in Q_R. Let $A = \{w \in V(Q_L) \mid d(u, w) \text{ is odd}\}$. We say that a vertex $x \in V(Q_n)$ *eliminates* a vertex $w \in A$ if $w = x$, or $w_R = x$, or w is surrounded by $F_L \cup \{u\}$ and x is a neighbor of w, or w_R is surrounded by $F_R \cup \{v\}$ and x is a neighbor of w_R. Thus, every vertex $w \in A$ that is not eliminated by any vertex from $F \cup \{v\}$ satisfies the claim. By Proposition 3.1, at most one vertex in A is surrounded by $F_L \cup \{u\}$ in Q_L, and at most one vertex $w \in A$

has the neighbor w_R surrounded by $F_R \cup \{v\}$ in Q_R. Hence, every vertex from $F \cup \{v\}$ eliminates at most one vertex from A. Therefore the claim holds as

$$|A| - |F| - 1 \geq 2^{n-2} - 2n + 3 \geq 1 \text{ for } n \geq 5.$$

Let $w \in V(Q_L)$ be a vertex satisfying the claim above. By induction, there is an (F_L, u, w)-path P in Q_L of length at least $2^{n-1} - 2|F_L| - 1$, and an (F_R, w_R, v)-path R in Q_R. Therefore, by adding the edge ww_R we obtain an (F, u, v)-path (P, R) of length at least $2^{n-1} - 2|F_L| - 1 + 2^{n-1} - 2|F_R| - 2 + 1 = 2^n - 2|F| - 2$.

Subcase 1.2: Vertex u or v has only 1 fault-free neighbor in its subcube.

Assume that u has the only fault-free neighbor u' in Q_L. Let v' be the vertex v if v has at least two fault-free neighbors in Q_R, otherwise let v' be the only fault-free neighbor of v in Q_R. Clearly, both u' and v' have at least two fault-free neighbors in their subcubes. By the previous case, there is an (F, u', v')-path P. Then, (u, P) if $v' = v$, or (u, P, v) if $v' \neq v$, is an (F, u, v)-path.

Case 2: $v \in V(Q_L)$.

Since both u and v have at least two neighbors uncovered by $F \cup \{u, v\}$, it follows that (F_L, u, v) is free in Q_L. See Figure 7 for an illustration.

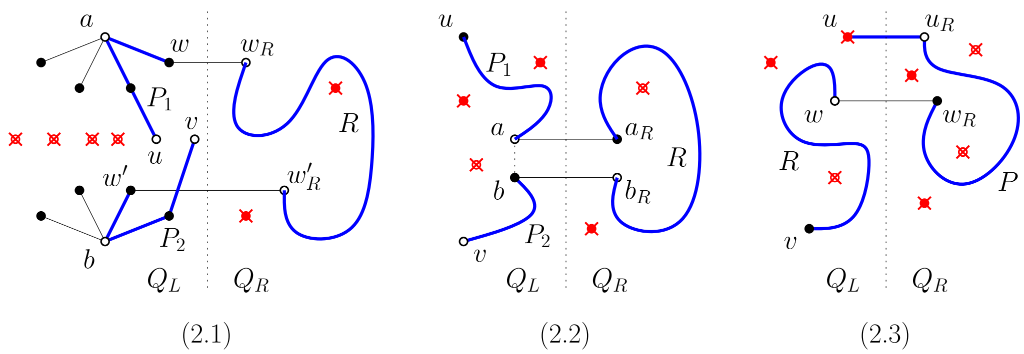

Figure 7: The construction of an (F, u, v)-path in Case 2 of Theorem 2.1.

Subcase 2.1: We have the exceptional configuration (2) in Q_L.

Assume that $a, b \in V(Q_L)$ are the vertices in the exceptional configuration (2). Let w and w' be some neighbors of a and b, respectively, such that w_R and w'_R are fault-free. Since $|F_R| \leq 2$, the triple (F_R, w_R, w'_R) is free in Q_R. Thus, by induction, there is (F_R, w_R, w'_R) path R in Q_R. Furthermore, there are disjoint fault-free paths P_1 between u and w, and P_2 between w' and v, both of length 3. Therefore, by adding the edges ww_R and $w'_R w'$ we obtain an (F, u, v)-path (P_1, R, P_2) of length at least $2^{n-1} - 2|F_R| - 2 + 2 \cdot 3 + 2 = 2^n - 2|F| - 2$.

Subcase 2.2: We do not have the exceptional configuration (2) *in* Q_L. *Moreover, at least one of* u_R *and* v_R *is faulty, or* $|F_R| \leq 2n-7$*, or* $d(u,v)$ *is odd.*

Applying induction we obtain an (F_L, u, v)-path P in Q_L. We claim that there is an edge ab on P so that the edge $a_R b_R \in E(Q_R)$ is fault-free and also (F_R, a_R, b_R) is free. At most $2|F_R|$ edges $a_R b_R \in E(Q_R)$ with ab on P are faulty. However, if at least one of u_R and v_R is faulty, it is less than $2|F_R|$ edges. Furthermore, by Proposition 3.4, we may assume that (F_R, a_R, b_R) is blocked for at most 2 fault-free edges $a_R b_R \in E(Q_R)$ with ab on P, otherwise we are done. However, if $|F_R| \leq 2n-7$, then by Proposition 3.3, (F_R, a_R, b_R) is blocked only for at most 1 fault-free edge $a_R b_R \in E(Q_R)$ with $ab \in E(P)$. Thus, some edge ab on P satisfying the claim exists as

$$\left.\begin{array}{l} E(P) - 2|F_R| - 1 \text{ for } d(u,v) \text{ even} \\ E(P) - 2|F_R| - 2 \text{ for } d(u,v) \text{ odd} \end{array}\right\} \geq 2^{n-1} - 2|F| - 3 \geq 2^{n-1} - 4n + 5 \geq 1 \text{ for } n \geq 5.$$

Hence by induction, there is an (F_R, a_R, b_R)-path R in Q_R of length at least $2^{n-1} - 2|F_R| - 1$. Therefore, by removing the edge ab and adding the edges aa_L, and $b_L b$ we obtain an (F, u, v)-path (P_1, R, P_2) of length at least $2^{n-1} - 2|F_L| - 2 + 2^{n-1} - 2|F_R| - 1 - 1 + 2 = 2^n - 2|F| - 2$ where P_1 and P_2 are the subpaths of $P \setminus \{ab\}$.

Subcase 2.3: Both u_R *and* v_R *are fault-free,* $|F_R| = 2n-6$*, and* $d(u,v)$ *is even.*

By Proposition 3.1, at most one of u_R and v_R is surrounded by F_R in Q_R. Assume that u_R is not surrounded by F_R in Q_R. We put $F'_L = F_L \cup \{u\}$, so $|F'_L| \leq 3 < 2n-6$. Note that v has at least two neighbors in Q_L that are not in F'_L since $d(u,v)$ is even. It follows for every $w \in V(Q_L)$ that if (F'_L, v, w) is blocked, then w is surrounded by $F'_L \cup \{v\}$.

We claim that there is a vertex $w \in V(Q_L)$ such that $d(v,w)$ is odd, both w and w_R are fault-free, (F'_L, v, w) is free in Q_L, and (F_R, u_R, w_R) is free in Q_R. Let $A = \{w \in V(Q_L) \mid d(u,w) \text{ is odd}\}$. By Proposition 3.2, (F_R, u_R, w'_R) is blocked for at most one vertex $w' \in A$. If that happens for some $w' \in A$, let $A' = A \setminus \{w'\}$, otherwise let $A' = A$.

We say that a vertex $x \in V(Q_n)$ *eliminates* a vertex $w \in A'$ if $w = x$, or $w_R = x$, or w is surrounded by $F'_L \cup \{v\}$ and x is a neighbor of w. Thus, every vertex $w \in A'$ that is not eliminated by any vertex from F satisfies the claim. By Proposition 3.1, at most one vertex in A is surrounded by $F'_L \cup \{v\}$. Hence every vertex from F eliminates at most one vertex from A. Therefore the claim holds as

$$|A'| - |F| \geq 2^{n-2} - 2n - 3 \geq 1 \text{ for } n \geq 5.$$

Hence by induction, there is an (F_R, u_R, w_R)-path P in Q_R of length at least $2^{n-1} - 2|F_R| - 1$. Furthermore, there is an (F'_L, w, v)-path R in Q_L that avoids u and has length at least $2^{n-1} - 2(|F_L| + 1) - 1$. Therefore, by adding the edges uu_R and $w_R w$, we obtain an (F, u, v)-path (u, P, R) of length at least $2^{n-1} - 2|F_R| - 1 + 2^{n-1} - 2|F_L| - 1 - 2 + 2 = 2^n - 2|F| - 2$. □

6 Long fault-free cycles

Let $D \subseteq [n]$ be a set of $d = |D|$ coordinates of Q_n. We can consider every vertex x of Q_n as a pair $x = (u, v)_D$ where $u \in \{0,1\}^{n-d}$ and $v \in \{0,1\}^d$ are projections of x on the coordinates of $[n] \setminus D$ and D, respectively. For $u \in \{0,1\}^{n-d}$ we denote by $Q_D(u)$ the d-dimensional *subcube* of Q_n induced by vertices $V_D(u) = \{(u, v)_D \mid v \in \{0,1\}^d\}$. In other words, $Q_D(u)$ is the subcube of Q_n with coordinates $[n] \setminus D$ fixed by u. The index D in $(u, v)_D$ is omitted whenever clear from the context.

Let F be a set of faulty vertices of Q_n. Recall that a cycle in Q_n is *long* if it has length at least $2^n - 2|F|$. For a set $D \subseteq [n]$ and $u \in \{0,1\}^{n-d}$ we define $F_D(u) = F \cap V_D(u)$. Assume that we want to find a long fault-free cycle in Q_n.

Our approach is based on subcube partitioning similar as in the work of Bruck et al. [1] where the hypercube is partitioned into subcubes so that each subcube contains a large fault-free component. However, instead of using the same partitioning as in [1], we apply recent results by Wiener [15] on edge multiplicity of traces in set systems which gives better bounds. We proceed as follows.

First, we find a set $D \subseteq [n]$ such that $|F_D(u)| \leq 2d - 4$ for every $u \in \{0,1\}^{n-d}$ where $d = |D|$. Then, for some Hamiltonian cycle $(u^1, u^2, \dots, u^{2^{n-d}}, u^{2^{n-d}+1} = u^1)$ of Q_{n-d} we choose in each subcube $Q_D(u^i)$ two appropriate vertices a^i and b^i such that $a^i b^{i+1} \in E(Q_n)$ for every $i \in [2^{n-d}]$. Next, applying Theorem 2.1 we find long fault-free paths between a^i and b^i in each subcube $Q_D(u^i)$. Finally, we glue these paths together and obtain a desired long fault-free cycle in Q_n. See Figure 8 for an illustration.

The crucial step is the determination of the set D. Although the following theorem by Wiener [15] was originally formulated for set systems, here we take the liberty to formulate it for vertices of the hypercube.

Theorem 6.1 (Wiener [15])**.** *Let F be a set of at least $2n$ vertices of Q_n, and let $d = \left\lceil \frac{n^2}{2|F|-n-2} \right\rceil$. Then, there exists a set $D \subseteq [n]$, $|D| = d$ such that $|F_D(u)| \leq d + 1$ for every $u \in \{0,1\}^{n-d}$.*

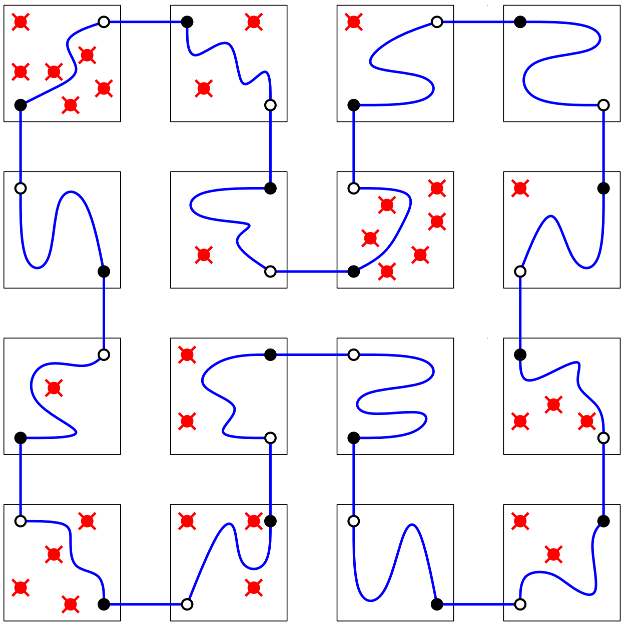

Figure 8: The construction of a long fault-free cycle in Theorem 2.2.

For the choice of vertices a^i and b^i we employ the following separate lemma. Recall that a triple (F, u, v) is blocked for $F \subseteq V(Q_n)$ and $u, v \in V(Q_n)$ if u is surrounded by $F \cup \{v\}$ or v is surrounded by $F \cup \{u\}$, otherwise (F, u, v) is free.

Lemma 6.2. *Let F be a set of faulty vertices of Q_n where $n \geq 5$, and let $D \subseteq [n]$ be such that $d = |D| = 5$ and $|F_D(u)| \leq 6$ for every $u \in \{0,1\}^{n-d}$. Let $(u^1, u^2, \dots, u^{2^{n-d}}, u^{2^{n-d}+1} = u^1)$ be a Hamiltonian cycle of Q_{n-d}. Then, there are fault-free vertices a^i and b^i in each $Q_D(u^i)$ such that*

- *$d(a^i, b^i)$ is odd,*
- *$(F_D(u^i), a^i, b^i)$ is free in $Q_D(u^i)$,*
- *$a^i b^{i+1} \in E(Q_n)$ where $b^{2^{n-d}+1} = b^1$,*

for every $i \in [2^{n-d}]$.

Proof. We determine vertices a^i and b^i in this order: a^1, b^2, $a^2, \dots, b^{2^{n-d}}$, $a^{2^{n-d}}$, $b^{2^{n-d}+1} = b^1$. Since u^i and u^{i+1} are neighbors in Q_{n-d}, every vertex in $Q_D(u^i)$ has one neighbor in $Q_D(u^{i+1})$. Let A and B be the bipartite classes of Q_n. We will choose $a^i = (u^i, v^i)$ from $A \cap V_D(u^i)$ and obtain $b^{i+1} = (u^{i+1}, v^i)$ from $B \cap V_D(u^{i+1})$. Thus $d(a^i, b^i)$ is odd and $a^i b^{i+1} \in E(Q_n)$.

There are 16 vertices in $A_i = A \cap V_D(u^i)$ since $Q_D(u^i)$ is isomorphic to Q_5. At most 6 of them are faulty since $|F_D(u^i)| \leq 6$. Furthermore, at most 6 of them have faulty neighbor in $Q_D(u^{i+1})$ since $|F_D(u^{i+1})| \leq 6$.

In each of the cases $i = 1$, $1 < i < 2^{n-d}$, and $i = 2^{n-d}$, we show that amongst the 4 remaining vertices of A_i, there are at most two vertices, denoted by x^i and y^i, that are not eligible for the choice of a^i.

Case $i = 1$. By Proposition 3.1, at most one vertex $x^1 \in A_1$ is surrounded by $F_D(u^1)$ in $Q_D(u^1)$. Furthermore, at most one vertex $y^1 \in A_1$ has the neighbor in $Q_D(u^2)$ surrounded by $F_D(u^2)$ in $Q_D(u^2)$.

Case $1 < i < 2^{n-d}$. By Proposition 3.2, $(F_D(u^i), x^i, b^i)$ is blocked in $Q_D(u^i)$ for at most one vertex $x^i \in A_i$. By Proposition 3.1, at most one vertex $y^i \in A_i$ has the neighbor in $Q_D(u^{i+1})$ surrounded by $F_D(u^{i+1})$ in $Q_D(u^{i+1})$.

Case $i = 2^{n-d}$. By Proposition 3.2, $(F_D(u^i), x^i, b^i)$ is blocked in $Q_D(u^i)$ for at most one vertex $x^i \in A_i$. Furthermore, at most one vertex $y^i \in A_i$ has the neighbor z in $Q_D(u^1)$ such that $(F_D(u^1), a^1, z)$ is blocked in $Q_D(u^1)$.

Hence, by choosing vertices a^i and b^i for every $i \in [2^{n-d}]$ such that

$$a^i = (u^i, v^i) \in A_i \setminus (\{x^i, y^i\} \cup F_D(u^i) \cup F_D^*(u^{i+1})) \text{ for some } v^i \in \{0,1\}^d,$$
$$b^{i+1} = (u^{i+1}, v^i) \text{ and } b^1 = b^{2^{n-d}+1},$$

where $F_D^*(u^{i+1})$ is the set of vertices of $Q_D(u^i)$ that have a faulty neighbor in $Q_D(u^{i+1})$, we obtain that both a^i and b^i are fault-free, and $(F_D(u^i), a^i, b^i)$ is free in $Q_D(u^i)$ for every $i \in [2^{n-d}]$. □

Now we are ready to prove Theorem 2.2.

Theorem 2.2. *Let F be a set of at most $\frac{n^2}{10} + \frac{n}{2} + 1$ faulty vertices of Q_n where $n \geq 15$. Then Q_n contains a long fault-free cycle.*

Proof. Let $F' \supseteq F$ be some set of exactly $\left\lfloor \frac{n^2}{10} + \frac{n}{2} + 1 \right\rfloor$ vertices of Q_n. Thus $|F'| \geq 2n$ as $n \geq 15$ and by Theorem 6.1, there is a set $D \subseteq [n]$ such that $d = |D| = 5$ and $|F_D(u)| \leq |F'_D(u)| \leq 6$ for every $u \in \{0,1\}^{n-d}$. Let $(u^1, u^2, \ldots, u^{2^{n-d}}, u^{2^{n-d}+1} = u^1)$ be some Hamiltonian cycle of Q_{n-d}.

By Lemma 6.2, there are fault-free vertices a^i and b^i in each $Q_D(u^i)$ such that $d(a^i, b^i)$ is odd, $(F_D(u^i), a^i, b^i)$ is free in $Q_D(u^i)$, and $a^i b^{i+1} \in E(Q_n)$ for every $i \in [2^{n-d}]$ where $b^{2^{n-d}+1} = b^1$.

Hence by Theorem 2.1, in each $Q_D(u^i)$ there is a fault-free path P_i between b^i and a^i of length at least $2^d - 2|F_D(u^i)| - 1$. Concatenating these paths with edges $a^i b^{i+1} \in E(Q_n)$ we obtain a fault-free cycle $(P_1, P_2, \ldots, P_{2^{n-d}}, b^1)$ of length at least

$$2^{n-d} \cdot 2^d - \sum_{i \in [2^{n-d}]} 2|F_D(u^i)| - 2^{n-d} + 2^{n-d} = 2^n - 2|F|.$$

□

Acknowledgements

The authors would like to thank Václav Koubek and Tomáš Dvořák for helpful discussion on this topic.

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