Line graphs of multigraphs and Hamilton-connectedness of claw-free graphs

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Abstract

We introduce a closure concept that turns a claw-free graph into the line graph of a multigraph while preserving its (non-)Hamilton-connectedness. As an application, we show that every 7-connected claw-free graph is Hamilton-connected, and we show that the well-known conjecture by Matthews and Sumner (every 4-connected claw-free graph is hamiltonian) is equivalent with the statement that every 4-connected claw-free graph is Hamilton-connected. Finally, we show a natural way to avoid the non-uniqueness of a preimage of a line graph of a multigraph, and we prove that the closure operation is, in a sense, best possible.

1 Notation and terminology

In this paper, by a *graph* we mean a finite simple undirected graph $G = (V(G), E(G))$; whenever we allow multiple edges we say that G is a *multigraph*.

For a vertex $x \in V(G)$, $d_G(x)$ denotes the *degree of* x *in* G, $N_G(x)$ denotes the *neighborhood of* x *in* G (i.e. $N_G(x) = \{y \in V(G) |\ xy \in E(G)\}$) and $N_G[x]$ denotes the *closed neighborhood of* x *in* G (i.e. $N_G[x] = N_G(x) \cup \{x\}$). For $x, y \in V(G)$, $\mathrm{dist}_G(x, y)$ denotes the *distance of* x, y *in* G. A *universal vertex*

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of G is a vertex that is adjacent to all other vertices of G. By a *clique* we mean a (not necessarily maximal) complete subgraph of G; $\alpha(G)$ denotes the *independence number* of G and $\kappa(G)$ denotes the (vertex) *connectivity* of G. By the *square* of a graph G we mean the graph G^2 with $V(G^2) = V(G)$ and $E(G^2) = \{xy \in V(G) | \operatorname{dist}_G(x, y) \leq 2\})$.

If G, H are (multi-)graphs, then $H \subset G$ or $H \overset{\text{IND}}{\subset} G$ means that H is a *subgraph* or an *induced subgraph* of G, respectively, and $H \simeq G$ stands for the *isomorphism* of H and G. The *induced subgraph* of G on a set $M \subset V(G)$ is denoted $\langle M \rangle_G$.

A path with endvertices a, b will be referred to as an (a, b)*-path*. If P is a path and $u \in V(P)$, then u^- and u^+ denotes the *predecessor* and *successor* of u on P. A path on k vertices is denoted P_k.

For a graph G and $a, b \in V(G)$, $p(G)$ denotes the length of a longest path in G, $p_a(G)$ the length of a longest path in G with one endvertex at $a \in V(G)$, and $p_{ab}(G)$ the length of a longest (a, b)-path in G. A graph G is *homogeneously traceable* if, for any $a \in V(G)$, G has a hamiltonian path with one endvertex at a (i.e., for any $a \in V(G)$, $p_a(G) = |V(G)|$), and G is *Hamilton-connected* if, for any $a, b \in V(G)$, G has a hamiltonian (a, b)-path (i.e., for any $a, b \in V(G)$, $p_{ab}(G) = |V(G)|$).

A *walk* (in G) is a sequence of vertices $u_1u_2 \dots u_k$ such that $u_iu_{i+1} \in E(G)$, $i = 1, \dots, k-1$. For a walk $J = u_1u_2 \dots u_k$ we denote $V(J) = \{u_1, u_2, \dots, u_k\}$ the corresponding set of vertices, and $|V(J)| = |\{u_1, u_2, \dots, u_k\}|$ (thus, $|V(J)| = k$ if and only if J is a path). Finally, G is *claw-free* if G does not contain an induced subgraph that is isomorphic to the *claw* $K_{1,3}$.

For further concepts and notations not defined here we refer the reader to [4].

2 Introduction

A vertex $x \in V(G)$ is *eligible* if $N_G(x)$ induces a connected noncomplete graph, and x is *simplicial* if the subgraph induced by $N_G(x)$ is complete. The *local completion* of G at a vertex x is the graph G_x^* obtained from G by adding all edges with both vertices in $N_G(x)$ (note that the local completion at x turns x into a simplicial vertex, and preserves the claw-free property of G).

The *closure* $\operatorname{cl}(G)$ of a claw-free graph G is the graph obtained from G by recursively performing the local completion operation at eligible vertices as long as this is possible. We say that G is *closed* if $G = \operatorname{cl}(G)$.

The following was proved in [12]

Theorem A [12]. *For every claw-free graph G:*

(*i*) cl(G) *is uniquely determined,*

(*ii*) cl(G) *is the line graph of a triangle-free graph,*

(*iii*) cl(G) *is hamiltonian if and only if* G *is hamiltonian.*

Note that the fact that cl(G) is a line graph can be seen e.g. also from the well-known Beineke's characterization of line graphs in terms of forbidden induced subgraphs.

Theorem B [1]. *A graph G is a line graph (of some graph) if and only if G does not contain a copy of any of the graphs in Figure 1 as an induced subgraph.*

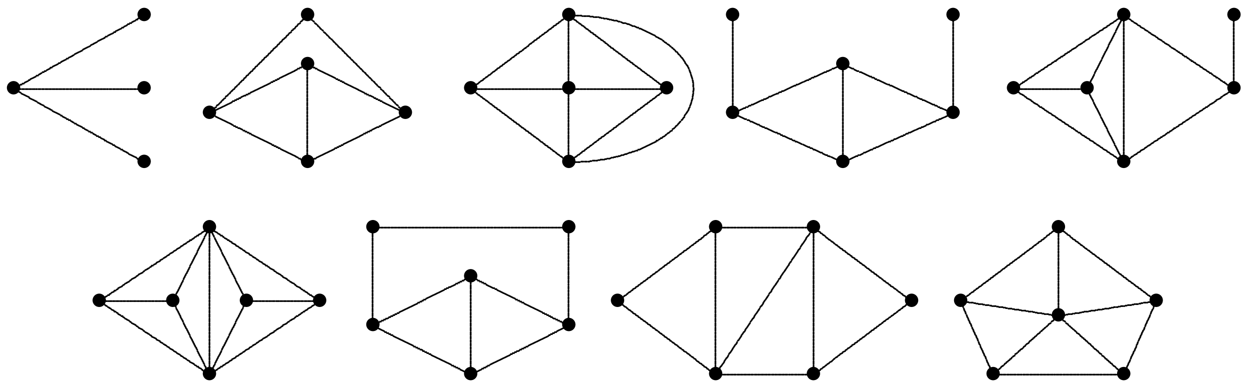

Figure 1

A class $\mathcal{C}$ is *stable* if $G \in \mathcal{C}$ implies $\mathrm{cl}(G) \in \mathcal{C}$. A graph property π is *stable* in a stable class $\mathcal{C}$ if, for any $G \in \mathcal{C}$, G has π if and only if cl(G) has π. Thus, Theorem A says that hamiltonicity is a stable property in the class of claw-free graphs.

Zhan [15] proved the following.

Theorem C [15]. *Every 7-connected line graph of a multigraph is Hamilton-connected.*

Using the fact that hamiltonicity is a stable property, combining Theorems A and C the following was obtained.

Theorem D [12]. *Every 7-connected claw-free graph is hamiltonian.*

However, the closure technique does not give a similar result for Hamilton-connectedness since the line graph of the graph H in Figure 2 shows that Hamilton-connectedness is not stable in 3-connected claw-free graphs (there

$G = L(H)$

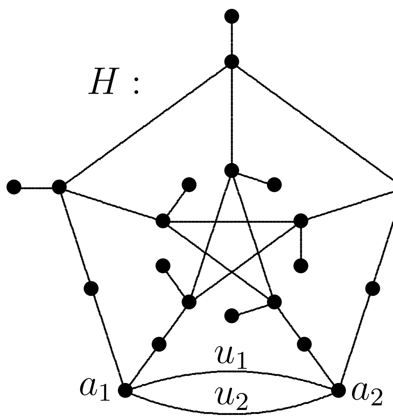

Figure 2

is no hamiltonian (u_1, u_2)-path in $L(H)$, where u_1, u_2 are the vertices of $L(H)$ that correspond to the edges u_1, u_2 in H).

The existence of a connectivity bound for Hamilton-connectedness in claw-free graphs was established by Brandt [5] who proved that every 9-connected claw-free graph is Hamilton-connected. This result was later on improved by Hu, Tian and Wei [8] as follows.

Theorem E [8]. *Every 8-connected claw-free graph is Hamilton-connected.*

In the same paper, Zhan's result (Theorem C) was improved as follows.

Theorem F [8]. *Let G be a 6-connected line graph of a multigraph with at most 29 vertices of degree 6. Then G is Hamilton-connected.*

On the other hand, the following conjectures by Matthews and Sumner (Conjecture G) and by Thomassen (Conjecture H) are still wide open.

Conjecture G [11]. *Every 4-connected claw-free graph is hamiltonian.*

Conjecture H [14]. *Every 4-connected line graph is hamiltonian.*

Note that Theorem A immediately implies that Conjectures G and H are equivalent. More equivalent versions of these conjectures (among others, on cycles in cubic graphs), can be found e.g. in [7].

Another equivalence was established by Kužel, Vrána and Xiong [10], who proved that Conjectures G and H are equivalent with the following statement.

Conjecture I [10]. *Every 4-connected line graph of a multigraph is Hamilton-connected.*

It is natural to pose the following question.

Conjecture J. *Every 4-connected claw-free graph is Hamilton-connected.*

For a similar reason as with the extension of Theorem D to Hamilton-connectedness, the closure technique as introduced in [12] does not establish the equivalence of Conjecture J with the previous ones.

In Section 4 we develop a closure concept for Hamilton-connectedness from which, as immediate applications, we obtain the following statements (see Theorems 15 and 17).

(i) *Every 6-connected claw-free graph with at most 29 vertices of degree 6 is Hamilton-connected.*

(ii) *Every 7-connected claw-free graph is Hamilton-connected.*

(iii) *Conjecture J is equivalent with Conjectures G, H and I.*

3 k-closure and structure of 2-closed graphs

The closure concept was extended in [3] as follows.

A vertex $x \in V(G)$ is *k-eligible* if its neighborhood induces a k-connected noncomplete graph, and the *k-closure* of G, denoted $\mathrm{cl}_k(G)$, is the graph obtained from G by recursively performing the local completion operation at k-eligible vertices as long as this is possible. A graph G is *k-closed* if $G = \mathrm{cl}_k(G)$.

A class $\mathcal{C}$ is *k-stable* if $G \in \mathcal{C}$ implies $\mathrm{cl}_k(G) \in \mathcal{C}$. A graph property π is *k-stable* in a k-stable class $\mathcal{C}$ if, for any $G \in \mathcal{C}$, G has π if and only if $\mathrm{cl}_k(G)$ has π.

Theorem K [3]. *For every claw-free graph G,*

(i) $\mathrm{cl}_k(G)$ *is uniquely determined,*

(ii) $\mathrm{cl}_2(G)$ *is homogeneously traceable if and only if G is homogeneously traceable,*

(iii) $\mathrm{cl}_3(G)$ *is Hamilton-connected if and only if G is Hamilton-connected.*

Thus, homogeneous traceability is 2-stable and hamilton-connectedness is 3-stable in the class of claw-free graphs.

The graph in Figure 3 has no hamiltonian (a, b)-path, the vertex x is 2-eligible, and there is a hamiltonian (a, b)-path in the local completion G^*_x of G at x. This shows that the property "having a hamiltonian (a, b)-path

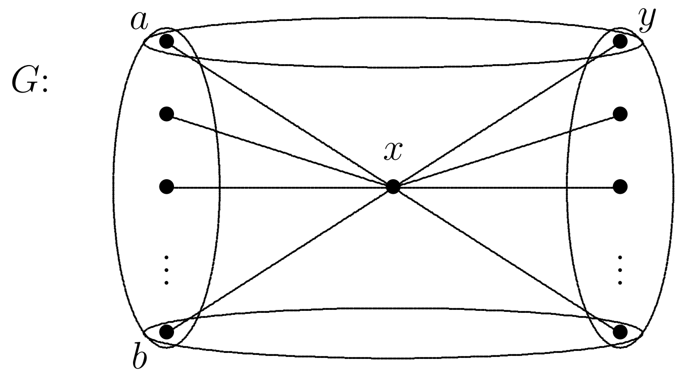

Figure 3

for given $a, b \in V(G)$" is not 2-stable. However, neither G nor its 2-closure are Hamilton-connected. This motivated the following conjecture.

Conjecture L [3]. *Hamilton-connectedness is 2-stable in the class of claw-free graphs.*

Note that in [9] the author claimed to give an infinite family of counterexamples to Conjecture L. However, this statement is not true, since it is not difficult to observe that the graphs constructed in [9] have similar behavior as the graphs in Figure 3 (i.e., they show that the property "having a hamiltonian (a, b)-path for given $a, b \in V(G)$" is not 2-stable, but do not disprove Conjecture L).

Affirmative answer to Conjecture L was given in [13].

Theorem M [13]. *Hamilton-connectedness is 2-stable in the class of claw-free graphs.*

A natural question is whether a 2-closure of a claw-free graph belongs to some "nice" class of graphs. It is easy to see that, in general, $\mathrm{cl}_2(G)$ is not a line graph, since e.g. the second or fourth graph in Figure 1 is an example of a 2-closed claw-free graph that is not a line graph. Thus, a next question is whether a 2-closure of a claw-free graph is a line graph of a multigraph.

Line graphs of multigraphs were characterized by Bermond and Meyer [2] (see also Zverovich [16]).

Theorem N [2]. *A graph G is a line graph* of a multigraph *if and only if G does not contain a copy of any of the graphs in Figure 4 as an induced subgraph.*

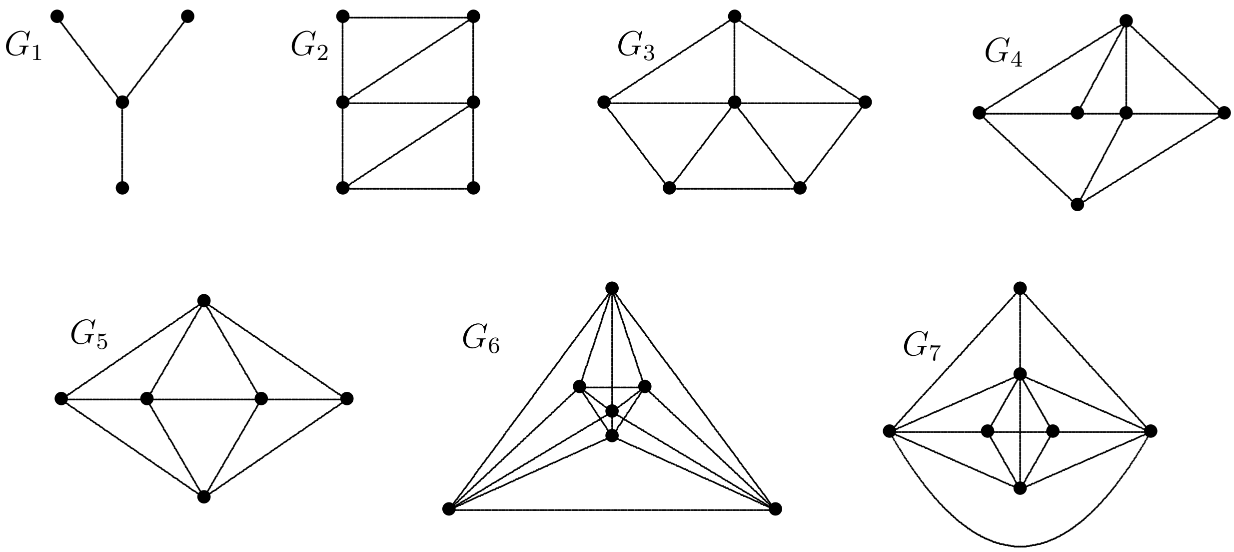

Figure 4

We see that, in general, $\mathrm{cl}_2(G)$ is not a line graph of a multigraph, since the graphs G_2 and G_4 of Figure 4 are 2-closed, i.e. they can be induced subgraphs in $\mathrm{cl}_2(G)$.

We now consider the structure of $\mathrm{cl}_2(G)$ in more detail. We include here only those results that are needed for introducing the closure concept in Section 4. Proofs and further necessary auxiliary results are postponed to Section 6.

Lemma 1. *Let G be a 2-closed claw-free graph, and let G_i, $i = 1, \dots, 7$ be the graphs from Figure 4. Then G is $\{G_1, G_3, G_5, G_6, G_7\}$-free.*

Thus, a 2-closed claw-free graph can contain only induced G_2 and/or G_4. In the rest of the paper we will keep the notation of these graphs as shown in Figure 5.

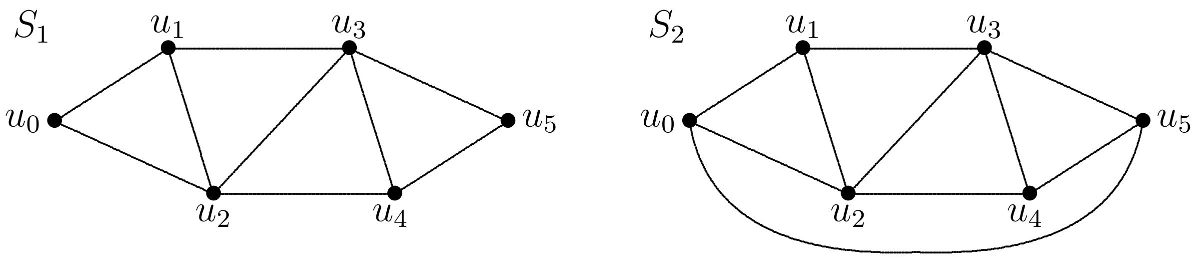

Figure 5

Let $J = u_0u_1 \dots u_{k+1}$ be a walk in G. We say that J is *good in G*, if $k \geq 4$, $J^2 \subset G$ and for any i, $0 \leq i \leq k-4$, $\langle\{u_i, u_{i+1}, \dots, u_{i+5}\}\rangle_G$ is isomorphic to S_1 or to S_2.

Similarly, a cycle $C \subset G$ is said to be *good in* G, if every set of six consecutive vertices of C induces in G the graph S_1 or S_2.

Lemma 2. *Let G be a 2-closed claw-free graph and $J = u_0u_1 \dots u_{k+1}$ a good walk in G, $k \geq 5$. Then $d_G(u_i) = 4$, $i = 3, \dots, k-2$.*

Thus, for $i = 3, \dots, k-2$, $\langle N_G(u_i) \rangle_G$ is a path of length 3 with vertices u_{i-2}, u_{i-1}, u_{i+1}, u_{i+2}.

Corollary 3. *Let G be a connected 2-closed claw-free graph and let $C \subset G$ be a good cycle in G. Then $G = C^2$.*

Corollary 3 specifically implies that a connected 2-closed claw-free graph either is isomorphic to the square of a cycle (and hence is trivially Hamilton-connected), or contains no good cycle. In the rest of the paper we restrict our observations to the second (nontrivial) case.

Let J be a good walk in G. We say that J is *maximal* if, for every good walk J' in G, J being a subsequence of J' implies $J = J'$.

Lemma 4. *Let G be a connected 2-closed claw-free graph that is not the square of a cycle, and let $J = u_0u_1 \dots u_{k+1}$ be a maximal good walk in G. Then $\langle N_G[u_1] \setminus \{u_3\} \rangle_G = \langle N_G[u_2] \setminus \{u_3, u_4\} \rangle_G$ and this subgraph is a clique.*

Note that symmetrically also $\langle N_G[u_k] \setminus \{u_{k-2}\} \rangle_G = \langle N_G[u_{k-1}] \setminus \{u_{k-2}, u_{k-3}\} \rangle_G$ is a clique.

Lemma 5. *Let G be a connected 2-closed claw-free graph that is not the square of a cycle, and let $J = u_0u_1 \dots u_{k+1}$ be a good walk in G. Then $u_1 \dots u_k$ is a path.*

Let J_i^k be the graphs in Figure 6. We set:
$\mathcal{J}_1 = \{J_1^k \mid k \geq 4\}$,
$\mathcal{J}_2 = \{J_2^k \mid k \geq 4\}$,
$\mathcal{J}_3 = \{J_3^k \mid k \geq 6\}$,
$\mathcal{J}_4 = \{J_4^k \mid k \geq 8\}$.

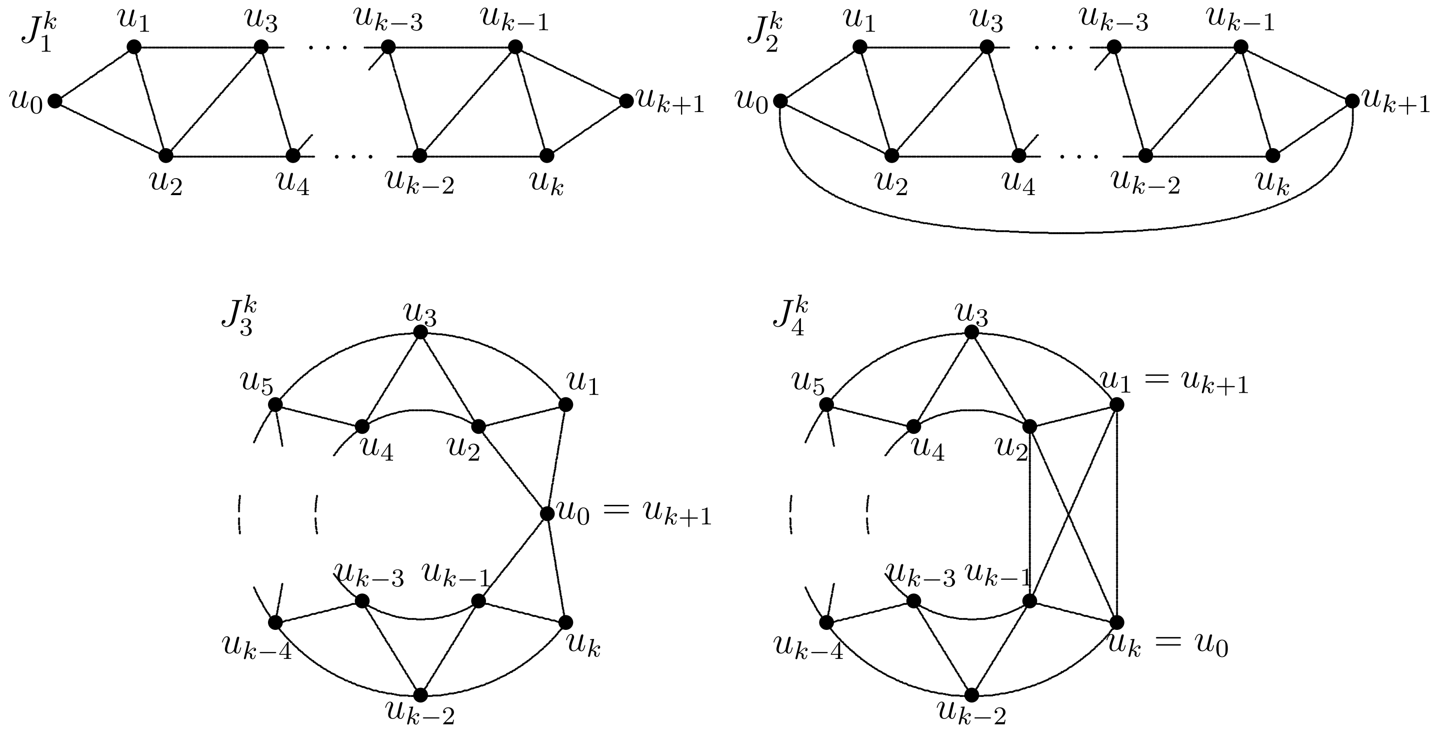

Figure 6

Our next lemma describes the structure of subgraphs induced by good walks.

Lemma 6. *Let G be a connected 2-closed claw-free graph that is not the square of a cycle, let $J = u_0u_1 \dots u_{k+1}$ be a maximal good walk in G, and let J be chosen such that*

$$|V(J)| = \min\{|\{x, u_1, \dots, u_k, y\}| \mid xu_1 \dots u_ky \text{ is a maximal good walk in } G\}.$$

Then

$$\langle V(J)\rangle_G \in \mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{J}_3 \cup \mathcal{J}_4.$$

The following lemma shows that the sets of interior vertices of maximal good walks in a 2-closed graph are vertex-disjoint.

Lemma 7. *Let G be a connected 2-closed claw-free graph that is not the square of a cycle and let $J^1 = u_0^1u_1^1 \dots u_{k+1}^1$, $J^2 = u_0^2u_1^2 \dots u_{k'+1}^2$ be maximal good walks in G such that $u_s^1 = u_t^2$ for some s, t, $1 \le s \le k$, $1 \le t \le k'$. Then*

(i) $\{u_1^1, \dots, u_k^1\} = \{u_1^2, \dots, u_{k'}^2\}$,

(ii) $k = k'$ *and* $u_i^1 = u_i^2$ *or* $u_i^1 = u_{k-i+1}^2$, $i = 1, \dots, k$.

4 Closure concept and Hamilton-connectedness

Before introducing the main concept of this paper, the closure operation, we first recall some helpful definitions and facts from [9].

Let $\mathcal{C}$ be a class of graphs and let $\mathcal{P}$ be a function on $\mathcal{C}$ such that, for any $G \in \mathcal{C}$, $\mathcal{P}(G) \subset 2^{V(G)}$ (i.e., $\mathcal{P}(G)$ is a set of subsets of $V(G)$). For any $X \subset V(G)$ let G^*_X denote the local completion of G at X, i.e. the graph with $V(G^*_X) = V(G)$ and $E(G^*_X) = E(G) \cup \{uv|\ u, v \in X\}$ (thus, the previous notation G^*_x means that, for a vertex $x \in V(G)$, we simply write G^*_x for $G^*_{N_G(x)}$).

We say that a graph F is a $\mathcal{P}$-*extension* of G, denoted $G \preceq F$, if there is a sequence of graphs $G_0 = G, G_1, \ldots, G_k = F$ such that $G_{i+1} = (G_i)^*_{X_i}$ for some $X_i \in \mathcal{P}(G_i)$. Clearly, for any graph G exists a $\preceq$-maximal $\mathcal{P}$-extension H, and in this case we say that H is a $\mathcal{P}$-closure of G. If a $\mathcal{P}$-closure is uniquely determined then it is denoted by $\mathrm{cl}_{\mathcal{P}}(G)$. Finally, a function $\mathcal{P}$ is *non-decreasing* (on a class $\mathcal{C}$), if, for any $H, H' \in \mathcal{C}$, $H \preceq H'$ implies that for any $X \in \mathcal{P}(H)$ there is an $X' \in \mathcal{P}(H')$ such that $X \subset X'$.

The following result was proved in [9]. For the sake of completeness, we include its (short) proof here.

Theorem O [9]. *If $\mathcal{P}$ is a non-decreasing function on a class $\mathcal{C}$, then, for any $G \in \mathcal{C}$, a $\mathcal{P}$-closure of G is uniquely determined.*

Proof. Let $H \neq H'$ be $\mathcal{P}$-closures of G, let $G = G_0, G_1, \ldots, G_k = H'$ be such that $G_{i+1} = (G_i)^*_{X_i}$ for some $X_i \in \mathcal{P}(G_i)$, and let s be a smallest integer such that $G_s \not\subset H$. Since $G_{s-1} \subset H$ and $\mathcal{P}$ is non-decreasing, there is $X \in \mathcal{P}(H)$ such that $X_{s-1} \subset X$. Since H is $\preceq$-maximal, we have $H^*_X = H$, a contradiction. ■

For a given graph G, let $\mathcal{C}_G$ denote the class of graphs with vertex set $V(G)$. The following two facts are easy to observe.

Lemma 8. *Let G be a graph.*

(*i*) *Let $\mathcal{P}$ be a non-decreasing function on $\mathcal{C}_G$, let $X \subset V(G)$, and for any $H \in \mathcal{C}_G$ set $\mathcal{P}^X(H) = \mathcal{P}(H) \cup \{N_H(x)|\ x \in X\}$. Then $\mathcal{P}^X$ is a non-decreasing function on $\mathcal{C}_G$.*

(*ii*) *For any integer $k \geq 1$, the function $\mathcal{P}_k(H) = \{N_H(x)|\ \langle N_H(x)\rangle_H$ is k-connected $\}$ is a non-decreasing function on $\mathcal{C}_G$.* ■

Consequently, for any graph G, integer $k \geq 1$ and a set $X \subset V(G)$, the function $\mathcal{P}_k^X$, defined (for any $H \in \mathcal{C}_G$) by $\mathcal{P}_k^X(H) = (\mathcal{P}_k)^X(H)$, is a non-decreasing function on $\mathcal{C}_G$.

Let now G be a connected claw-free graph that is not the square of a cycle and let $J_1, \ldots, J_t$ be all maximal good walks in $\mathrm{cl}_2(G)$. For any $J_i = u_0^i u_1^i \ldots u_{k+1}^i$ set

$X_i = \{u_1^i, \ldots, u_{r-1}^i\} \cup \{u_{r+2}^i \ldots u_{2r}^i\}$ if $k = 2r$

or

$X_i = \{u_1^i, \ldots, u_{r-1}^i\} \cup \{u_{r+3}^i \ldots u_{2r+1}^i\}$ if $k = 2r + 1$,

respectively, and set $X = \cup_{i=1}^t X_i$ (note that the sets X_i are pairwise disjoint by Lemma 7). Then, by Lemma 8, the function $\mathcal{P}^M(H) = \mathcal{P}_2^X(H)$ is a non-decreasing function on $\mathcal{C}_G$. The corresponding $\mathcal{P}^M$-closure of G (which is unique by Lemma 8) will be called the *multigraph closure* (or simply *M-closure*) of G and denoted $\mathrm{cl}^M(G)$. If G is the square of a cycle, we define $\mathrm{cl}^M(G)$ as the complete graph on $V(G)$. If $G = \mathrm{cl}^M(G)$ then we say that G is *M-closed*.

Theorem 9. *Let G be a connected claw-free graph and let $\mathrm{cl}^M(G)$ be the M-closure of G. Then*

(*i*) *$\mathrm{cl}^M(G)$ is uniquely determined,*
(*ii*) *there is a multigraph H such that $\mathrm{cl}^M(G) = L(H)$,*
(*iii*) *for every $a \in V(G)$, $p_a(\mathrm{cl}^M(G)) = p_a(G)$,*
(*iv*) *$\mathrm{cl}^M(G)$ is Hamilton-connected if and only if G is Hamilton-connected.*

Proof. If $G = C^2$ for some cycle C then the statement is trivial, hence we suppose that G is not the square of a cycle. Part (*i*) then follows immediately from Lemma 8, and part (*ii*) follows immediately from Lemma 1, from the construction of $\mathrm{cl}^M(G)$, from Lemma 25 and from Theorem N. ■

Before proving parts (*iii*) and (*iv*) of Theorem 9, we first show that if G is not the square of a cycle, then $\mathrm{cl}^M(G)$ can be equivalently constructed by the following algorithm.

Algorithm 10. *Let G be a connected claw-free graph that is not the square of a cycle.*

1. *Set $G_1 = \mathrm{cl}_2(G)$, $i := 1$.*
2. *If G_i contains a good walk, then*
   (*a*) *choose a maximal good walk $J = u_0 u_1 \ldots u_{k+1}$,*
   (*b*) *set $G_{i+1} = \mathrm{cl}_2((G_i)^*_{u_1 u_k})$,*
   (*c*) *$i := i + 1$ and go to (2).*
3. *Set $\overline{G} = G_i$.*

Proposition 11. *Let G be a connected claw-free graph that is not the square of a cycle and let $\overline{G}$ be the graph constructed by Algorithm 10. Then $\overline{G} = \mathrm{cl}^M(G)$.*

Proof. By Lemma 28, Algorithm 10 closes all vertices with neighborhood in some $\mathcal{P}^M(G_i)$, hence $\mathrm{cl}^M(G) \subset \overline{G}$. By Lemma 25, every vertex with neighborhood in some $\mathcal{P}^M(G_i)$ is closed by Algorithm 10. Hence $\overline{G}$ is a special case of one possible construction of $\mathcal{P}^M(G)$ and, by Theorem 9(i), $\overline{G} = \mathrm{cl}^M(G)$. ■

Proof of parts (iii), (iv) of Theorem 9 now immediately follows from Proposition 27.

■

Let T_1, T_2, T_3 be the graphs in Figure 7. It is easy to observe that if $G = L(H)$ and $x \in V(G)$ is 2-eligible, then the edge $x_1x_2 \in E(H)$, corresponding to x, is contained in a copy of T_i for some i, $1 \leq i \leq 3$, such that $d_{T_i}(x_1) = d_{T_i}(x_2) = 3$ (since these are the situations when the neighborhood of x is not a clique). However, the converse is not true in general, unless x_1 and/or x_2 have an appropriate neighbor outside. More specifically, it is straightforward to verify the following observation.

Proposition 12. *Let G be a claw-free graph and let T_1, T_2, T_3 be the graphs shown in Figure 7. Then G is M-closed if and only if there is a multigraph H such that $G = L(H)$ and H does not contain a subgraph S (not necessarily induced) with any of the following properties:*

(i) $S \simeq T_1$,

(ii) $S \simeq T_2$ *and there is a* $u \in V(H) \setminus V(S)$ *such that* $|N_H(u) \cap \{x_1, x_2\}| = 1$,

(iii) $S \simeq T_3$ *and there are* $u_1, u_2 \in V(H) \setminus V(S)$ *such that* $u_1 \neq u_2$ *and* $u_ix_i \in E(H)$, $i = 1, 2$

(where x_1, x_2 are the only vertices in S with $d_S(x_i) = 3$). ■

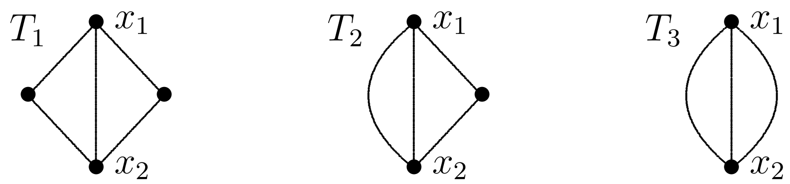

Figure 7

A well-known drawback of line graphs of multigraphs is the fact that there can be multigraphs H_1, H_2 such that $H_1 \not\simeq H_2$ but $L(H_1) \simeq L(H_2)$

(i.e., the "preimage" is not uniquely determined). However, this problem can be avoided by a slight modification of an approach given in [16]. Namely, we show that the preimage $H = L_M^{-1}(G)$ of a line graph G of a multigraph is uniquely determined under a (very natural) additional assumption that simplicial vertices in G correspond to edges in H with one vertex of degree 1 (called *pendant edges*).

The *basic graph* of a multigraph H is the graph with the same vertex set, in which two vertices are adjacent if and only if they are adjacent in H. A *multitriangle* (*multistar*) is a multigraph such that its basic graph is a triangle (star). The *center* of a multistar S with m edges is the vertex $x \in V(S)$ with $d_S(x) = m$ (for $|V(S)| = 2$ we choose the center arbitrarily), and all other vertices of S are its *leaves*. An induced multistar S in H is *pendant* if none of its leaves has a neighbor in $V(G) \setminus V(S)$, and similarly a multitriangle T is pendant if exactly one of its vertices (called the *root*) has neighbors in $V(G) \setminus V(S)$. We will use the following operations introduced in [16].

Operation A. Choose a pendant multistar in H and identify all its leaves.

Operation B. Choose a pendant multitriangle H with vertices $\{v, x, y\}$ and root v, delete all edges joining v and x, and add the same number of edges between v and y.

Now, for a multigraph H, $AB(H)$ denotes the multigraph obtained by recursively repeating operations A and B. The following result was proved in [16].

Theorem P [16]. *Let H, H' be connected multigraphs such that $L(H) \simeq L(H')$. Then $AB(H) = AB(H')$ except that one of H, H' is a multitriangle and the other one is a nonisomorphic multitriangle or a multistar.*

We will need one more operation.

Operation C. Choose a pendant multistar in H and replace every leaf of degree $k \geq 2$ by k leaves of degree 1.

Similarly as before, let $BC(H)$ denote the multigraph obtained from a multigraph H by recursively repeating operations B and C. Theorem P then easily implies the following result.

Theorem 13. *Let G be a connected line graph of a multigraph. Then there is, up to an isomorphism, a uniquely determined multigraph $H = L_M^{-1}(G)$ such that a vertex $e \in V(G)$ is simplicial in G if and only if the corresponding edge $e \in E(H)$ is a pendant edge in H.*

Proof. Let $G = L(H)$. It is easy to see that every edge $e \in E(H)$ corresponding to a simplicial vertex $e \in V(G)$ is in a pendant multitriangle or in a pendant multistar. Thus, $BC(H)$ has the required properties. Uniqueness follows from Theorem P. ■

Note that if, specifically, G is a line graph of a graph, then the multigraph preimage $L_M^{-1}(G)$ of G, given by Theorem 13, and the obvious line graph preimage $L^{-1}(G)$ can be different. For example, for the graph T_1 of Figure 7, $L_M^{-1}(T_1)$ and $L^{-1}(T_1)$ are shown in Figure 8.

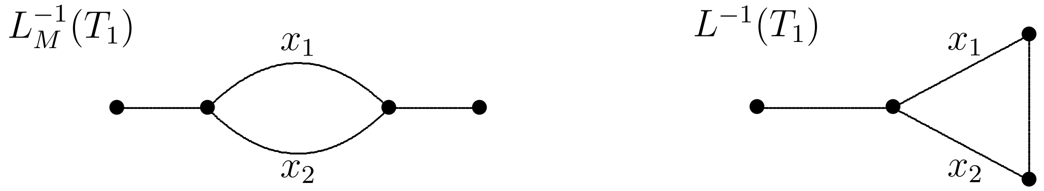

Figure 8

The following result shows that, with the use of the (uniquely determined) preimage $L_M^{-1}(G)$ of a line graph of a multigraph G, Proposition 12 can be simplified.

Proposition 14. *Let G be a claw-free graph and let T_1, T_2, T_3 be the graphs shown in Figure 7. Then G is M-closed if and only if G is a line graph of a multigraph and $L_M^{-1}(G)$ does not contain a subgraph (not necessarily induced) isomorphic to any of the graphs T_1, T_2 or T_3.*

Proof. If $L_M^{-1}(G)$ does not contain any of T_1, T_2, T_3, then clearly the conditions (*i*), (*ii*) and (*iii*) of Proposition 12 are satisfied and hence G is M-closed by Proposition 12.

Conversely, suppose that G is M-closed and let H be a multigraph given by Proposition 12. Then clearly any T_2 or T_3 not satisfying (*ii*) or (*iii*) is turned by Operations B and/or C into a star, hence $BC(H)$ does not contain any of T_1, T_2, T_3. ■

5 Applications and sharpness

Combining Theorems F and 9(*iv*), we immediately obtain the following result.

Theorem 15. *Every 6-connected claw-free graph with at most 29 vertices of degree 6 is Hamilton-connected.*

Proof. If G is a counterexample to Theorem 15, then $H = \mathrm{cl}^M(G)$ is a counterexample to Theorem C. ■

Corollary 16. *Every 7-connected claw-free graph is Hamilton-connected.* ■

Similarly, Theorem 9(*iv*) immediately implies the following result.

Theorem 17. *Conjecture J is equivalent with Conjectures G, H and I.*

Proof. Conjecture J implies Conjecture I since every line graph (of a multigraph) is claw-free. Conversely, if G is a counterexample to Conjecture J, then $H = \mathrm{cl}^M(G)$ is a counterexample to Conjecture I.

■

We conclude by showing that the closure operation $\mathrm{cl}^M(G)$ is, in a sense, best possible; more specifically, there is no closure operation that turns a 3-connected line graph of a multigraph into a line graph (of a graph) and preserves Hamilton-connectedness.

If $\mathcal{C}$ is a class of graphs, then by a *closure on* $\mathcal{C}$ we mean a mapping $\mathfrak{cl} : \mathcal{C} \to \mathcal{C}$ such that, for any $G \in \mathcal{C}$, $V(G) = V(\mathfrak{cl}(G))$ and $E(G) \subset E(\mathfrak{cl}(G))$. Let $\mathcal{L}_k$ denote the class of k-connected line graphs (of graphs) and let $\mathcal{L}_k^M$ denote the class of k-connected line graphs of multigraphs.

Theorem 18. *There is no closure* $\mathfrak{cl}$ *on* $\mathcal{L}_3^M$ *such that* $\mathfrak{cl} : \mathcal{L}_3^M \to \mathcal{L}_3$ *and Hamilton-connectedness is stable under* $\mathfrak{cl}$*.*

Proof. Let H be the multigraph shown in Figure 2 and let $G = L(H)$. Then G is not Hamilton-connected, and the vertices of G that correspond to edges of H adjacent to some of the vertices a_1, a_2, induce in G a subgraph F isomorphic to the sixth graph in Figure 1. Thus, for any closure $\mathfrak{cl}$: $\mathcal{L}_3^M \to \mathcal{L}_3$, $\mathfrak{cl}(G)$ contains at least one edge joining two nonadjacent vertices of F. However, adding any such edge turns G into a graph that is Hamilton-connected.

■

6 Proofs and lemmas

Lemma 19. *Let* G *be a claw-free graph,* $x \in V(G)$*, let* $y \in V(G)$ *be a cutvertex of* $\langle N_G(x)\rangle_G$ *and let* K_1*,* K_2 *be components of* $\langle N_G(x)\rangle_G - y$*. Then (up to a relabeling of* K_1, K_2*),*

(*i*) $\langle V(K_1) \cup \{y\}\rangle_G$ *is a clique and* K_2 *is a clique,*

(*ii*) *if* $H \subset \langle N_G(x)\rangle_G$ *is 2-connected noncomplete, then* $H \subset \langle V(K_2) \cup \{y\}\rangle_G$.

Proof. If (*i*) fails, then $\alpha(\langle N_G(x)\rangle_G) \geq 3$ and x is a center of an induced claw, a contradiction. Part (*ii*) follows immediately from (*i*). ■

Corollary 20. *Let* G *be a claw-free graph,* $x \in V(G)$*, let* $H \overset{IND}{\subset} \langle N_G(x)\rangle_G$ *be a 2-connected graph containing two distinct pairs of independent vertices. Then* $\langle N_G(x)\rangle_G$ *is 2-connected.*

Proof follows immediately from Lemma 19. ■

Corollary 21. *Let* G *be a 2-closed claw-free graph,* $H \subset G$ *(not necessarily induced),* $H \simeq S_1$*. If* $\{u_\ell u_{\ell+3} \mid \ell = 0, 1, 2\} \cap E(G) = \emptyset$*, then*

(*i*) *either* $H \overset{IND}{\subset} G$,

(*ii*) *or* $H + u_0u_5 \overset{IND}{\subset} G$ *(and* $H + u_0u_5 \simeq S_2$*).*

Proof. If $u_\ell u_{\ell+4} \in E(G)$ for some $\ell \in \{0, 1\}$, then $u_{\ell+2}$ is 2-eligible by Corollary 20, a contradiction. ■

Lemma 22. *Let G be a 2-closed claw-free graph, $x \in V(G)$, $H \overset{IND}{\subset} \langle N_G(x)\rangle_G$ 2-connected, $u, v \in V(H)$ independent. Then u or v is a cutvertex of $\langle N_G(x)\rangle_G$.*

Proof. Since G is 2-closed and u, v are independent, $\langle N_G(x)\rangle_G$ cannot be 2-connected. If $\langle N_G(x)\rangle_G$ is disconnected, then, for an arbitrary vertex w in the component of $\langle N_G(x)\rangle_G$ not containing H, $\langle\{x, u, v, w\}\rangle_G \simeq K_{1,3}$, a contradiction. Hence $\kappa(\langle N_G(x)\rangle_G) = 1$. Rest of the proof follows from Lemma 19. ■

Proof of Lemma 1. Each of the graphs G_i, $i \in \{1, 3, 5, 6, 7\}$, contains a vertex x_i satisfying the assumptions of Corollary 20, i.e. such that x_i is 2-eligible in any claw-free graph G such that $G_i \overset{\text{IND}}{\subset} G$. Hence none of the G_i can be an induced subgraph of a 2-closed graph. ■

Lemma 23. *Let G be a 2-closed claw-free graph, $H \overset{IND}{\subset} G$, $H \simeq S_1$ or $H \simeq S_2$. Then there is no vertex $z \in V(G)\backslash V(H)$ such that $\{u_1, u_3\} \subset N_G(z)$ or $\{u_2, u_3\} \subset N_G(z)$ (and, symmetrically, neither $\{u_2, u_4\} \subset N_G(z)$).*

Proof. 1. We first show that there is no $z \in V(G) \setminus V(H)$ such that $\{u_1, u_2, u_3, u_4\} \subset N_G(z)$. Let, to the contrary, $z \in V(G) \setminus V(H)$ and $u_i \in N_G(z)$ for $i = 1, 2, 3, 4$. Then $\langle\{u_1, u_2, u_3, u_4\}\rangle_G$ is a 2-connected subgraph of $\langle N_G(z)\rangle_G$ and u_1, u_4 are independent. By Lemma 22, u_1 or u_4 is a cutvertex of $\langle N_G(z)\rangle_G$.

Suppose u_4 is a cutvertex of $\langle N_G(z)\rangle_G$ (the other case is symmetric), and let $w \in N_G(z)$ be in the component of $\langle N_G(z)\rangle_G - u_4$ not containing u_1, u_2 and u_3. Since $\langle\{u_4, u_5, u_2, w\}\rangle_G \not\simeq K_{1,3}$, we have $u_5 w \in E(G)$. Then $\langle\{u_2, u_3, u_5, w, z\}\rangle_G$ is a 2-connected subgraph of $\langle N_G(u_4)\rangle_G$ containing two distinct pairs of independent vertices, hence u_4 is 2-eligible by Corollary 20, a contradiction.

2. We show that there is no $z \in V(G) \setminus V(H)$ such that $\{u_1, u_2, u_3\} \subset N_G(z)$ or $\{u_2, u_3, u_4\} \subset N_G(z)$. Let, to the contrary, $\{u_1, u_2, u_3\} \subset N_G(z)$ (the second case is symmetric). By part 1 of the proof, $zu_4 \notin E(G)$ and from $\langle\{u_2, u_0, z, u_4\}\rangle_G \not\simeq K_{1,3}$ we have $zu_0 \in E(G)$. Then $\langle\{u_0, u_2, u_3, z\}\rangle_G$ is 2-connected, u_0, u_3 are independent and, by Lemma 22, either u_0 or u_3 is a cutvertex of $\langle N_G(u_1)\rangle_G$. Choose a vertex w in the component of $\langle N_G(u_1)\rangle_G - u_0$ ($\langle N_G(u_1)\rangle_G - u_3$) not containing u_2 and z, respectively.

(*i*) If u_0 is a cutvertex of $\langle N_G(u_1)\rangle_G$, then $\langle\{w, u_0, u_1, u_2, u_3, u_4\}\rangle_G$ is isomorphic to S_1 or S_2 and we have a contradiction with part 1 of the proof (for the vertex z).

(*ii*) If u_3 is a cutvertex of $\langle N_G(u_1)\rangle_G$, then from $\langle\{u_3, w, u_2, u_5\}\rangle_G \not\simeq K_{1,3}$ we have $wu_5 \in E(G)$, but then $\langle N_G(u_3)\rangle_G$ contains a 2-connected induced subgraph with two distinct pairs of independent vertices. By Corollary 20, u_3 is 2-eligible, a contradiction.

3. a) Let now $\{u_1, u_3\} \subset N_G(z)$ (but $u_2z \notin E(G)$). From $\langle\{u_3, z, u_2, u_5\}\rangle_G \not\simeq K_{1,3}$ we have $zu_5 \in E(G)$, but then again u_3 is 2-eligible by Corollary 20, a contradiction.

b) The case $\{u_2, u_4\} \subset N_G(z)$ is symmetric.

c) Finally, if $\{u_2, u_3\} \subset N_G(z)$ (but $u_1z \notin E(G)$), then from $\langle\{u_3, z, u_1, u_4\}\rangle_G \not\simeq K_{1,3}$ we have $zu_4 \in E(G)$, which is not possible by part 2 of the proof. ■

Corollary 24. *Let G be a 2-closed claw-free graph, $H \overset{IND}{\subset} G$, $H \simeq S_1$ or $H \simeq S_2$. Then*

(*i*) *both $\langle N_G[u_1] \setminus \{u_2, u_3\}\rangle_G$ and $\langle N_G[u_2] \setminus \{u_3, u_4\}\rangle_G$ are cliques,*

(*ii*) *$N_G[u_2] \setminus \{u_3, u_4\} \subset N_G[u_1] \setminus \{u_3\}$,*

(*iii*) *the only neighbor of u_4 in $N_G(u_2)$ is u_3.*

Note that also symmetrically $\langle N_G[u_4] \setminus \{u_3, u_2\}\rangle_G$ and $\langle N_G[u_3] \setminus \{u_2, u_1\}\rangle_G$ are cliques.

Proof. (*i*) If $\langle N_G[u_1] \setminus \{u_2, u_3\}\rangle_G$ is not a clique, then there is a $z \in N_G(u_1)$ such that $zu_0 \notin E(G)$, but then by Lemma 23 $\langle\{u_1, z, u_0, u_3\}\rangle_G \simeq K_{1,3}$, a contradiction. The proof for $\langle N_G[u_2] \setminus \{u_3, u_4\}\rangle_G$ is symmetric.

(*ii*) By (*i*), every neighbor of u_2 is adjacent to u_1.

(*iii*) If $z \in N_G(u_2)$, $z \neq u_3$, is adjacent to u_4, then $z \notin V(H)$ since H is induced, but this contradicts Lemma 23. ■

Lemma 25. *Let G be a claw-free graph, $F \subset V(G)$, $F = \{u_0, u_1, u_2, u_3, u_4, u_5\}$. If F induces S_1 or S_2 in $\mathrm{cl}_2(G)$, then there are vertices $v_0, v_5 \in V(G)$ such that the set $\{v_0, u_1, u_2, u_3, u_4, v_5\}$ induces S_1 or S_2 in G.*

Proof. Let $\mathrm{cl}_2(G) = G^*_{x_1\ldots x_k}$, set $G_i = G^*_{x_1\ldots x_i}$, $i = 1, \ldots, k$ (i.e., $G_k = \mathrm{cl}_2(G)$), and let $F = \{u_0, u_1, u_2, u_3, u_4, u_5\}$ be such that $\langle F\rangle_{G_k} \simeq S_1$ or $\langle F\rangle_{G_k} \simeq S_2$. The proof then follows by induction from the following fact.

If $v_0, v_5 \in V(G)$ are such that $\{v_0, u_1, u_2, u_3, u_4, v_5\}$ induces S_1 or S_2 in G_{i+1} for some i, $1 \leq i \leq k-1$, then there are $w_0, w_5 \in V(G)$ such that $\{w_0, u_1, u_2, u_3, u_4, w_5\}$ induces S_1 or S_2 in G_i.

Thus, suppose that $\{v_0, u_1, u_2, u_3, u_4, v_5\}$ induces S_1 or S_2 in $G_{i+1} = (G_i)^*_{x_i}$, and set $B = E(G_{i+1}) \setminus E(G_i)$.

Since x_i is adjacent to both vertices of all edges in B and F induces S_1 or S_2 in $G_k = \mathrm{cl}_2(G)$, by Lemma 23, $B \cap \{u_1u_3, u_2u_3, u_2u_4\} = \emptyset$. Since $\langle N_{G_k}(x_i)\rangle_{G_k}$ is a clique, and by symmetry, we can suppose that $B \subset \{v_0u_1, v_0u_2, u_1u_2\}$. If $u_1u_2 \in B$, then $\langle\{u_3, u_1, u_2, u_5\}\rangle_{G_i}$ is a claw; hence $u_1u_2 \in E(G_i)$ and $|B| \leq 2$. If x_i is adjacent in G_i to both u_1 and u_2, then $\{x_i, u_1, u_2, u_3, u_4, u_5\}$ induces S_1 or S_2 in G_i, we set $w_0 = x_i$, $w_5 = v_5$ and we are done. Hence it remains to consider the case when x_i is adjacent in G_i to at most one of u_1, u_2 and, consequently, $|B| = 1$. But then for $B = \{v_0u_1\}$ we have $\langle\{u_2, v_0, u_1, u_4\}\rangle_{G_i} \simeq K_{1,3}$ and for $B = \{v_0u_2\}$ we have $\langle\{u_2, x_i, u_1, u_4\}\rangle_{G_i} \simeq K_{1,3}$, a contradiction. ■

Proof of Lemma 2. Let $d_G(u_i) \geq 5$ for some i, $3 \leq i \leq k-2$, and let $w \in V(G)$ be a neighbor of u_i, $w \notin \{u_{i-2}, u_{i-1}, u_{i+1}, u_{i+2}\}$. By Lemma 23 and since J is good, we have $wu_{i-2} \notin E(G)$ and $wu_{i+2} \notin E(G)$. From $\langle\{u_i, w, u_{i-2}, u_{i+2}\}\rangle_G \not\simeq K_{1,3}$ we then have $u_{i-2}u_{i+2} \in E(G)$, contradicting the fact that J is good. ■

Proof of Corollary 3. If $|V(C)| \leq 6$, then C cannot be good, hence $|V(C)| \geq 7$. Then, by Lemma 2, all vertices of C are of degree 4 in G, implying $C^2 = G$. ■

Proof of Lemma 4. By Corolary 24(i), $\langle N_G[u_2] \setminus \{u_3, u_4\}\rangle_G$ is a clique and by Corollary 24(ii), $N_G[u_2] \setminus \{u_3, u_4\} \subset N_G[u_1] \setminus \{u_3\}$. Thus, it remains to show that $N_G[u_1] \setminus \{u_3\} \subset N_G[u_2] \setminus \{u_3, u_4\}$. If this is not the case, then there is a vertex $x \in V(G)$ such that $xu_1 \in E(G)$ and $xu_2 \notin E(G)$. By Corollary 24(i) then $xu_0 \in E(G)$ and, by Corollary 21, $J' = xu_0u_1 \dots u_{k+1}$ is a good walk in G, contradicting the maximality of J. ■

Proof of Lemma 5. Suppose that $u_i = u_j$ for some i, j, $1 \leq i < j \leq k$, and choose i, j such that $j - i$ is minimum. Then $u_i \dots u_{j-1}u_j$ is a cycle, and by the minimality of $j - i$, $u_{i+1} \neq u_{j-1}$.

1. Let first $3 \leq j \leq k-2$. Then, by Lemma 2, $\langle N_G(u_j)\rangle_G \simeq P_4$.

If $2 \leq i \leq k-2$, then also the neighborhood of u_i in J^2 is a P_4, and these neighborhoods coincide. Since $u_{i+1} \neq u_{j-1}$, we have $u_{i+1} = u_{j+1}$, from which $u_{i+2} = u_{j+2}$, $u_{i-1} = u_{j-1}$ and $u_{i-2} = u_{j-2}$. Then $u_i \dots u_{j-1}u_j$ is a good cycle, a contradiction by Corollary 3.

If $i = 1$, then the equality $u_0 = u_{j-1}$ follows from $u_2 = u_{j+1}$ and from the equality of neighborhoods, and the cycle $u_i \dots u_{j-1}u_j$ is good by Corollary 21.

2. The case $3 \leq i \leq k-2$ is symmetric.

3. Thus, it remains to consider the possibility $i \in \{1, 2\}$, $j \in \{k-1, k\}$. This specifically implies that for every good walk $J = u_0 u_1 \dots u_{k+1}$ we have $k \leq |V(G)| + 2$, hence for every good walk J there is a maximal good walk J' such that J is a subsequence of J'. Hence we can without loss of generality suppose that J is maximal. We distinguish 4 cases.

a) $i = 1$, $j = k - 1$. Then, by Lemma 4 and by the fact that J is good, $\langle N_G(u_1) \rangle_G$ consists of a clique and one edge while $\langle N_G(u_{k-1}) \rangle_G$ consists of a clique and a P_3, a contradiction.

b) $i = 2$, $j = k$. This case is symmetric to the previous one.

c) $i = 2$, $j = k - 1$. Then the only possible vertices of degree 1 in $\langle N_G(u_2) \rangle_G$ are u_0 and u_4, and, in $\langle N_G(u_{k-1}) \rangle_G$ only u_{k-3} and u_{k+1}. Since $u_{k-3} \neq u_4$ (by the choice of i and j), we have $u_{k+1} = u_4$, and hence $u_{k-3} = u_0$. Since clearly $k \geq 5$, we have $d_G(u_3) = 4$ and u_3 is the only common neighbor od u_2, u_4, but then, since u_k is a common neighbor of $u_{k-1} = u_2$ and $u_{k+1} = u_4$, necessarily $u_k = u_3$ and we are in Case 2.

d) $i = 1$, $j = k$. The only universal vertex in $\langle N_G(u_1) \rangle_G$ is u_2 and in $\langle N_G(u_k) \rangle_G$ is u_{k-1}. Hence $u_2 = u_{k-1}$, contradicting the choice of i, j. ■

Lemma 26. *Let G be a connected 2-closed claw-free graph that is not the square of a cycle, $J = u_0 u_1 \dots u_{k+1}$ a maximal good walk in G, $u \in V(G)$, $u \notin \{u_0, u_1, u_2, u_3, u_4\}$, such that $uu_1 \in E(G)$ or $uu_2 \in E(G)$. Then:*

(i) both $uu_1 \in E(G)$ and $uu_2 \in E(G)$,

(ii) $uu_1 \dots u_{k+1}$ is a good walk in G,

(iii) if $u \in V(J)$, then $k \geq 6$ and $u \in \{u_{k-1}, u_k, u_{k+1}\}$.

Proof. *(i)* follows immediately from Lemma 4.

(ii), *(iii)* If $u \notin V(J)$, then Lemma 23 implies $uu_3 \notin E(G)$ and we are done by Corollary 21. Hence suppose $u \in V(J)$. Since $uu_2 \in E(G)$ and J is good, necessarily $u = u_j$ for some $j \geq 7$, implying $k \geq 6$. Since $d_G(u_3) = 4$ (by Lemma 2), $uu_3 \notin E(G)$ and hence $uu_1 \dots u_{k+1}$ is good by Corollary 21. Since $d_G(u_j) = 4$ for $3 \leq j \leq k-2$ (by Lemma 2), we have $u \in \{u_{k-1}, u_k, u_{k+1}\}$. ■

Proof of Lemma 6. First observe that by Lemma 2 the only edges to be considered are those between u_0, u_1, u_2 and u_{k-1}, u_k, u_{k+1}.

Case 1: J is not a path. Since $u_1, \dots, u_k$ is a path by Lemma 5, the only possibilities are $u_0 \in \{u_{k-1}, u_k, u_{k+1}\}$, and, symmetrically, $u_{k+1} \in \{u_0, u_1, u_2\}$ (note that $k \geq 6$ by Lemma 26).

a) $u_0 = u_{k-1}$. By Lemma 4, $\langle\{u_1, u_2, u_0, u_{k+1}, u_k\}\rangle_G$ is a clique (not excluding the possibility that $u_{k+1} \in \{u_1, u_2\}$). Then $\langle\{u_1, \ldots, u_k\}\rangle_G \in \mathcal{J}_4$ (since all edges between u_1, u_2 and u_{k-1}, u_k are present and no other edges are possible by Lemma 2), and hence for $u_{k+1} \in \{u_1, u_2\}$ we have $\langle V(J)\rangle_G = \langle\{u_1, \ldots, u_k\}\rangle_G \in \mathcal{J}_4$ and we are done, otherwise we have a contradiction with the minimality of J.

b) $u_0 = u_k$. Then similarly, by Lemma 4, $\langle\{u_1, u_2, u_k, u_{k-1}, u_{k+1}\}\rangle_G$ is a clique and then, as before, for $u_{k+1} \in \{u_1, u_2\}$ we obtain $\langle V(J)\rangle_G = \langle\{u_1, \ldots, u_k\}\rangle_G \in \mathcal{J}_4$, and otherwise we have a contradiction with the minimality of J.

c) $u_0 = u_{k+1}$. Then the only possible edges to be considered are the edges between u_1, u_2 and u_{k-1}, u_k. By Lemma 26, either $\{u_1u_k, u_1u_{k+1}, u_2u_k, u_2u_{k+1}\} \subset E(G)$, or $\{u_1u_k, u_1u_{k+1}, u_2u_k, u_2u_{k+1}\} \cap E(G) = \emptyset$. In the first case we have $\langle V(J) \setminus \{u_0\}\rangle_G = \langle\{u_1, \ldots, u_k\}\rangle_G \in \mathcal{J}_4$, contradicting the minimality of J, otherwise $\langle V(J)\rangle_G \in \mathcal{J}_3$.

Case 2: J is a path. By Lemma 26, either $\{u_1u_{k+1}, u_2u_{k+1}\} \subset E(G)$, or $\{u_1u_{k+1}, u_2u_{k+1}\} \cap E(G) = \emptyset$. In the first case, the walk $J - u_0 = u_{k+1}u_1u_2 \ldots u_ku_{k+1}$ is good in G, contradicting the minimality of J. Hence $u_1u_{k+1}, u_2u_{k+1} \notin E(G)$, and, symmetrically, $u_0u_{k-1}, u_0u_k \notin E(G)$.

It remains to consider the edges between u_1, u_2 and u_{k-1}, u_k. Again, by Lemma 26, either all of them or none of them are present. In the first case, the walk $J - \{u_0, u_{k+1}\} = u_ku_1u_2 \ldots u_{k-1}u_ku_1$ is good in G, contradicting the minimality of J; in the second case we have $\langle V(J)\rangle_G \in \mathcal{J}_1$ if $u_0u_{k+1} \notin E(G)$ and $\langle V(J)\rangle_G \in \mathcal{J}_2$ if $u_0u_{k+1} \in E(G)$. ∎

Proof of Lemma 7. If $3 \leq s \leq k-2$ or $3 \leq t \leq k'-2$, then the statement follows immediately by Lemma 2 (for $\{s, t\} \cap \{1, 2\} \neq \emptyset$ we use the equality of neighborhoods of the vertices $u_3^1 = u_3^2$, and symmetrically for $s \in \{k-1, k\}$ or $t \in \{k'-1, k'\}$).

It remains to consider the cases when $s \in \{1, 2, k-1, k\}$ and $t \in \{1, 2, k'-1, k'\}$. By symmetry, it is sufficient to suppose $s, t \in \{1, 2\}$ (otherwise we relabel one or both walks).

1. Let $u_1^1 = u_2^2$. By Lemma 4, $\langle N_G(u_1^1)\rangle_G$ consists of a clique and an edge, while $\langle N_G(u_2^2)\rangle_G$ consists of a clique and a P_3, a contradiction. Hence $u_1^1 \neq u_2^2$ and, symmetrically, $u_1^2 \neq u_2^1$.

2. Suppose that $u_2^1 = u_2^2$. By Lemma 4, at most two vertices in $\langle N_G(u_2^i)\rangle_G$ can be of degree 1, namely, u_0^i and u_4^i, $i = 1, 2$. We distinguish two subcases.

a) $u_4^1 = u_4^2$. The only neighbor of u_4^i in $\langle N_G(u_2^i)\rangle_G$ is the vertex u_3^i, $i = 1, 2$; hence $u_3^1 = u_3^2$. By Lemma 23, u_1^i is the only neighbor of u_3^i in $\langle N_G(u_2^i)\rangle_G$, distinct from u_4^i, $i = 1, 2$, hence also $u_1^1 = u_1^2$. For $k = k' = 4$ we thus have $u_j^1 = u_j^2$, $j = 1, 2, 3, 4$; otherwise (i.e. if $k \geq 5$ or $k' \geq 5$) the statement follows from $u_3^1 = u_3^2$ by the beginning of the proof.

b) $u_0^1 = u_4^2$ (and hence $u_4^1 = u_0^2$). Similarly as in a) we have $u_1^1 = u_3^2$. The vertex u_0^2 is of degree 1 in $\langle N_G(u_2^2)\rangle_G$ (since $u_0^2 = u_4^1$ and u_4^1 is of degree 1), hence $u_3^1 = u_1^2$. But then the vertices $u_3^1 = u_1^2$ and $u_4^1 = u_0^2$ have a common neighbor u_5^1 and $u_2^1 u_5^1 \notin E(G)$, contradicting the fact that, by Lemma 4, $N_G[u_1^2] \setminus \{u_3^2\} = N_G[u_2^2] \setminus \{u_3^2, u_4^2\}$.

3. Finally, let $u_1^1 = u_1^2$. By Lemma 23, the only universal vertex in $\langle N_G(u_1^i)\rangle_G$ is u_2^i, $i = 1, 2$. Hence $u_2^1 = u_2^2$ and we are back in Case 2. ■

Proposition 27. *Let G be a connected 2-closed claw-free graph that is not the square of a cycle and let $J = u_0 u_1 \dots u_{k+1}$ be a maximal good walk in G. Then*

(i) for every $a \in V(G)$, $p_a(G^*_{u_1 u_k}) = p_a(G)$,

(ii) the graph $G^*_{u_1 u_k}$ is Hamilton-connected if and only if G is Hamilton-connected.

Proof. In the proof of Proposition 27 we will need the following result by Brandt et al. (see [6], Proposition 3.2).

Proposition Q [6]. *Let x be an eligible vertex of a claw-free graph G, G'_x the local completion of G at x, and a, b two distinct vertices of G. Then for every longest (a, b)-path $P'(a, b)$ in G'_x there is a path P in G such that $V(P) = V(P')$ and P admits at least one of a, b as an endvertex. Moreover, there is an (a, b)-path $P(a, b)$ in G such that $V(P) = V(P')$ except perhaps in each of the following two situations (up to symmetry between a and b):*

(i) There is an induced subgraph $H \subset G$ isomorphic to the graph S in Figure 9 such that both a and x are vertices of degree 4 in H. In this case G contains a path P_b such that b is an endvertex of P and $V(P_b) = V(P')$. If, moreover, $b \in V(H)$, then G contains also a path P_a with endvertex a and with $V(P_a) = V(P')$.

(ii) $x = a$ and $ab \in E(G)$. In this case there is always both a path P_a in G with endvertex a and with $V(P_a) = V(P')$ and a path P_b in G with endvertex b and with $V(P_b) = V(P')$.

Let G and $J = u_o u_1 \dots u_{k+1}$ satisfy the assumptions of Proposition 27 and let S be the graph of Figure 9. For simplicity, set $G' = G^*_{u_1}$ and $G'' = (G')^*_{u_k} = G^*_{u_1 u_k}$. We show the following.

<u>Claim 27.1.</u> *There is no set $M \subset V(G)$ satisfying either of the following conditions:*

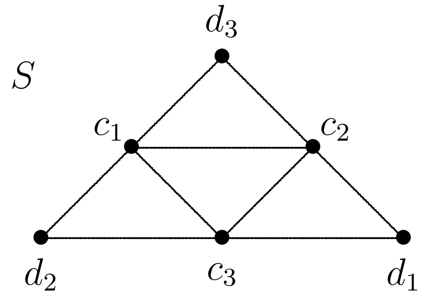

Figure 9

(*i*) $\langle M\rangle_G \simeq S$ *and* $d_{\langle M\rangle_G}(u_1) = 4$ *or* $d_{\langle M\rangle_G}(u_2) = 4$,
(*ii*) $\langle M\rangle_{G'} \simeq S$ *and* $d_{\langle M\rangle_{G'}}(u_k) = 4$ *or* $d_{\langle M\rangle_{G'}}(u_{k-1}) = 4$.

Proof of Claim 27.1. Suppose there is such a set $M \subset V(G)$.

(*i*) If $d_{\langle M\rangle_G}(u_1) = 4$, then $\langle N_{\langle M\rangle_G}(u_1)\rangle_G \simeq P_4$, but, by Lemma 4, $\langle N_G(u_1)\rangle_G$ consists of a clique and an edge, a contradiction.

Suppose that $d_{\langle M\rangle_G}(u_2) = 4$, let e.g. $u_2 = c_1$ (see Figure 9). Then $\langle N_{\langle M\rangle_G}(u_2)\rangle_G$ is a P_4 with vertices d_2, c_1, c_2, d_1. By Lemma 4, the only possible induced P_4 in $\langle N_G(u_2)\rangle_G$ is $xu_1u_3u_4$, where $x \in N_G(u_2) \setminus \{u_3, u_4\}$, but then $u_1 = c_1$ or $u_1 = c_2$ and we are in the previous case.

(*ii*) Let first $J \notin \mathcal{J}_4$. Since $\langle N_G(u_k)\rangle_G = \langle N_{G'}(u_k)\rangle_{G'}$, and for $k \geq 5$ also $\langle N_G(u_{k-1})\rangle_G = \langle N_{G'}(u_{k-1})\rangle_{G'}$, the proof is symmetric to the proof in (*i*) in these cases. It remains to consider the case $d_{\langle M\rangle_{G'}}(u_{k-1}) = 4$ for $k = 4$. Then $\langle N_{G'}(u_3)\rangle_{G'}$ can be covered by two cliques K_1, K_2, where $\{u_0, u_1, u_2\} \subset V(K_1)$ and $\{u_4, u_5\} \subset V(K_2)$, and hence the only possible induced P_4 is xu_2u_4y for $x \in V(K_1)$ and $y \in V(K_2)$. This again leads to the previous case.

Secondly, if $J \in \mathcal{J}_4$, then $k \geq 8$, we have $\langle N_{G'}(u_k)\rangle_{G'} = \langle N_G(u_k) \cup \{u_3\}\rangle_{G'}$ and then, by Lemma 4 and by the definition of G', $\langle N_{G'}(u_k)\rangle_{G'}$ consists of a clique and an edge, a contradiction. □

By Claim 27.1, the case (*i*) of Proposition Q is not possible. From this, again by Proposition Q, we conclude that:

- for any $a \in V(G)$, $p_a(G'') = p_a(G)$, i.e., statement (*i*) of Proposition 27 holds,
- if the statement (*ii*) of Proposition 27 fails, i.e. if $p_{ab}(G') \neq p_{ab}(G)$ or $p_{ab}(G'') \neq p_{ab}(G')$, then we have the situation described in case (*ii*) of Proposition Q, i.e. $ab \in E(G)$ and $x \in \{a, b\}$ (where $x = u_1$ or $x = u_k$, respectively).

Suppose that $p_{ab}(G') \neq p_{ab}(G)$. Then $u_1 \in \{a, b\}$. Let $\tilde{G}$ denote the local completion of G at u_2. Since $N_G(u_1) \subset N_G(u_2)$ by Lemma 4, we have $E(G') \subset E(\tilde{G})$, and hence for any pair $a, b \in V(G)$ for which $p_{ab}(G') \neq p_{ab}(G)$

also $p_{ab}(\tilde{G}) \neq p_{ab}(G)$. Thus, by Proposition Q, $u_2 \in \{a, b\}$. Hence we conclude that if $p_{ab}(G') \neq p_{ab}(G)$, then $\{a, b\} = \{u_1, u_2\}$.

Symmetrically, if $p_{ab}(G'') \neq p_{ab}(G')$, then $\{a, b\} = \{u_{k-1}, u_k\}$ (since the argument for u_1, u_2 used only the statements of Lemma 4 and of Proposition Q and these remain true also in G'). In the latter case (i.e., $\{a, b\} = \{u_{k-1}, u_k\}$), we observe that $G'' = G^*_{u_1 u_k} = G^*_{u_k u_1}$. The proof for $G^*_{u_k}$ is then symmetric to the proof for G' and $\{a, b\} = \{u_1, u_2\}$, and the proof for $G^*_{u_k u_1}$ (i.e. for the local completion of $G^*_{u_k}$ at u_1) follows by Proposition Q. Hence it is sufficient to prove the statement for u_1, u_2.

Consider the following statements:

(a) G' is Hamilton-connected,

(b) G contains a hamiltonian (a, b)-path for all pairs $a, b \in V(G)$ except possibly $\{a, b\} = \{u_1, u_2\}$,

(c) G' contains a hamiltonian (u_2, u_3)-path,

(d) G contains a hamiltonian (u_1, u_2)-path,

(e) G is Hamilton-connected.

By the previous discussion, (a) $\Rightarrow$ (b). Obviously (a) $\Rightarrow$ (c) and (b) $\wedge$ (d) $\Rightarrow$ (e). Thus, in order to show that (a) $\Rightarrow$ (e) (i.e. to finish the proof of Proposition 27), it is sufficient to show that (c) $\Rightarrow$ (d).

Claim 27.2. *If G' contains a hamiltonian (u_2, u_3)-path, then G contains a hamiltonian (u_1, u_2)-path.*

Proof of Claim 27.2. Let P' be a hamiltonian (u_2, u_3)-path in G'. We first show that P' can be chosen such that $P' \subset G$.

By Lemma 4, every edge in $E(G') \setminus E(G)$ contains the vertex u_3. Thus, if P' contains an edge in $E(G') \setminus E(G)$, then this is the edge $u_3^- u_3$. If $u_2^+ = u_1$, we set $P' := u_2 u_3^- \overleftarrow{P'} u_1 u_3$ (since $u_2 u_3^- \in E(G)$ by Lemma 4); for $u_2^+ \neq u_1$ we replace in P' the path $u_1^- u_1 u_1^+$ by the edge $u_1^- u_1^+$ and the edge $u_3^- u_3$ by the path $u_3^- u_1 u_3$, i.e. we set $P' := u_2 P' u_1^- u_1^+ P' u_3^- u_1 u_3$ (the edges we need are in G again by Lemma 4). Thus, in the rest of the proof we suppose that P' is a hamiltonian (u_2, u_3)-path in G and we construct a hamiltonian (u_1, u_2)-path P in G.

If $u_1 = u_2^+$, then we set $P = u_1 P' u_3 u_2$, and if $u_1 \notin \{u_2^+, u_3^-\}$, then we set $P = u_1 u_3 \overleftarrow{P'} u_1^+ u_1^- \overleftarrow{P'} u_2$ (note that $u_1^- u_1^+ \in E(G)$ by Lemma 4). Thus, we can suppose that $u_1 = u_3^-$. For $u_2^+ = u_4$ we then set $P = u_1 \overleftarrow{P'} u_4 u_3 u_2$, hence we can further suppose that $u_2^+ \neq u_4$. Now, if $u_4 u_5 \in E(P')$ (which, by Lemma 2, necessarily occurs if $k \geq 6$), then for $u_5 = u_4^+$ we set $P = u_1 \overleftarrow{P'} u_5 u_3 u_4 \overleftarrow{P'} u_2$ and for $u_4 = u_5^+$ we set $P = u_1 \overleftarrow{P'} u_4 u_3 u_5 \overleftarrow{P'} u_2$.

Thus, it remains to consider the following situation: $u_1 = u_3^-$, $u_2^+ \neq u_4$, $u_4u_5 \notin E(P')$ and $4 \leq k \leq 5$.

If $k = 4$, then $u_3, u_4, u_5, u_4^-, u_4^+$ are in a clique (by Lemma 4) and we replace $u_4u_4^+$ by $u_4u_3u_4^+$, i.e. we set $P = u_1P'u_4^+u_3u_4P'u_2$.

Finally, if $k = 5$, then $u_4, u_5, u_4^-, u_4^+, u_5^-, u_5^+$ are in a clique (again by Lemma 4) and we set $P = u_1P'u_5^+u_5^-P'u_4^+u_5u_3u_4P'u_2$ if $P' = u_2P'u_4P'u_5P'u_1u_3$, and $P = u_1P'u_4u_3u_5u_4^-P'u_5^+u_5^-P'u_2$ if $P' = u_2P'u_5P'u_4P'u_1u_3$. ■

Lemma 28. *Let G be a connected 2-closed claw-free graph that is not the square of a cycle, $J_1 = u_0u_1 \ldots u_{k+1}$, $J_2 = v_0v_1 \ldots v_{p+1}$ two maximal good walks in G, $\{u_1 \ldots u_k\} \neq \{v_1 \ldots v_p\}$, and let $G' = \mathrm{cl}_2(G^*_{v_1v_p})$. Then either $\langle V(J_1)\rangle_{G'}$ is a clique, or there are vertices w_0, w_{k+1} such that $w_0u_1 \ldots u_kw_{k+1}$ is a maximal good walk in G'.*
If moreover $p \geq 6$, then also either $\langle V(J_2)\rangle_{G'}$ is a clique, or $v_1 \ldots v_p$ is a maximal good walk in G'.

Proof. First note that, by Lemma 7, $\{u_1 \ldots u_k\} \cap \{v_1 \ldots v_p\} = \emptyset$. Let $G_0, G_1, \ldots, G_t$ be a sequence of graphs such that $G_0 = G^*_{v_1v_p}$, $G_{i+1} = (G_i)^*_{z_i}$ for some z_i that is 2-eligible in G_i, $i = 0, 1, \ldots, t-1$, and $G_t = G'$. Set $J_1' = \{u_3, \ldots, u_{k-2}\}$, $J_2' = \{v_4, \ldots, v_{p-3}\}$ and let j be the smallest integer such that at least one of the following holds:

(*i*) there is a vertex $w \in J_1' \cup J_2'$ such that $d_{G_j}(w) > 4$,

(*ii*) J_1 or $v_1 \ldots v_p$ is not good in G_j.

Thus, there is an edge $e \in E(G_j) \setminus E(G_{j-1})$ such that either

(*i'*) e has one vertex at some $w \in J_1' \cup J_2'$, or

(*ii'*) e joins some vertices u_i, u_{i+p} or v_i, v_{i+p} for $3 \leq p \leq 5$

(such an edge will be referred to as a *bad edge*).

If $j = 0$, then a bad edge is obtained by local completion at v_1 or at v_p. Then clearly $v_1 \ldots v_p$ remains good, and (*i'*) is not possible since neither v_1 nor v_p can be adjacent in G_{j-1} to any $w \in J_1' \cup J_2'$. Hence the bad edge has both vertices in $V(J_1)$. But, for v_1, all edges in $E((G_{j-1})^*_{v_1}) \setminus E(G_{j-1})$ contain v_3, hence the existence of a bad edge implies $v_3 \in V(J_1)$, contradicting Lemma 7. The argument for v_p is symmetric.

Hence $j \geq 1$, i.e. a bad edge is obtained by closing a 2-eligible vertex. We prove the statement for the case when the bad edge has at least one vertex w in $V(J_1)$; the proof for a bad edge with both vertices in $v_1 \ldots v_p$ is the same.

We first verify the following two observations.

(∗) *If $\langle V(J_1)\rangle_{G_t}$ is not a clique, then every vertex $w \in \{u_3, \ldots, u_{k-2}\}$ has in G_j no neighbors outside $V(J_1)$.*

<u>Proof.</u> Suppose $(*)$ fails and let $w = u_\alpha$ have a neighbor outside $V(J_1)$. Then w has in G_{j-1} a 2-eligible neighbor z, and, by the choice of j, $z \in \{u_{\alpha-2}, u_{\alpha-1}, u_{\alpha+1}, u_{\alpha+2}\}$. Also by the choice of j, $z \notin \{u_3, \dots, u_{k-2}\}$ (since $d_{G_{j-1}}(z) = 4$ and any additional edge in $\langle N_{G_{j-1}}(z)\rangle_{G_{j-1}}$ would violate (ii). Thus, by symmetry, it remains to consider the cases $z \in \{u_1, u_2\}$.

If $k \geq 6$, then u_2 cannot be 2-eligible in G_{j-1} since u_4 is of degree 1 in $\langle N_{G_{j-1}}(u_2)\rangle_{G_{j-1}}$, and similarly with u_3 being of degree 1 in $\langle N_{G_{j-1}}(u_1)\rangle_{G_{j-1}}$ for $k \geq 5$. Since clearly $k \neq 4$ (otherwise there is nothing to do), it remains to consider the case $k = 5$ and $z = u_2$. However, in this case, if u_2 happens to be 2-eligible, then it is easy to see that $\langle V(J_1)\rangle_{G_t}$ is a clique. □

$(**)$ *If* $\langle V(J_1)\rangle_{G_t}$ *is not a clique, then no vertex* u_i, $1 \leq i \leq k$, *is 2-eligible in* G_{j-1}.

<u>Proof.</u> We first consider the case $i \in \{1, 2\}$. If u_1 is 2-eligible in G_{j-1} and $k = 4$ or if u_2 is 2-eligible in G_{j-1} and $k \leq 5$, then, by Lemma 4, $\langle V(J_1)\rangle_{G_t}$ is a clique. In all remaining cases, by $(*)$ and by the choice of j, u_i has a neighbor of degree 1 in $\langle N_{G_{j-1}}(u_i)\rangle_{G_{j-1}}$, $i = 1, 2$, hence u_i cannot be 2-eligible. Symmetrically, $i \notin \{k-1, k\}$.

Hence $3 \leq i \leq k-2$. Then $\langle N_{G_{j-1}}(u_i)\rangle_{G_{j-1}}$ contains a path P that is not in G. By the choice of j, P has no interior vertices, hence P is an edge. But then P is a bad edge in G_{j-1}, a contradiction. □

By the assumption, there is an edge $xy \in E(G_j) \setminus E(G_{j-1})$ such that xy is a bad edge in G_j. By $(*)$ and $(**)$, there are the following two cases.

<u>Case 1:</u> $x \in \{u_1, u_2\}$, $y \in \{u_{k-1}, u_k\}$ and xy is obtained by closing a vertex $z \notin V(J_1)$ that is 2-eligible in G_{j-1}. Then, by Lemma 4, $\{u_1, u_2, u_{k-1}, u_k\} \subset N_{G_{j-1}}(z)$. Since closing at z creates a bad edge, $(k-1) - 2 \leq 4$, i.e. $k \leq 7$. But then, for any k, $4 \leq k \leq 7$, $V(J_1)$ contains a vertex that is 2-eligible in G_j, implying $\langle V(J_1)\rangle_{G_t}$ is a clique.

<u>Case 2:</u> $k = 4$, $x = u_0$, $y \in \{u_3, u_4\}$ or $k = 5$, $x = u_0$, $y = u_4$ (or, symmetrically, $k = 4$, $x = u_5$, $y \in \{u_1, u_2\}$ or $k = 5$, $x = u_5$, $y = u_2$). Then, using Lemma 4, $\langle V(J_1)\rangle_{G_t}$ is again a clique. ■

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