# Graphs with bounded tree-width and large odd-girth are almost bipartite

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**Abstract**

We prove that for every $k$ and every $\varepsilon > 0$, there exists $g$ such that every graph with tree-width at most $k$ and odd-girth at least $g$ has circular chromatic number at most $2 + \varepsilon$.

## 1 Introduction

It has been a challenging problem to prove the existence of graphs of arbitrary high girth and chromatic number [2]. On the other hand, graphs with large girth that avoid a fixed minor are known to have low chromatic number (in particular, this applies to graphs embedded on a fixed surface). More precisely, as Thomassen observed [8], a graph that avoids a fixed minor and has large girth is 2-degenerate, and hence 3-colorable. Further, Galluccio, Goddyn and Hell [3] proved the following theorem, which essentially states that graphs with large girth that avoid a fixed minor are almost bipartite.

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**Theorem 1** (Galluccio, Goddyn and Hell, 2001)**.** *For every graph $H$ and every $\varepsilon > 0$, there exists an integer $g$ such that the circular chromatic number of every $H$-minor free graph of girth at least $g$ is at most $2 + \varepsilon$.*

A natural way to weaken the girth-condition is to require the graphs to have high odd-girth (the *odd-girth* is the length of a shortest odd cycle). However, Young [9] constructed 4-chromatic projective graphs with arbitrary high odd-girth. Thus, the high odd-girth requirement is not sufficient to ensure 3-colorability, even for graphs embedded on a fixed surface. Klostermeyer and Zhang [4], though, proved that the circular chromatic number of every planar graph of sufficiently high odd-girth is arbitrarily close to 2. In particular, the same is true for $K_4$-minor free graphs, i.e. graphs with tree-width at most 2. We prove that the conclusion is still true for any class of graphs of bounded tree-width, which answers a question of Pan and Zhu [6, Question 6.5] also appearing as Question 8.12 in the survey by Zhu [10].

**Theorem 2.** *For every $k$ and every $\varepsilon > 0$, there exists $g$ such that every graph with tree-width at most $k$ and odd-girth at least $g$ has circular chromatic number at most $2 + \varepsilon$.*

Motivated by tree-width duality, Nešetřil and Zhu [5] proved the following theorem.

**Theorem 3** (Nešetřil and Zhu, 1996)**.** *For every $k$ and every $\varepsilon > 0$, there exists $g$ such that every graph $G$ with tree-width at most $k$ and homomorphic to a graph $H$ with girth at least $g$ has circular chromatic number at most $2 + \varepsilon$.*

To see that Theorem 2 implies Theorem 3, observe that if $G$ has an odd cycle of length $g$, then $H$ has an odd cycle of length at most $g$.

# 2 Notation

A $(p, q)$-*coloring* of a graph is a coloring of the vertices with colors from the set $\{0, \ldots, p-1\}$ such that the colors of any two adjacent vertices $u$ and $v$ satisfy $q \le |c(u) - c(v)| \le p - q$. The *circular chromatic number* $\chi_c(G)$ of a graph $G$ is the infimum (and it can be shown to be the minimum) of the ratios $p/q$ such that $G$ has a $(p, q)$-coloring. For every finite graph $G$, it holds that $\chi(G) = \lceil \chi_c(G) \rceil$ and there is $(p, q)$-coloring of $G$ for every $p$ and $q$ with $p/q \ge \chi_c(G)$. In particular, the circular chromatic number of $G$ is at most $2 + 1/k$ if and only if $G$ is homomorphic to a cycle of length $2k + 1$. The reader is referred to the surveys by Zhu [10, 11] for more information about circular colorings.

A *p-precoloring* is a coloring $\varphi$ of a subset $A$ of vertices of a graph $G$ with colors from $\{0, \ldots, p-1\}$, and its *extension* is a coloring of the whole graph $G$ that coincides with $\varphi$ on $A$. The following lemma can be seen as a corollary of a theorem of Albertson and West [1, Theorem 1], and it is the only tool we use from this area.

**Lemma 4.** *For every $p$ and $q$ with $2 < p/q$, there exists $d$ such that any $p$-precoloring of vertices with mutual distances at least $d$ of a bipartite graph $H$ extends to a $(p, q)$-coloring of $H$.*

A *$k$-tree* is a graph obtained from a complete graph of order $k+1$ by adding vertices of degree $k$ whose neighborhood is a clique. The *tree-width* of a graph $G$ is the smallest $k$ such that $G$ is a subgraph of a $k$-tree. Graphs with tree-width at most $k$ are also called *partial $k$-trees*.

A *rooted partial $k$-tree* is a partial $k$-tree $G$ with $k+1$ distinguished vertices $v_1, \ldots, v_{k+1}$ such that there exists a $k$-tree $G'$ that is a supergraph of $G$ and the vertices $v_1, \ldots, v_{k+1}$ form a clique in $G'$. We also say that the partial $k$-tree is *rooted* at $v_1, \ldots, v_{k+1}$. If $G$ is a partial $k$-tree rooted at $v_1, \ldots, v_{k+1}$ and $G'$ is a partial $k$-tree rooted at $v'_1, \ldots, v'_{k+1}$, then the graph $G \oplus G'$ obtained by identifying $v_i$ and $v'_i$ is again a rooted partial $k$-tree (identify the cliques in the corresponding $k$-trees).

Fix $p$ and $q$. If $G$ is a rooted partial $k$-tree, then $\mathcal{F}(G)$ is the set of all $p$-precolorings of the $k+1$ distinguished vertices of $G$ that can be extended to a $(p, q)$-coloring of $G$.

The next lemma is a standard application of results in the area of graphs of bounded tree-width [7].

**Lemma 5.** *Let $k$ and $N$ be positive integers such that $N \geq k+1$. If $G$ is a partial $k$-tree with at least $3N$ vertices, then there exist partial rooted $k$-trees $G_1$ and $G_2$ such that $G$ is isomorphic to $G_1 \oplus G_2$ and $G_1$ has at least $N+1$ and at most $2N$ vertices.*

If $G$ is a partial $k$-tree rooted at $v_1, \ldots, v_{k+1}$, then its *type* is a $(k+1) \times (k+1)$ matrix $M$ such that $M_{ij}$ is the length of the shortest path between the vertices $v_i$ and $v_j$. If there is no such path, $M_{ij}$ is equal to $\infty$. Any matrix $M$ that is a type of a partial rooted $k$-tree satisfies the triangle inequality (setting $\infty + x = \infty$ for any $x$). A symmetric matrix $M$ whose entries are non-negative integers and $\infty$ (and zeroes only on the main diagonal) that satisfies the triangle inequality is a *type*. A type is *bipartite* if $M_{ij} + M_{jk} + M_{ik} \equiv 0 \mod 2$ for any three finite entries $M_{ij}$, $M_{jk}$ and $M_{ik}$. Two bipartite types $M$

and $M'$ are *compatible* if $M_{ij}$ and $M'_{ij}$ have the same parity whenever both of them are finite. We define a binary relation on bipartite types as follows: $M \preceq M'$ if and only if $M$ and $M'$ are compatible and $M_{ij} \leq M'_{ij}$ for every $i$ and $j$. Note that the relation $\preceq$ is a partial order.

We finish this section with the following lemma. Its straightforward proof is included to help us in familiarizing with the just introduced notation.

**Lemma 6.** *Let $G^1$ and $G^2$ be two bipartite rooted partial $k$-trees with types $M^1$ and $M^2$ such that there exists a bipartite type $M^0$ with $M^0 \preceq M^1$ and $M^0 \preceq M^2$. Then the types $M^1$ and $M^2$ are compatible, $G^1 \oplus G^2$ is a bipartite rooted partial $k$-tree and its type $M$ satisfies $M^0 \preceq M$.*

*Proof.* The types $M^1$ and $M^2$ are compatible: if both $M^1_{ij}$ and $M^2_{ij}$ are finite, then $M^0_{ij}$ is finite and has the same parity as $M^1_{ij}$ and $M^2_{ij}$. Hence, the entries $M^1_{ij}$ and $M^2_{ij}$ have the same parity.

Let $M$ be the type of $G^1 \oplus G^2$. Note that it does not hold in general that $M_{ij} = \min\{M^1_{ij}, M^2_{ij}\}$. We show that $M^0 \preceq M$ which will also imply that $G^1 \oplus G^2$ is bipartite since $M^0$ is a bipartite type. Consider a shortest path $P$ between two distinguished vertices $v_i$ and $v_{i'}$ and split $P$ into paths $P_1, \ldots, P_\ell$ delimited by distinguished vertices on $P$. Note that $\ell \leq k$ since $P$ is a path. Let $j_0 = i$ and let $j_i$ be the index of the end-vertex of $P_i$ for $i \in \{1, \ldots, \ell\}$. In particular, $j_\ell = i'$. Each of the paths $P_1, \ldots, P_\ell$ is fully contained in $G^1$ or in $G^2$ (possibly in both if it is a single edge). Since $M^0 \preceq M^1$ and $M^0 \preceq M^2$, the length of $P_i$ is at least $M^0_{j_{i-1}j_i}$, and it has the same parity as $M^0_{j_{i-1}j_i}$. Since $M^0$ is a bipartite type (among others, it satisfies the triangle inequality), the length of $P$, which is $M_{ii'}$, has the same parity as $M^0_{j_0 j_\ell} = M^0_{ii'}$ and is at least $M^0_{ii'}$. This implies that $M^0 \preceq M$. $\square$

# 3 The Main Lemma

In this section, we prove a lemma which forms the core of our argument. To this end, we first prove another lemma that asserts that for every $k$, $p$ and $q$, the set of types of all bipartite rooted partial $k$-trees forbidding a fixed set of $p$-precolorings from extending (and maybe some other precolorings, too) has always a maximal element. We formulate the lemma slightly differently to facilitate its application.

**Lemma 7.** *For every $k$, $p$ and $q$, there exists a finite number of (bipartite) types $M^1, \ldots, M^m$ such that for any bipartite rooted partial $k$-tree $G$ with type $M$, there exists a bipartite rooted partial $k$-tree $G'$ with type $M^i$ for some $i \in \{1, \ldots, m\}$ such that $\mathcal{F}(G') \subseteq \mathcal{F}(G)$ and $M \preceq M^i$.*

*Proof.* Let $d \geq 2$ be the constant from Lemma 4 applied for $p$ and $q$. Let $M^1, \ldots, M^m$ be all bipartite types with entries from the set $\{1, \ldots, D^{(k+1)^2}\} \cup \{\infty\}$ where $D = 4d$. Thus, $m$ is finite and does not exceed $(D^{(k+1)^2} + 1)^{k(k+1)/2}$.

Let $G$ be a bipartite rooted partial $k$-tree with type $M$. If $M$ is one of the types $M^1, \ldots, M^m$, then there is nothing to prove (just choose $i$ such that $M = M^i$). Otherwise, one of its entries is finite and exceeds $D^{(k+1)^2}$.

For $i \in \{1, \ldots, (k+1)^2\}$, let $J^i$ be the set of all positive integers between $D^{i-1}$ and $D^i - 1$ (inclusively). Let $i_0$ be the smallest integer such that no entry of $M$ is contained in $J^{i_0}$. Since $M$ has at most $k(k+1)/2$ different entries, such an index $i_0$ exists. Note that if $i_0 = 1$, then Lemma 4 implies that $\mathcal{F}(G)$ contains all possible $p$-precolorings, and the sought graph $G'$ is the bipartite rooted partial $k$-tree composed of $k+1$ isolated vertices, with the all-$\infty$ type.

Two vertices $v_i$ and $v_j$ at which $G$ is rooted are *close* if $M_{ij}$ is at most $D^{i_0-1}$. The relation $\approx$ of being close is an equivalence relation on $v_1, \ldots, v_{k+1}$. Indeed, it is reflexive and symmetric by the definition, and we show now that it is transitive. Suppose that $M_{ij}$ and $M_{jk}$ are both at most $D^{i_0-1}$. Then, the distance between $v_i$ and $v_k$ is at most $M_{ij} + M_{jk} \leq 2D^{i_0-1} - 2 \leq D^{i_0} - 1$ since $D \geq 2$. Consequently, by the choice of $i_0$, the distance between $v_i$ and $v_k$ is at most $D^{i_0-1} - 1$ and thus $v_i \approx v_k$.

Let $C_1, \ldots, C_\ell$ be the equivalence classes of the relation $\approx$. Note that $C_1, \ldots, C_\ell$ is a finer partition than that given by the equivalence relation of being connected.

Since $G$ is bipartite, we can partition its vertices into two color classes, say red and blue. For every $i \in \{1, \ldots, \ell\}$, contract the closed neighborhood of a vertex $v$ if $v$ is a blue vertex and its distance from any vertex of $C_i$ is at least $D^{i_0-1}$ and keep doing so as long as such a vertex exists. Observe that the resulting graph is uniquely defined. After discarding the components that do not contain the vertices of $C_i$, we obtain a bipartite partial $k$-tree $G_i$ rooted at the vertices of $C_i$: it is bipartite as we have always contracted closed neighborhoods of vertices of the same color (blue) to a single (red) vertex, and its tree-width is at most $k$ since the tree-width is preserved by contractions. Moreover, the distance between any two vertices of $C_i$ has not decreased since any path between them through any of the newly arising vertices has length at least $2D^{i_0-1} - 2 \geq D^{i_0-1}$.

Now, let $G'$ be the bipartite rooted partial $k$-tree obtained by taking the disjoint union of $G_1, \ldots, G_\ell$. The type $M'$ of $G'$ can be obtained from the type of $G$: set $M'_{ij}$ to be $M_{ij}$ if the vertices $v_i$ and $v_j$ are close, and $\infty$ otherwise. Thus, $M'$ is one of the types $M^1, \ldots, M^m$ and $M \preceq M'$. It remains to show that $\mathcal{F}(G') \subseteq \mathcal{F}(G)$.

Let $c \in \mathcal{F}(G')$ be a $p$-precoloring that extends to $G'$, and recall that $D \geq 4$. For $i \in \{1, \ldots, \ell\}$, let $A_i$ be the set of all red vertices at distance at most $D^{i_0-1}$ and all blue vertices at distance at most $D^{i_0-1} - 1$ from $C_i$, and let $R_i$ be the set of all red vertices at distance $D^{i_0-1} - 1$ or $D^{i_0-1}$ from $C_i$. Set $B_i = A_i \setminus R_i$ ($B_i$ is the "interior" of $A_i$ and $R_i$ its "boundary"). The extension of $c$ to $G_i$ naturally defines a coloring of all vertices of $A_i$: $G_i$ is the subgraph of $G$ induced by $A_i$ with some red vertices of $R_i$ identified (two vertices of $R_i$ are identified if and only if they are in the same component of the graph $G - B_i$).

Let $H$ be the following auxiliary graph obtained from $G$: remove the vertices of $B = B_1 \cup \cdots \cup B_\ell$ and, for $i \in \{1, \ldots, \ell\}$, identify every pair of vertices of $R_i$ that are in the same component of $G - B$. Let $R$ be the set of vertices of $H$ corresponding to some vertices of $R_1 \cup \cdots \cup R_\ell$. Precolor the vertices of $R$ with the colors given by the colorings of $G_i$ (note that two vertices of $R_i$ in the same component of $G - B_i$ are also in the same component of $G - B$, so this is well-defined). The graph $H$ is bipartite as only red vertices have been identified. The distance between any two precolored vertices is at least $d$: consider two precolored vertices $r$ and $r'$ at distance at most $d - 1$. Let $i$ and $i'$ be such that $r \in R_i$ and $r' \in R_{i'}$. If $i = i'$, then $r$ and $r'$ are in the same component of $G - B$ and thus $r = r'$. If $i \neq i'$ then by the definition of $R_i$ and $R_{i'}$, the vertex $r$ is in $G$ at distance at most $D^{i_0-1}$ from some vertex $v$ of $C_i$ and $r'$ is at distance at most $D^{i_0-1}$ from some vertex $v'$ of $C_{i'}$. So, the distance between $v$ and $v'$ is at most $2D^{i_0-1} + d < D^{i_0} - 1$. Since $M$ has no entry from $J^{i_0}$, the vertices $v$ and $v'$ must be close and thus $i = i'$, a contradiction.

Since the distance between any two precolored vertices is at least $d$, the precoloring extends to $H$ by Lemma 4 and in a natural way it defines a coloring of $G$. We conclude that every $p$-precoloring that extends to $G'$ also extends to $G$ and thus $\mathcal{F}(G') \subseteq \mathcal{F}(G)$. □

We now prove our main lemma, which basically states that there is only a finite number of bipartite rooted partial $k$-trees that can appear in a minimal non-$(p, q)$-colorable graph with tree-width $k$ and a given odd girth.

**Lemma 8.** *For every $k$, $p$ and $q$, there exist a finite number $m$ and bipartite rooted partial $k$-trees $G^1, \ldots, G^m$ with types $M^1, \ldots, M^m$ such that for any bipartite rooted partial $k$-tree $G$ with type $M$ there exists $i$ such that $\mathcal{F}(G^i) \subseteq \mathcal{F}(G)$ and $M \preceq M^i$.*

*Proof.* Let $M^1, \ldots, M^m$ be the types from Lemma 7. We define the graph $G^i$ as follows: for every $p$-precoloring $c$ that does not extend to a bipartite partial rooted $k$-tree with type $M^i$, fix any partial rooted $k$-tree $G^i_c$ with type $M^i$ such that $c$ does not extend to $G^i_c$. Set $G^i = \bigoplus_c G^i_c$, where $c$ runs over all such $p$-precolorings. If the above sum of partial $k$-trees is non-empty, then the type $M$ of $G^i$ is $M^i$. Indeed, $M \preceq M^i$ by the definition of $G^i$, and Lemma 6 implies that $M^i \preceq M$. If all the $p$-precolorings of the $k+1$ vertices in the root extend to each partial $k$-tree of type $M^i$, then let $G^i$ be the graph consisting of $k+1$ isolated vertices. This happens in particular for the all-$\infty$ type.

Let us verify the statement of the lemma. Let $G$ be a bipartite rooted partial $k$-tree and let $M$ be the type of $G$. If $\mathcal{F}(G)$ is composed of all $p$-precolorings, the sought graph $G^i$ is the one composed of $k+1$ isolated vertices. Hence, we assume that $\mathcal{F}(G)$ does not contain all $p$-precolorings, i.e., there are $p$-precolorings that do not extend to $G$. By Lemma 7, there exists a bipartite rooted partial $k$-tree $G'$ with type $M'$ such that $M \preceq M' = M^i$ for some $i$ and $\mathcal{F}(G') \subseteq \mathcal{F}(G)$. For every $p$-precoloring $c$ that does not extend to $G'$ (and there exists at least one such $p$-precoloring $c$), some graph $G^i_c$ has been glued into $G^i$. Hence, $\mathcal{F}(G^i) \subseteq \mathcal{F}(G') \subseteq \mathcal{F}(G)$. Since the type of $G^i$ is $M^i$, the conclusion of the lemma follows. □

# 4 Proof of Theorem 2

We are now ready to prove Theorem 2, which is recalled below.

**Theorem 2.** *For every $k$ and every $\varepsilon > 0$, there exists $g$ such that every graph with tree-width at most $k$ and odd-girth at least $g$ has circular chromatic number at most $2+\varepsilon$.*

*Proof.* Fix $p$ and $q$ such that $2 < p/q \leq 2+\varepsilon$. Let $G^1, \ldots, G^m$ be the bipartite partial $k$-trees from Lemma 8 applied for $k$, $p$ and $q$. Set $N$ to be the largest order of the graphs $G^i$ and set $g$ to be $3N$. We assert that each partial $k$-tree with odd-girth $g$ has circular chromatic number at most $p/q$. Assume that this is not the case and let $G$ be a counterexample with the fewest vertices.

The graph $G$ has at least $3N$ vertices (otherwise, it has no odd cycles and thus it is bipartite). By Lemma 5, $G$ is isomorphic to $G_1 \oplus G_2$, where $G_1$ and $G_2$ are rooted partial $k$-trees and the number of vertices of $G_1$ is between $N+1$ and $2N$. By the choice of $g$, the graph $G_1$ has no odd cycle and thus it is a bipartite rooted partial $k$-tree. By Lemma 8, there exists $i$ such that $\mathcal{F}(G^i) \subseteq \mathcal{F}(G_1)$ and $M_1 \preceq M^i$ where $M_1$ is the type of $G_1$ and $M^i$ is the type of $G^i$. Let $G'$ be the partial $k$-tree $G^i \oplus G_2$.

First, $G'$ has fewer vertices than $G$ since the number of vertices of $G^i$ is at most $N$ and the number of vertices of $G_1$ is at least $N+1$. Second, $G'$ has no $(p,q)$-coloring: if it had a $(p,q)$-coloring, then the corresponding

$p$-precoloring of the $k+1$ vertices shared by $G^i$ and $G_2$ would extend to $G_1$ since $\mathcal{F}(G^i) \subseteq \mathcal{F}(G_1)$ and thus $G$ would have a $(p,q)$-coloring, too. Finally, $G'$ has no odd cycle of length at most $g$: if it had such a cycle, replace any path between vertices $v_j$ and $v_{j'}$ of the root of $G^i$ with a path of at most the same length between them in $G_1$ (recall that $M_1 \preceq M^i$). If such paths for different pairs of $v_j$ and $v_{j'}$ on the considered odd cycle intersect, take their symmetric difference. In this way, we obtain an Eulerian subgraph of $G = G_1 \oplus G_2$ with an odd number of edges such that the number of its edges does not exceed $g$. Consequently, this Eulerian subgraph has an odd cycle of length at most $g$, which violates the assumption on the odd-girth of $G$. We conclude that $G'$ is a counterexample with less vertices than $G$, a contradiction. □

We end by pointing out that the approach used yields an upper bound of $3(k+1) \cdot 2^{2^{p^{k+1}}((4d)^{(k+1)^2}+1)^{k^2}}$ for the smallest $g$ such that all graphs with tree-width at most $k$ and odd-girth at least $g$ have circular chromatic number at most $p/q$, whenever $p/q > 2$. More precisely, the value of $N$ cannot exceed $(k+1) \cdot 2^{2^{p^{k+1}}((4d)^{(k+1)^2}+1)^{k^2}}$. To see this, we consider all pairs $P = (C, M)$ where $C$ is a set of $p$-precolorings of the root and $M$ is a type such that there is a bipartite rooted partial $k$-tree of type $M$ to which no coloring of $C$ extends. Let $n_P$ be the size of a smallest such partial $k$-tree. We obtain a sequence of at most $2^{p^{k+1}} \times \left((4d)^{(k+1)^2}+1\right)^{k^2}$ integers. The announced bound follows from the following fact: if the sequence is sorted in increasing order, then each term is at most twice the previous one.

Indeed, consider the tree-decomposition of the partial $k$-tree $G_P$ chosen for the pair $P$. If the bag containing the root has a single child, then we delete a vertex of the root, and set a vertex in the single child to be part of the root. We obtain a partial $k$-tree to which some $p$-precolorings of $C$ do not extend. Thus, $n_P \leq 1 + n_{P'}$ for some pair $P'$ and $n_{P'} < n_P$. If the bag containing the root has more than one child, then $G_P$ can be obtained by identifying the roots of two smaller partial $k$-trees $G$ and $G'$. By the minimality of $G_P$, the orders of $G$ and $G'$ are $n_{P_1}$ and $n_{P_2}$ for two pairs $P_1$ and $P_2$ such that $n_{P_i} < n_P$ for $i \in \{1,2\}$. This yields the stated fact, which in turn implies the given bound, since the smallest element of the sequence is $k+1$.

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