# POINTFREE ASPECTS OF THE $T_D$ AXIOM OF CLASSICAL TOPOLOGY

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Abstract. Abstraction from the condition defining $T_D$-spaces leads to the following notion in an arbitrary frame $L$: a filter $F$ in $L$ is called *slicing* if it is prime and there exist $a, b \in L$ such that $a \notin F$, $b \in F$, and $a$ is covered by $b$. This paper deals with various aspects of these slicing filters. As a first step, we present several results about the original $T_D$ condition. Next, concerning slicing filters, we show they are completely prime and characterize them in various ways. In addition, we prove for the frames $\mathfrak{O}X$ of open subsets of a space $X$ that every slicing filter is an open neighbourhood filter $\mathcal{U}(x)$ and $X$ is $T_D$ iff every $\mathcal{U}(x)$ is slicing. Further, for $\mathbf{Top}_D$ and $\mathbf{Frm}_D$ the categories of $T_D$ spaces and their continuous maps, and all frames and those homomorphisms whose associated spectral maps preserve the completely prime elements, respectively, we show that the usual contravariant functors between **Top** and **Frm** induce analogous functors here, providing a dual equivalence between $\mathbf{Top}_D$ and the subcategory of $\mathbf{Frm}_D$ given by the $T_D$-spatial frames (not coinciding with the spatial ones). In addition, we show that $\mathbf{Top}_D$ is mono-coreflective in a suitable subcategory of **Top**. Finally, we provide a comparison between $T_D$-separation and sobriety showing they may be viewed, in some sense, as mirror images of each other.

## Introduction

This paper is motivated by the familiar topological separation axiom

$T_D$: Every point $x$ has an open neighbourhood $U$ such that $U \smallsetminus \{x\}$ is open,

originally introduced by Aull and Thron [1] as a condition notable because it is strictly between $T_0$ and $T_1$. On the other hand, and perhaps more importantly, there is the result that, for $T_D$-spaces $X$ and $Y$, $X \cong Y$ whenever their lattices of open sets are isomorphic ([5, 11]). Here, this will be seen as an immediate consequence of the contravariant adjointness we establish between the category of $T_D$-spaces and their continuous maps, and a suitable subcategory of the category of all frames.

The tool for this is the following general concept, meaningfull in pointfree topology, that is, for arbitrary frames $L$, abstracted from the

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condition which $T_D$ imposes on the open neighbourhood filters of the points: a filter $F$ in $L$ is called *slicing* if it is prime and there exist $a, b \in L$ such that $a \notin F$, $b \in F$, and $a$ is covered by $b$. The aim of this paper is to investigate the properties of these slicing filters and their general rôle.

As a first step, we characterize the $T_D$-points of a space $X$, that is, the $x \in X$ satisfying the condition which $T_D$ requires of all points, in a variety of ways (2.3.2). Next, turning to the slicing filters in an arbitrary frame, we show they are completely prime (2.6.1) and then characterize them among all completely prime filters in several ways (2.6.2). Further, for the frame $\mathfrak{O}X$ of open subsets of a space $X$, we show that every slicing filter is the open neighbourhood filter $\mathcal{U}(x)$ of some $T_D$-point $x \in X$ (2.7.1) and $X$ is $T_D$ iff every $\mathcal{U}(x)$ is slicing (2.7.2).

Next, we consider the category $\mathbf{Top}_D$ of all $T_D$-spaces and their continuous maps and the category $\mathbf{Frm}_D$ of all frames and those homomorphisms $h : L \to M$ for which the spectral map $p \mapsto h_*(p)$ takes any *completely prime* $p \in M$ to a *completely prime* element in $L$ (see 1.5, 3.1.1.2; the right adjoint $h_*$ of a standard homomorphism $h$ just takes *primes* to *primes*). We show that there are contravariant functors $\mathbf{Top}_D \to \mathbf{Frm}_D$ and $\mathbf{Frm}_D \to \mathbf{Top}_D$, adjoint on the right, restricting the usual functors between $\mathbf{Top}$ and $\mathbf{Frm}$ (3.5.1) which induces a dual equivalence between $\mathbf{Top}_D$ and the subcategory of $\mathbf{Frm}_D$ given by the $T_D$-spatial frames. In addition, we obtain that $\mathbf{Top}_D$ is monocoreflective in the subcategory of $\mathbf{Top}$ given by the continuous maps $f : X \to Y$ for which $\mathfrak{O}f : \mathfrak{O}Y \to \mathfrak{O}X$ belongs to $\mathbf{Frm}_D$ (3.7.2).

Finally, we compare $T_D$-separation with sobriety, based on the relation $R(X, Y)$ between spaces which says that "$X$ is a proper subspace of $Y$ such that $(U \mapsto U \cap Y) : \mathfrak{O}Y \to \mathfrak{O}X$ is an isomorphism". We show that the two properties may be regarded, in some sense, as mirror images of each other (($X$ is sober iff $\forall Y, \neg R(X, Y)$, $Y$ is $T_D$ iff $\forall X, \neg R(X, Y)$, see 4.3).

## 1. Preliminaries

**1.1. Frames and their homomorphisms.** Recall that a *frame* is a complete lattice $L$ satisfying the distributivity law

$$a \wedge \bigvee B = \bigvee \{a \wedge b \mid b \in B\}$$

for all $a \in L$ and $B \subseteq L$. A *frame homomorphism* $h : L \to M$ preserves arbitrary joins (including the bottom 0) and all *finitary* meets (including the top 1). As usual, the resulting category will be denoted by

$$\mathbf{Frm}.$$

If $X$ is a topological space we have the frame $\mathfrak{O}X$ of its open sets, and if $f : L \to M$ is a continuous map then $\mathfrak{O}f = (U \mapsto f^{-1}[U]) : \mathfrak{O}Y \to \mathfrak{O}X$

is a frame homomorphism. Thus we have a contravariant functor

$$\mathfrak{O} : \mathbf{Top} \to \mathbf{Frm}.$$

**1.2. Primes and complete primes.** An element $p \neq 1$ of a frame $L$ is said to be *prime* (alternatively, also called *meet-irreducible*) if

$$p = a \wedge b \quad \Rightarrow \quad p = a \ \text{ or } \ p = b$$

$$(\text{equivalently,} \ \ p \geq a \wedge b \quad \Rightarrow \quad p \geq a \ \text{ or } \ p \geq b),$$

and *completely prime* if

$$p = \bigwedge S \quad \Rightarrow \quad p \in S$$

for any subset $S \subseteq L$.

**1.3. Right adjoints to frame homomorphism.** Since a frame homomorphism $h : L \to M$ preserves all suprema it has a right adjoint

$$h_* : M \to L$$

(satisfying $h(a) \leq b$ iff $a \leq h_*(b)$). The following is a well-known

**Fact.** *If $p$ is prime in $M$ then $h_*(p)$ is prime in $L$.*

(Indeed, if $a \wedge b \leq h_*(p)$ then $h(a) \wedge h(b) \leq p$ and hence, say, $h(a) \leq p$ and $a \leq h_*(p)$.)

**1.4. Spectrum.** Among the various ways of defining the spectrum $\Sigma L$ of a frame we have the following

$$\Sigma L = (\{p \in L \mid P \text{ prime}\}, \{\Sigma_a \mid a \in L\})$$

where $\Sigma_a = \{p \mid a \not\leq p\}$: further, for a frame homomorphism $h : L \to M$ we have and the map $\Sigma h = (p \mapsto h_*(p)) : \Sigma M \to \Sigma L$. The functor

$$\Sigma : \mathbf{Frm} \to \mathbf{Top}$$

thus obtained and the $\mathfrak{O}$ from 1.1 are adjoint on the right.

**1.5. Prime and completely prime filters.** A filter $F$ in a frame $L$ is *prime* if it is proper and

$$a \vee b \in F \quad \Rightarrow \quad a \in F \ \text{ or } \ b \in F,$$

and *completely prime* if for any subset $S \subseteq L$,

$$\bigvee S \in F \quad \Rightarrow \quad S \cap F \neq \emptyset.$$

**1.5.1.** For any completely prime filter $F$ we have a prime element

$$a = \bigvee \{x \mid x \notin F\}. \tag{$*$}$$

It will be called the *associated prime* of $F$. Note that, on the other hand, if $a$ is a prime element then

$$F = \{x \mid x \not\leq a\} \tag{$**$}$$

is a completely prime filter and

*the formulas* (∗) *and* (∗∗) *constitute a one-one correspondence between the prime elements and the completely prime filters in a frame* $L$*.*

**1.5.2. The fixed filters.** We will use the notation

$$\mathcal{U}(x) = \{U \in \mathfrak{O}X \mid x \in U\}$$

for the *fixed filters* in $\mathfrak{O}X$. Note that each $\mathcal{U}(x)$ is completely prime, the associated prime element of $\mathcal{U}(x)$ being $X \smallsetminus \overline{\{x\}}$.

**1.6. Special topological spaces.** Since the restriction to $T_0$-spaces has no impact on the point-free aspects of spaces, that is, their frames of open sets, we assume throughout that all spaces are $T_0$.

Recall that a space $X$ is *sober* if the open sets $X \smallsetminus \overline{\{x\}}$ are precisely the prime elements of $\mathfrak{O}X$. Equivalently,

*$X$ is sober iff each completely prime filter in $\mathfrak{O}X$ is fixed*

(see [6, 5, 7]). A space $X$ is $T_D$ if

$$\forall x \in X\ \exists \text{ open } U \ni x \text{ such that } U \smallsetminus \{x\} \text{ is open.}$$

This notion was introduced independently in [1] and [5] (see also [11]).

**1.7. $T_D$-spatial frames.** Recall that a frame is *spatial* if it is isomorphic with some $\mathfrak{O}X$. If there is a $T_D$-space $X$ with this property we call $L$ *$T_D$-spatial.*

For more about frames see [7] or [9].

For the basic categorical facts used here see the standard texts, e.g., [8].

## 2. $T_D$-points and slicing filters

**2.1.** An element $a$ in a poset is said to be *covered* by $b$, notation

$$a \lessdot b,$$

if $a < b$ and $a \leq x \leq b$ implies $a = x$ or $x = b$. We say that $a$ is covered if it is covered by some $b$.

**2.1.1. Lemma.** 1. *In a semilattice, if a prime element $a$ is covered by $b$ then for each $x > a$ one has $x \geq b$.*

2. *In a complete lattice, if $a$ is completely prime then it is covered; further, if $a$ is prime and covered then it is completely prime.*

*Proof.* 1. We have $a \leq b \wedge x \leq b$; by the primeness, $a \neq b \wedge x$, hence $b \wedge x = b$, that is, $b \leq x$.

2. $a < b = \bigwedge\{c \mid a < c\}$ ($a = b$ is excluded by complete primeness). Thus, if $a < c \leq b$ then $b \leq c \leq b$, showing $a \lessdot b$.

If $a = \bigwedge S$ and if it is prime and covered by $b$ then $b \leq s$ for any $s \in S$ that is not $a$, and hence some of the $s \in S$ is equal to $a$. □

**2.1.2. Lemma.** 1. *In the frame* $\mathfrak{O}X$ *we have* $U \lessdot V$ *iff* $U = V \smallsetminus \{x\}$ *for some* $x \in X$.

2. *If* $X \smallsetminus \overline{\{x\}} \lessdot V$ *then* $V = (X \smallsetminus \overline{\{x\}}) \cup \{x\}$.

*Proof.* 1. Suppose $x, y \in V \smallsetminus U$, $x \neq y$. Choose an open set $W$ such that, say $x \notin W \ni y$. Then

$$U \subsetneq U \cup (W \cap V) \subsetneq V$$

(since $x \notin U \cup (W \cap V) \ni y$), a contradiction.

2. By 1, $V = (X \smallsetminus \overline{\{x\}}) \cup \{y\}$ with $y \notin X \smallsetminus \overline{\{x\}}$, that is, $y \in \overline{\{x\}}$. Thus, $x \in V$ and hence $(X \smallsetminus \overline{\{x\}}) \cup \{x\} \subseteq V$ so that

$$((X \smallsetminus \overline{\{x\}}) \cup \{x\}) \smallsetminus \{y\} = X \smallsetminus \overline{\{x\}},$$

showing $x = y$, and the statement follows. □

**2.2.** A point $x \in X$ is said to be a $T_D$*-point* if there is an open $U \ni x$ such that $U \smallsetminus \{x\}$ is open. In particular, then, $X$ is a $T_D$-space iff all its points are $T_D$.

**2.2.1. Proposition.** $x \in X$ *is s* $T_D$*-point iff the set* $(X \smallsetminus \overline{\{x\}}) \cup \{x\}$ *is open.*

*Proof.* If $(X \smallsetminus \overline{\{x\}}) \cup \{x\} = U$ is open take $V = X \smallsetminus \overline{\{x\}} = U \smallsetminus \{x\}$. On the other hand, let $x$ be a $T_D$-point, with $U \ni x$, $U$ open and $U \smallsetminus \{x\}$ open. Then $U \smallsetminus \{x\} = U \smallsetminus \overline{\{x\}}$. Now $(X \smallsetminus \overline{\{x\}}) \cup \{x\}$ is a neighbourhood of all the $y \in X \smallsetminus \overline{\{x\}}$. But it is a neighbourhood of $x$ as well: we have $x \in U = (U \smallsetminus \overline{\{x\}}) \cup \{x\} \subseteq (X \smallsetminus \overline{\{x\}}) \cup \{x\}$. □

**2.2.2. Corollary.** 1. *If* $x \in X$ *is a* $T_D$*-point then it is a* $T_D$*-point in every subspace* $Y \subseteq X$ *such that* $x \in Y$.

2. *Consequently the subspace*

$$\{x \in X \mid x \textit{ is a } T_D\textit{-point in } X\} \subseteq X$$

*is a* $T_D$*-space.*

**2.3.1. Observation.** *In any space* $X$, $X \smallsetminus \overline{\{x\}}$ *is a prime in* $\mathfrak{O}X$.

**2.3.2. Proposition.** *Let* $x$ *be a point in a space* $X$. *Then the following statements are equivalent.*

1. $x$ *is a* $T_D$*-point.*
2. $X \smallsetminus \overline{\{x\}}$ *is completely prime in* $\mathfrak{O}X$.
3. $X \smallsetminus \overline{\{x\}}$ *is covered in* $\mathfrak{O}X$.

*Proof.* (1)⇒(2). Let $X \smallsetminus \overline{\{x\}} = \bigwedge \mathcal{S}$ and let $x$ be a $T_D$-point. Since $X \smallsetminus \overline{\{x\}}$ is prime then by 2.1.1 and 2.1.2, each $U \in \mathcal{S}$ is either $X \smallsetminus \overline{\{x\}}$ or $(X \smallsetminus \overline{\{x\}}) \cup \{x\}$. But it cannot always be the latter.

(2)⇒(3) by 2.2.1, 2.

(3)⇒(1) by 2.2.2. □

**2.4. Lemma.** *Let $a \lessdot b$ in a distributive lattice and let $a \vee c < b \vee c$. Then $a \vee c \lessdot b \vee c$.*

*Proof.* Let $a \vee c \leq x \leq b \vee c$. Then $a = a \wedge x \leq b \wedge x \leq b$ and hence either $a = b \wedge x$ and then

$$a \vee c = (b \wedge x) \vee c = (b \vee c) \wedge (x \vee c) \geq (b \vee c) \wedge x = x,$$

or $b = b \wedge x$, that is, $b \leq x$, and since $c \leq x$ we have

$$b \vee c \leq x \leq b \vee c. \qquad \square$$

**2.5.** For an ideal $A$ and a filter $B$ in a distributive lattice $L$ define the filter

$$A \uparrow B = \{x \mid \exists a \in A \ \exists b \in B, \ a \vee x \leq b\}$$

and extend the definition for arbitrary subsets $A, B$ by considering $A \uparrow B = \mathsf{Idl} A \uparrow \mathsf{Flt} B$. In particular

$$a \uparrow b = \{a\} \uparrow \{b\} = \{x \mid a \vee x \geq b\}.$$

This operation is due to Ball, [2] (see also [3]).

**2.5.1. Observations.** 1. *$a \uparrow b$ is always non-empty, and it is proper iff $a \ngeq b$.*

2. *$a \uparrow b = a \uparrow (a \vee b)$.*

Consequently, since we will be interested in proper filters we will always assume that $a \ngeq b$ and in view of 2 we will typically work with $a < b$.

**2.5.2. Lemma.** *Let $F$ be a completely prime filter in a frame and $p$ its associated prime. Assume $F = a \uparrow b$ so that*

$$x \nleq p \quad \textit{iff} \quad a \vee x \geq b.$$

*Then*

$$a \uparrow b = p \uparrow (b \vee p)$$

*and*

$$p \lessdot b \vee p.$$

*Proof.* As is familiar, $p \neq 1$ and $a \vee a = a \vee 0 \ngeq b$, and hence $a \notin F$, and

$$(*) \qquad a \leq p.$$

Since $p \leq p$ we have

$$(**) \qquad p = a \vee p \ngeq b.$$

By Observation 2.5.1, for the first statement it suffices to prove that

$$a \uparrow b = p \uparrow b.$$

If $a \vee x \geq b$ then $p \vee x \geq b$ by $(*)$; if $a \vee x \ngeq b$ then $x \leq p$ and $p \vee x = p \ngeq b$ by $(**)$.

Further, let $p \leq x \leq b \vee p$. If $p < x$ we have $x \nleq p$ and hence $p \vee x \geq b$, and finally $x = p \vee x \geq b \vee p$. $\square$

**2.6.** A *slicing filter* on a frame $L$ is a prime filter $F$ for which there exist $a \lessdot b$ such that

$$a \notin F \ni b.$$

**2.6.1. Lemma.** *Any slicing filter is completely prime.*

*Proof.* Let $a \notin F \ni b$ for some $a \lessdot b$. $\bigvee S \in F$ implies that

$$a \vee (b \wedge \bigvee S) = \bigvee \{a \vee (b \wedge s) \mid s \in S\} \in F.$$

For any $s \in S$ we have

$$a \leq a \vee (b \wedge s) \leq b$$

and hence

(i) either $a = a \vee (b \wedge s)$, that is, $b \wedge s \leq a$,
(ii) or $a \vee (b \wedge s) = b$.

If all the $s$ followed (i) we had $b \wedge \bigvee S \leq a$ contradicting $a \notin F$. Thus, there exists an $s \in S$ such that $b = a \vee (b \wedge s) \in F$, and since $F$ is prime and $a \notin F$, $s \geq b \wedge s \in F$. □

**2.6.2. Proposition.** *Let $F$ be a prime filter in a frame. Then the following statements are equivalent.*

(1) *$F$ is a slicing filter.*
(2) *$F$ is a completely prime filter and the associated prime is covered.*
(3) *$F$ is a completely prime filter and the associated prime is completely prime.*
(4) *$F$ is a completely prime filter of the form $a{\uparrow}b$.*

*Proof.* (1)⇒(2): By 2.6.1, $F$ is completely prime. Let $a \lessdot b$, $a \notin F \ni b$, and let $p$ be the element associated with $F$. Then $p < p \vee b$ since $b \leq p$ contradicts $p \notin F$. On the other hand, $a \leq p$ since $a \notin F$, and we have $p = a \vee p < b \vee p$, and $p \lessdot b \vee p$ by lemma 2.4.

(2)⇔(3) by lemma 2.1.1.

(2)⇒(4): Take an $F = \{x \mid x \nleq p\}$ with $p \lessdot b$. If $x \in T$ (that is, $x \nleq p$) then $p \vee x$ and $b$ are in $F$ and hence $(p \vee x) \wedge b = b$ (the meet is not $\leq p$) so that $p \vee x \geq b$, and $x \in p{\uparrow}b$.

On the other hand, if $x \in a{\uparrow}b$ $(= p{\uparrow}b$, recall 1.2.3) then $x \vee a \geq b \in F$, and as $F$ is prime, $x \in F$.

(4)⇒(1) immediately follows from 2.2.2. □

**2.6.3. Notes.** 1. The statement (4) is a reformulation of a property used in characterizing $T_D$ spaces in [10].

2. The statement (3) restricts the correspondence from 1.5.1:

> *the formulas* $(*)$ *and* $(**)$ *constitute a one-one correspondence between the* completely prime *elements and the* slicing *filters in a frame $L$.*

**2.7.1. Proposition.** *In an arbitrary ($T_0$) space,*

1. *every slicing filter in $\mathfrak{O}X$ is fixed, and*
2. *the fixed filter $\mathcal{U}(x)$ is slicing iff there is a $U \in \mathcal{U}(x)$ such that $U \smallsetminus \{x\}$ is open.*

*Proof.* 1. Let $W$ be the prime open set associated with a slicing filter $\mathcal{F}$. Let $W \lessdot V$ and let $x \in X$ be the point for which $W = V \smallsetminus \{x\}$ (recall 2.1.2, 2.6.2(2)). We have $W = V \smallsetminus \overline{\{x\}}$ and hence $W = V \cap (X \smallsetminus \overline{\{x\}})$ and as it is prime, $W = X \smallsetminus \overline{\{x\}}$. Now

$$\mathcal{F} = \{U \in \mathfrak{O}X \mid U \nsubseteq X \smallsetminus \overline{\{x\}}\} = \{U \mid U \cap \overline{\{x\}} \neq \emptyset\} = \mathcal{U}(x).$$

2. $\mathcal{U}(x)$ is a slicing filter iff there are $V \notin \mathcal{U}(x) \ni U$ such that $V \lessdot U$. By 2.1.2 this amounts to $V = U \smallsetminus \{x\}$. □

**2.7.2. Corollary.** *A space $X$ is $T_D$ iff each fixed filter $\mathcal{U}(x)$ in $\mathfrak{O}X$ is slicing.*

## 3. $T_D$-spectrum and $T_D$-coreflection

**3.1.** The category of all $T_D$-spaces and continuous maps will be denoted by

$$\mathbf{Top}_D.$$

Further, a frame homomorphism $h : L \to M$ will be called a *$D$-homomorphism* if for each completely prime $p$ in $M$, the right adjoint image $h_*(p)$ is completely prime in $L$. The category of all frames and $D$-homomorphisms will be denoted by

$$\mathbf{Frm}_D.$$

**3.1.1. Remarks.** 1. Note that $\mathbf{Top}_D$ is obtained from $\mathbf{Top}$, the category of all topological spaces, by reducing the class of objects and keeping the specification of morphisms intact while in $\mathbf{Frm}_D$ we keep all the objects and reduce the class of morphisms.

2. As is familiar, for each frame homomorphism $h$ the $h_*$ preserves prime elements. This is because $h$ preserves meets, see 1.3; for the same reason, if $h$ is a complete homomorphism then $h_*$ preserves completely prime elements. This may create the impression that being $D$-homomorphism is a strong requirement akin to completeness, but this is not the case: consider the following examples.

(a) The condition is, of course, satisfied trivially if there are no completely prime elements in $M$. This is trivial.

(b) A more interesting case is that of regular frames (coresponding to regular spaces: they are characterized by the property that each element $x$ is the join $\bigvee\{y \mid y \prec x\}$ where $y \prec x$ means that $y^* \vee x$ for the pseudocomplement $y^*$ of $y$) for which we have the following familiar

**Fact.** *In a regular frame each prime element is maximal, thus covered by 1, and hence completely prime.*

(Indeed, let $a$ be prime and let $a < b = \bigvee\{x \mid x \prec b\}$ by regularity. Then there is an $x \not\leq a$ such that $x^* \vee b = 1$. Now $x \wedge x^* = 0 \leq a$, hence by primeness, $x^* \leq a$, and $b = a \vee b \geq x^* \vee b = 1$.)

Thus,

*each frame homomorphism $h : L \to M$ with regular $L$ is a $D$-homomorphism.*

(c) Another case will be presented right away in 3.2.

**3.2.** Consider the standard functor $\mathfrak{O} : \mathbf{Top} \to \mathbf{Frm}$. For a continuous map $f : X \to Y$ we have $(\mathfrak{O}f)_*(U) = Y \smallsetminus \overline{f[X \smallsetminus U]}$ and hence

$$(\mathfrak{O}f)_*(X \smallsetminus \overline{\{x\}}) = Y \smallsetminus \overline{\{f(x)\}}.$$

By 2.3.2, the completely prime elements in $\mathfrak{O}X$ for a $T_D$ space $X$ are precisely the $X \smallsetminus \overline{\{x\}}$. Consequently, we can view the restriction of $\mathfrak{O}$ to $T_D$-spaces as a functor

$$\mathfrak{O} : \mathbf{Top}_D \to \mathbf{Frm}_D.$$

In the sequel the symbol $\mathfrak{O}$ will be used in this sense.

**3.3. The space $\Phi L$.** For any frame $L$ define $\Phi L$ as the subspace of $\Sigma L$ (recall 1.4) of all *completely prime* elements, with $\Phi_a = \Sigma_a \cap \Phi L = \{p \text{ completely prime} \mid a \not\leq p\}$ as the open sets. In particular then

$$(*)\quad \Phi_0 = \emptyset,\quad \Phi_1 = \Phi L,\quad \Phi_{a \wedge b} = \Phi_a \cap \Phi_b \text{ and } \Phi_{\bigvee S} = \bigcup\{\Phi_a \mid a \in S\}.$$

**3.3.1. Lemma.** *For each $p \in \Phi L$, $\Phi L \smallsetminus \overline{\{p\}} = \Phi_p$.*

*Proof.* $\Phi L \smallsetminus \overline{\{p\}} = \bigcup\{\Phi_a \mid p \notin \Phi_a\} = \bigcup\{\Phi_a \mid a \leq p\} = \Phi_p$ by $(*)$ in 3.3. □

**3.3.2. Proposition.** *$\Phi L$ is a $T_D$-space.*

*Proof.* Recall 2.2.1. We have to prove that each $\{p\} \cup (\Phi L \smallsetminus \overline{\{p\}})$ ($= \{p\} \cup \Phi_p$ by 3.3.1) is open; that is, it (also) is a neighbourhood of $p$. Let $p \lessdot a$. Then $p \in \Phi_a$. We will show that $\Phi_a \subseteq \{p\} \cup \Phi_p$. Indeed, if $q \notin \{p\} \cup \Phi_p$ then $p < q$, hence by 2.1.1, $a \leq q$, and hence $q \notin \Phi_a$. □

**3.3.3.** Furthermore, the correspondence $L \mapsto \Phi L$ extends to a functor

$$\Phi : \mathbf{Frm}_D \to \mathbf{Top}_D$$

amounting to the restriction of the spectrum functor $\Sigma$. That is, $\Phi h(p) = h_*(p)$ and one has

$$(\Phi h)^{-1}[\Phi_a] = \Phi_{h(a)}$$

so that $\Phi h$ is continuous.

**3.4.** For any frame $L$, define a mapping (as in the case of the standard spectrum)

$$\delta_L : L \to \mathfrak{O}\Phi L$$

by setting $\delta_L(a) = \Phi_a$. By $(*)$ in 3.3, $\delta_L$ is a frame homomorphism.

**3.4.1. Lemma.** *For each completely prime $p \in L$ we have*

$$(\delta_L)_*(\Phi_p) = p.$$

*Proof.* We have $\Phi_a \subseteq \Phi_p$ iff $a \le p$: $a \le p$ implies $\Phi_a \subseteq \Phi_p$, and if $a \nleq p$ then $p \in \Phi_a \smallsetminus \Phi_p$ and hence $\Phi_a \nsubseteq \Phi_p$.

Now $a \le (\delta_L)_*(\Phi_p)$ iff $\Phi_a \subseteq \Phi_p$. Hence for any $a$, $a \le (\delta_L)_*(\Phi_p)$ iff $a \le p$, and the statement follows. □

**3.4.2. Lemma.** *The $\delta_L$ constitute a natural transformation $\delta : \mathrm{Id} \to \mathfrak{O}\Phi$.*

*Proof.* By 3.3.1 and 3.4.1 each $\delta_L$ is a $D$-homomorphism. As in the case of the usual spectrum we have $\mathfrak{O}\Phi(h)(\delta_L(a)) = (h_*)^{-1}[\Phi_a] = \Phi_{h(a)} = \delta_M(h(a))$. □

**3.5.** For a $T_D$-space $X$ (where the $X \smallsetminus \overline{\{x\}}$ are completely prime) define

$$\pi_X : X \to \Phi\mathfrak{O}X, \quad x \mapsto X \smallsetminus \overline{\{x\}}.$$

**Proposition.** *The mappings $\pi_X$ constitute a natural equivalence $\pi : \mathrm{Id} \to \Phi\mathfrak{O}$.*

*Proof.* First, $\pi_X$ is onto and one-one since for a $T_D$-space, $X \smallsetminus \overline{\{x\}}$ are precisely the completely primes, and if $x \ne y$ then $\overline{\{x\}} \ne \overline{\{y\}}$ (our spaces are $T_0$).

Next, $\pi_X^{-1}[\Phi_U] = \{x \mid U \nsubseteq X \smallsetminus \overline{\{x\}}\} = \{x \mid x \in U\} = U$, proving that $\pi_X$ is a homeomorphism.

Finally, if $f : X \to Y$ is a continuous map we have

$$\Phi\mathfrak{O}f(\pi_X(x)) = (\mathfrak{O}f)^*(X \smallsetminus \overline{\{x\}}) = Y \smallsetminus \overline{\{f(x)\}} = \pi_Y(f(x)),$$

that is, $\pi$ is natural. □

**3.5.1. Proposition.** *The contravariant functors $\Phi : \mathbf{Frm}_D \to \mathbf{Top}_D$ and $\mathfrak{O} : \mathbf{Top}_D \to \mathbf{Frm}_D$ are adjoint on the right with the adjunction maps $\delta_L : L \to \mathfrak{O}\Phi L$ and $\pi_X : X \to \Phi\mathfrak{O}X$ where the latter is a homeomorphism for each $X$.*

*Proof.* As in the case of the standard spectrum, consider the composition

$$\mathfrak{O}X \xrightarrow{\delta_{\mathfrak{O}X}} \mathfrak{O}\Phi\mathfrak{O}X \xrightarrow{\mathfrak{O}\pi_X} \mathfrak{O}X.$$

We have

$$\mathfrak{O}\pi_X(\delta_{\mathfrak{O}X}(U) = \pi_X^{-1}(\Phi_U) = \{x \mid X \smallsetminus \overline{\{x\}} \in \Phi_U\} = \\ = \{x \mid U \nsubseteq X \smallsetminus \overline{\{x\}}\} = \{x \mid x \in U\} = U.$$

Similarly for

$$\Phi L \xrightarrow{\pi_{\Phi L}} \Phi\mathfrak{O}\Phi L \xrightarrow{\Phi\delta_L} \Phi L.$$

We have

$$\Phi(\delta_L)(\pi_{\Phi L}(p)) = (\delta_L)_*(\Phi L \smallsetminus \overline{\{p\}}) = (\delta_L)_*(\Phi_p) = p$$

by 3.3.1 and 3.4.1. □

**3.5.2.** In particular, since $\pi$ is a natural equivalence,

*$\Phi$ and $\mathfrak{O}$ induce a dual equivalence between* $\mathbf{Top}_D$ *and the subcategory of* $\mathbf{Frm}_D$ *given by the $T_D$-spatial frames* (recall 1.7).

**3.5.3.** The following fact is related to a result from Bruns, [5].

**Proposition.** *A frame $L$ is $T_D$-spatial if and only if any proper interval $\{x \mid a \le x \le b\}$ in $L$ contains a covered pair.*

*Proof.* Let $L = \mathfrak{O}X$ for some $T_D$-space $X$ and $U \subsetneq Y$ in $\mathfrak{O}X$. Then, for $x \in V \smallsetminus U$, $(V \smallsetminus \overline{\{x\}}) \cup \{x\}$ is open since $(X \smallsetminus \overline{\{x\}}) \cup \{x\}$ is open by 2.2.1; further, $U \subseteq X \smallsetminus \overline{\{x\}}$ as $x \notin U$ and hence

$$U \subseteq V \cap (X \smallsetminus \overline{\{x\}}) = (V \smallsetminus \overline{\{x\}}) \lessdot (V \smallsetminus \overline{\{x\}}) \cup \{x\} \subseteq V,$$

exhibiting a covered pair between $U$ and $V$.

Conversely, given the condition in question, we have to show that $\delta_L$ is an isomorphism, and for this it is enough to prove $\Phi_a \nleq \Phi_b$ whenever $a < b$. Now, by our hypothesis, we have $u, v \in L$ such that $a \le u \lessdot v \le b$, hence the homomorphism

$$h : L \to \{u, v\} \cong \mathbf{2}, \quad s \mapsto (s \vee u) \wedge v,$$

and therefore the completely prime filter

$$F = \{s \in L \mid h(s) = v\} = \{s \in L \mid s \vee u \ge v\} = u \uparrow v$$

(recall 2.5). Finally, $F$ is a slicing filter by 2.6.2, and since $a \notin F$ and $b \in F$ this shows for the (completely!) prime $p$ associated with $F$ that $p \notin \Phi_a$ and $p \in \Phi_b$. □

**3.6.** Consider the inverse $j_X : \Phi\mathfrak{O}X \to X$ to the adjunction homeomorphism $\pi_X$, given by the formula

$$j_X(X \smallsetminus \overline{\{x\}}) = x.$$

Now let us extend this definition to a general space $X$ (recall that our spaces are $T_0$ and hence $j_X$ is well-defined).

For an open set $U$ in $X$ we have

$$j_X^{-1}[U] = \{X \smallsetminus \overline{\{x\}} \mid x \in U\} = \{X \smallsetminus \overline{\{x\}} \mid U \nsubseteq X \smallsetminus \overline{\{x\}}\} = \Phi_U$$

so that the $j_X^{-1}[U]$ are precisely the open subsets of $\Phi\mathfrak{O}X$. Thus

*$j_X$ ia a homeomorphic embedding of (the $T_D$-space) $\Phi\mathfrak{O}X$ onto the subspace of $X$ constituted by all the $T_D$-points.*

This suggests something like a $T_D$-coreflection. But $\mathbf{Top}_D$ is not coreflective in **Top**: it is not closed under quotients, not even if we restrict ourselves to the $T_0$-case, as is seen in the following

**3.6.1. Example.** Let $\mathbb{N}$ be the set of natural numbers, $\omega \notin \mathbb{N}$. Consider

$$X = (\mathbb{N} \times \{0,1\}) \cup \{\omega\}$$

endowed by the following topology:

$$U \subseteq X \text{ is open if } \begin{cases} (\exists n,\ (n,1,) \in U) \ \Rightarrow\ \omega \in U, \ \text{ and} \\ \omega \in U \ \Rightarrow\ \exists k\ (n \geq k \ \Rightarrow\ (n,0) \in U). \end{cases}$$

Thus in particular $\mathbb{N} \times \{0\}$, $(\mathbb{N} \times \{0\}) \cup \{\omega\}$ and each $(\mathbb{N} \times \{0\}) \cup \{\omega\} \cup \{(n,1)\}$ are open, and we see that $X$ is $T_D$.

Now define $q : X \to Y = \mathbb{N} \times \{\omega\}$ by setting $q(n,i) = n$ and $q(\omega) = \omega$. In the corresponding quotient topology on $Y$, $U$ is open iff either it is $\emptyset$ or contains $\omega$ and all $n \geq k$ for some $k$. Thus, $Y$ is still $T_0$, but it is not $T_D$: $\omega$ is not a $T_D$-point.

**3.7.** Let $\mathcal{A}$ be a general category and let $\mathcal{C}$ be a full subcategory of $\mathcal{A}$. Let us have to each object $A \in \mathcal{A}$ assigned an $A^\circ \in \mathcal{C}$ and a monomorphism

$$\iota_A : A^\circ \to A$$

such that

$$\text{for } C \in \mathcal{C},\ C^\circ = C \text{ and } \iota_C = \mathrm{id}_C, \text{ in particular } A^{\circ\circ} = A^\circ.$$

Define a subcategory $\mathcal{A}^\circ$ of $\mathcal{A}$ as follows.

- the class of objects of $\mathcal{A}^\circ$ is the same as that of $\mathcal{A}$, and
- $f : A \to B$ from $\mathcal{A}$ is a morphism in $\mathcal{A}^\circ$ iff there exists an $f^\circ : A \to B$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \iota_A}\uparrow & & \uparrow{\scriptstyle \iota_B} \\ A^\circ & \xrightarrow{f^\circ} & B^\circ \end{array}$$

  commutes.

We have the immediate

**Observation.** 1. *The correspondence $f \mapsto f^\circ$ constitutes a functor $\mathcal{A}^\circ \to \mathcal{A}$, and $\iota = (\iota_A)_A$ is a natural transformation.*

2. *$\mathcal{C}$ is a full subcategory of $\mathcal{A}^\circ$.*

3. *$\mathcal{C}$ is a mono-coreflective subcategory of $\mathcal{A}^\circ$ with coreflection maps $\iota_A : A^\circ \to A$*

**3.7.1.** This applies to the situation in 3.6. Let us specify more concretely the subcategory of **Top** appearing as the $\mathcal{A}^\circ$ above.

**Lemma.** *There exists a continuous $f^\circ : \Phi\mathfrak{O}X \to \Phi\mathfrak{O}Y$ such that*

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ {\scriptstyle j_X}\uparrow & & \uparrow{\scriptstyle j_Y} \\ \Phi\mathfrak{O}X & \xrightarrow{\;f^\circ\;} & \Phi\mathfrak{O}Y \end{array}$$

*commutes iff $\mathfrak{O}f$ is a $D$-homomorphism.*

*Proof.* $(\mathfrak{O}f)_*(X \smallsetminus \overline{\{x\}}) = Y \smallsetminus \overline{\{f(x)\}}$. Thus, by 2.3.2, the completely prime elements are sent by $(\mathfrak{O}f)_*$ to completely prime ones iff the image of a $T_D$-point is a $T_D$-point, that is, iff $f$ can be restricted to the subspaces of $T_D$ points. □

**3.7.2. Conclusion.** Define

$${}_D\mathbf{Top}$$

as the category of all ($T_0$) spaces and the continuous maps $f$ such that $\mathfrak{O}f$ is a $D$-homomorphism. Then

$\mathbf{Top}_D$ *is a (mono-)coreflective subcategory of* ${}_D\mathbf{Top}$.

## 4. $T_D$ versus sobriety

**4.1. Lemma.** *Let $j$ be the identical embedding $X \smallsetminus \{x\} \to X$. Then $\mathfrak{O}j$ is an isomorphism iff $x$ is not a $T_D$-point.*

*Proof.* We have $\mathfrak{O}j(U) = U \smallsetminus \{x\}$. $\mathfrak{O}j$ is trivially onto, and it is one-one if and only if $U \smallsetminus \{x\}$ is never open unless $x \notin U$. □

**4.2. Observation.** *For $Z \subseteq Y \subseteq X$ let $k : Z \to Y$ and $j : Y \to X$ be the identical embedding and put $l = j \cdot k$. If $\mathfrak{O}l$ is an isomorphism then both $\mathfrak{O}j$ and $\mathfrak{O}k$ are isomorphisms.*

($\mathfrak{O}j \cdot \mathfrak{O}k$ is one-one, hence $\mathfrak{O}k$ is, which makes it an isomorphism; then $\mathfrak{O}j = \mathfrak{O}l(\mathfrak{O}k)^{-1}$.)

**4.3.** For any spaces $X$ and $Y$ consider the relation

$R(X,Y)$ : *$X$ is a proper subspace of $Y$ such that the identical embedding $X \to Y$ induces an isomorphism $\mathfrak{O}Y \to \mathfrak{O}X$.*

Then, sobriety and $T_D$ have the following parallel characterizations.

**Proposition.** *$X$ is $\begin{Bmatrix} \text{sober} \\ T_D \end{Bmatrix}$ iff $\neg(\exists Y)$ $\begin{Bmatrix} R(X,Y) \\ R(Y,X) \end{Bmatrix}$.*

*Proof.* (1) If $X$ is not sober then $R(X,Y)$ for the sobrification $Y \supsetneq X$. Conversely, given$Y$ such that $R(X,Y)$, take any $y \in Y \smallsetminus X$ and let

$$\mathcal{F} = \{U \cap X \mid y \in U \in \mathfrak{O}Y\}.$$

Then $\mathcal{F}$ is a completely prime filter in $\mathfrak{O}X$, given $R(X,Y)$, which is not fixed in $X$ (since all our spaces are $T_0$), showing $X$ is not sober.

(2) If $X$ is not $T_D$ then there exists a non-$T_D$-point $x \in X$, and by 4.1 we have $R(X,Y)$ for $Y = X \smallsetminus \{x\}$. Conversely, given $Y$ such that $R(X,Y)$, take any $x \in X \smallsetminus Y$ and consider the identical embeddings

$$Y \to X \smallsetminus \{x\} \to X.$$

Then $R(Y,X)$ implies $R(X \smallsetminus \{x\}, X)$ by 4.2, and by 4.1 $x$ is not a $T_D$-point, showing $X$ is not $T_D$. $\square$

**4.4.1. Corollary.** *If $X \subsetneq Y$ and $\mathfrak{O}j_{XY}$ is an isomorphism then $X$ is not sober and $Y$ is not $T_D$.*

In particular we obtain the well known fact that

*a non-trivial sobrification is never $T_D$.*

**4.5.** In [4], Berger proved (a.o.) that hereditarily sober spaces are $T_D$. In fact we have more.

**Corollary.** *If for each $x \in X$ the space $X \smallsetminus \{x\}$ is sober then $X$ is $T_D$.*

(Indeed, for any a non-$T_D$-point $x \in X$ we have $R(X \smallsetminus \{x\}, X)$, contradicting the sobriety of $X \smallsetminus \{x\}$ by 4.3.)

**4.6.** The non-fixed completely prime filters are (up to isomorphism) precisely the individual points $y$ that one can *add* and keep the $\mathfrak{O}(X \cup \{y\})$ isomorphic with $\mathfrak{O}X$.

The non-$T_D$-points are precisely the individual points $x$ that one can *subtract* and keep the $\mathfrak{O}(X \smallsetminus \{x\})$ isomorphic with $\mathfrak{O}X$.

The symmetry between sobriety and the $T_D$ property (which has been expressed already in 4.3) cannot be stretched much further, though: while we can add all the completely non-fixed prime filters to $X$ to obtain the *maximal* $Y$ with $R(X,Y)$, namely the sobrification, we do not generally have a *minimal* $X \subseteq Y$ with $R(X,Y)$. We would have to remove, first, all the non-$T_D$-points, which could result in the void subspace.

**4.6.3. Example.** Let $X$ be any order-dense subset of the set $\mathbb{R}$ of reals, endowed with the topology consisting of $\emptyset$ and all

$$>x = \{u \mid x < y\}.$$

In $X$, obviously, none of the points is $T_D$ (removing any point from $>x$ destroys the openness).

Since $T_D$-spatiality implies the existence of covered elements (recall 3.5.3) our $\mathfrak{O}X$ is also an example of a spatial frame that is not $T_D$-spatial.

**4.7.** In point-free topology one uses the concept of *subfitness* (if $a < b$ then $a \vee c < 1 = b \vee c$ for some $c$) as a weaker counterpart of $T_1$. The question naturally arises whether there is some relation between the subfitness of $\mathfrak{O}X$ and the $T_D$-property of $X$. There is not. We have

**Proposition.** *For $T_0$-spaces $X$, $\mathfrak{O}X$ subfit neither implies nor is implied by $X$ being $T_D$.*

*Proof.* ($\not\Rightarrow$) Let $Y$ be any $T_1$-space and $X \supseteq Y$ its sobrification. Then $\mathfrak{O}X \cong \mathfrak{O}Y$, and since $\mathfrak{O}Y$ is trivially subfit the same holds for $\mathfrak{O}X$. Suppose now that $X$ is $T_D$. Then every neighbourhood filter $\mathcal{U}(x)$, $x \in X$, is a slicing filter of $\mathfrak{O}X$, and by the relation between $X$ and $Y$ it follows that every completely prime filter of $\mathfrak{O}Y$ is slicing. That, however, implies that every completely prime filter of $\mathfrak{O}Y$ is fixed (2.6.1), saying $Y$ is sober – which is known not to hold for every $Y$.

($\not\Leftarrow$) Let $X$ be the unit interval $[0,1]$ with the usual downsets as open sets. Then $X$ is $T_D$ but it is not subfit since for any $0 \le \delta < \mu < 1$ and for any downset $U \subseteq [0,1]$ such that $(\downarrow \mu) \cup U = [0,1]$ we also have $(\downarrow \delta) \cup U = [0,1]$. $\square$

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