Algebraic proof of Brooks' theorem*

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Abstract

We give a proof of Brooks' theorem as well as its list coloring extension using the algebraic method of Alon and Tarsi.

1 Introduction

One of the most famous theorems on graph colorings is Brooks' theorem [3] which asserts that every connected graph G with maximum degree Δ is Δ-colorable unless G is an odd cycle or a complete graph. Brooks' theorem has been extended in various directions, e.g., its list version can be found in [8], also see [4].

A non-integer relaxation of the chromatic number $\chi(G)$ is the circular chromatic number $\chi_c(G)$, which was introduced in [7] and attracted a considerable amount of interest of researchers (see two recent surveys [9, 11] on circular colorings by Zhu). Classical and circular colorings are closely related, in particular, it holds that $\chi(G) = \lceil \chi_c(G) \rceil$. An analogous equality is not true for their list counterparts. The circular list chromatic number is always at least the list chromatic number decreased by one but it is not upper-bounded by any function of the list chromatic number [10].

Circular list colorings seem to be of surprising difficulty, e.g., Norine [5] only recently proved that the list chromatic number of even cycles is equal to two. In his proof, he has successfully applied the algebraic method of Alon

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and Tarsi from [2]. For another application of this method to circular list colorings, see [6]. It seems natural to ask whether this approach can also be used to prove the variant of Brooks' theorem for circular list colorings, which is still not known [10]. The natural first step towards this goal is finding an algebraic proof of the classical Brooks' theorem. We fulfil this goal in this short note. We also believe that an algebraic proof of Brooks' theorem is of its own interest independent of possible applications to circular list colorings.

The algebraic method of Alon and Tarsi is based on studying of properties of a certain graph polynomial. This polynomial is closely related to the existence of special orientations with bounded in-degrees. We summarize this relation in the next theorem.

Theorem 1 (Alon and Tarsi [2])**.** *Let G be a graph whose edges are oriented in such a way that the maximum in-degree of G is at most k. If the numbers of even and odd Eulerian subgraphs of G differ, then G is list $(k+1)$-colorable.*

Let us remind that an *even Eulerian subgraph* is a spanning subgraph of G with even number of edges such that each vertex has the same in-degree and out-degree. Similarly, an odd Eulerian subgraph is such a subgraph with an odd number of edges. Even and odd Eulerian subgraphs do not need be connected and can contain isolated vertices. Theorem 1 has been successfully applied to several coloring problems, e.g., showing that planar bipartite graphs are list 3-colorable [2]. We refer the reader to [1] for further applications.

2 Structural lemma

In this section, we give a structural lemma which allows us to apply the algebraic technique of Alon and Tarsi in our proof of Brooks' theorem.

Lemma 2. *Let G be a connected Δ-regular graph. If G is neither an odd cycle nor a complete graph, then G contains an even cycle C with at most one chord.*

Proof. We prove the lemma in a series of four claims.

Claim 1. *The graph G contains an even cycle C that does not induce a complete graph.*

If G is not 2-connected, let H be an end-block of the block-decomposition of G, v the cut-vertex of G contained in H and w any vertex of H not adjacent to v. If G is 2-connected, set H to be G and let v and w be arbitrary two vertices of H. By Menger's theorem, there are two vertex disjoint paths between v and w in H. Consider two such shortest paths and let C_{vw} be the cycle formed by them. Observe that both v and w have exactly two neighbors on C_{vw} (otherwise, the two paths could be shortened).

If C_{vw} is even, the statement follows. Hence, we can assume that C_{vw} is odd. Consequently, $G \neq C_{vw}$ and thus $\Delta \geq 3$. As the degree of w in H is $\Delta \geq 3$, there is a neighbor x of w not in C_{vw}. Let P be the shortest path from x to a vertex z on C_{vw} avoiding the vertex w. The existence of P follows from the fact that H is 2-connected. Let P_1 and P_2 be the two paths of C_{vw} delimited by w and z. By symmetry, we can assume that P_1 contains v. If the cycle formed by P_1, P and the edge wx is even, it is the desired cycle C (it contains both v and w). Otherwise, the cycle formed by P_2, P and wx is even. If the vertex z were adjacent to both w and x, then the lengths of P_2 and P would be equal to one (by the choice of C and P) and thus the length of the cycle formed by P_2, P and wx would be three. Hence, the vertex z is not adjacent to both w and x and the cycle formed by P_2, P and wx is the sought one.

Claim 2. *Let $C = v_1 \ldots v_\ell$ be the shortest cycle with the properties given in Claim 1. No vertex of C is adjacent to all other vertices on C.*

If $\ell = 4$, C has a single chord and the claim follows. Assume that $\ell \geq 6$. Suppose that the vertex v_1 is adjacent to all the vertices $v_2, \ldots, v_\ell$. By the choice of C, both the cycle $v_1 \ldots v_{\ell-2}$ and $v_1 v_4 \ldots v_\ell$ induce complete graphs. Hence, one of the vertices v_2 or v_3, say v_i, is not adjacent to at least one of the vertices $v_{\ell-1}$ and v_ℓ, say v_j. However, the cycle $v_1 v_i v_4 v_j$ is then a shorter cycle satisfying the properties of Claim 1.

Claim 3. *Let $C = v_1 \ldots v_\ell$ be the shortest cycle with the properties given in Claim 1. Each vertex of C is incident with at most one chord.*

Assume that v_1 is adjacent to vertices v_a and v_b, $3 \leq a < b \leq \ell - 2$. If $b - a$ is even, the cycle $v_1 v_a \ldots v_b$ is even and thus it must induce a complete graph in G by the choice of C. In particular, the vertices v_1 and v_{a+1} are adjacent. By Claim 2, v_1 is not adjacent to a vertex v_i. By symmetry, we can assume that $3 \leq i < a$. However, either the cycle $v_1 \ldots v_a$ or the cycle $v_1 \ldots v_{a+1}$ is even, shorter than C and does not induce a complete graph (the vertices v_1 and v_i are non-adjacent).

If $b - a$ is odd, then either $a - 1$ or $\ell + 1 - b$ is odd. By symmetry, we can assume that $a - 1$ is even. Since the cycle $v_1 \ldots v_b$ is even, it must induce a complete subgraph of G by the choice of C. Consequently, the cycle $v_1 v_{b-1} v_b \ldots v_\ell$ is an even cycle which does not induce a complete subgraph (the vertex v_1 is not adjacent to at least one of the vertices $v_{b+1}, \ldots, v_{\ell-2}$).

Claim 4. *Let $C = v_1 \dots v_\ell$ be the shortest cycle with the properties given in Claim 1. The cycle C contains at most one chord.*

Suppose that C has at least two chords, say $v_a v_b$ and $v_c v_d$. We distinguish two cases based on the fact whether the two chords cross. If they do not cross, we can assume without loss of generality that $1 \le a < b < c < d \le \ell$. If $b-a$ is odd, then the cycle $v_a \dots v_b$ is a shorter even cycle not inducing a complete graph (the vertices v_a and v_{a+2} are not adjacent). Hence, $b-a$ is even, and similarly $d-c$ is even. Consequently, the cycle $v_1 \dots v_a v_b \dots v_c v_d \dots v_\ell$ is an even cycle which does not induce a complete graph in G.

Assume now that the two chords cross. By symmetry, we can assume that $1 \le a < c < b < d \le \ell$. The vertices v_a, v_b, v_c and v_d split the cycle C into four parts; let n_{xy}, $x \in \{a, b\}$ and $y \in \{c, d\}$, be the number of vertices of C between v_x and v_y. If not all n_{xy} have the same parity, then there are two consecutive parts (viewed in the order they correspond to the parts of C) with different parities, say the parities of n_{ac} and n_{bc} are different. Then, the cycle $v_a \dots v_c \dots v_b$ is an even cycle satisfying the properties of Claim 1 which is shorter than C. Since this is impossible, we can assume that the parities of all the four numbers n_{xy} are the same. Consider now the even cycle $v_a \dots v_c v_d \dots v_b$. If this cycle induced a complete graph, then all $v_a v_c$, $v_b v_c$, $v_a v_d$ and $v_b v_d$ would be edges of C, the length of C would be four and C would induce a complete graph of order four. Since this is impossible, the cycle $v_a \dots v_c v_d \dots v_b$ is the sought even cycle. □

3 Main result

Before presenting the algebraic proof of Brooks' theorem, we need to recall a simple folklore structural result on ordering vertices of a connected graph. We include its proof for completeness.

Lemma 3. *Let G be a connected graph and v an arbitrary vertex of G. The vertices of G can be ordered in such a way that every vertex except for v is preceded by at least one of its neighbors.*

Proof. Consider an arbitrary spanning tree T of G and root it at v. The vertices of G are ordered in the following way: the first vertex is the root v followed by all its children (vertices of the second level of T). Then, all vertices of the third level are listed, then all vertices of the forth level, etc. Since each vertex except for v is preceded by its parent, the obtained ordering has the desired property. □

We are now ready to give the algebraic proof of Brooks' theorem.

Theorem 4. *Let G be a connected graph with maximum degree Δ. If G is neither a complete graph nor an odd cycle, then G is list Δ-colorable.*

Proof. Assume first that the graph G has a vertex of degree less than Δ and fix an ordering $v_1, \dots, v_n$ of the vertices of G as in Lemma 3 with v_1 being a vertex of degree less than Δ. Every edge $v_i v_j$, $i < j$, is oriented from v_j to v_i. Observe that there are at most $\Delta - 1$ edges coming to v_1 as the degree of v_1 is at most $\Delta - 1$ and there are at most $\Delta - 1$ edges coming to each v_i, $i \neq 1$, since at least one neighbor of v_i precedes v_i in the ordering. Since the maximum in-degree of the constructed orientation is at most $\Delta - 1$ and its only Eulerian subgraph is the empty one, Theorem 1 implies that G is Δ-choosable.

We now consider the case that G is Δ-regular. By Lemma 2, the graph G contains an even cycle C with at most one chord. Let ℓ be the length of C. Contract C to a single vertex w and apply Lemma 3 to the resulting graph. We obtain an ordering $v_1, \dots, v_n$ of the vertices of G by replacing w in the order given by Lemma 3 with the vertices of C, inserted in an arbitrary order. The edges of C are oriented in a cyclic way. Every edge $v_i v_j$, $i < j$, that is not contained in C, is now oriented from v_j to v_i (this rule also applies to the chord of C if it exists). Observe that the maximum in-degree of the resulting orientation is $\Delta - 1$.

There are always two even Eulerian subgraphs of the constructed orientation, the empty one and the one formed by the cycle C. If C contains a chord e, there is also an Eulerian subgraph formed by the chord and one of the two parts delimited by e, which is either even or odd. Since there are no other Eulerian subgraphs, the numbers of odd and even Eulerian subgraphs must differ. Theorem 1 yields the statement. □

References

[1] N. Alon: *Combinatorial Nullstellensatz*, Combin. Probab. Comput. **8** (1999), 7–29.

[2] N. Alon, M. Tarsi: *Colorings and orientations of graphs*, Combinatorica **12** (1992), 125–134.

[3] R. L. Brooks: *On colouring the nodes of a network*, Proc. Cambridge Phil. Soc. **37** (1941), 194–197.

[4] A. V. Kostochka, M. Stiebitz, B. Wirth: *The colour theorems of Brooks and Gallai extended*, Discrete Math. **162** (1996), 299–303.

[5] S. Norine: *On two questions about circular choosability*, J. Graph Theory 58 (2008), 261–269.

[6] S. Norine, T.-L. Wong, X. Zhu: *Circular choosability via combinatorial Nullstellensatz*, J. Graph Theory **59** (2008), 190–204.

[7] A. Vince: *Star chromatic number*, J. Graph Theory **12** (1988), 551–559.

[8] V. G. Vizing: *Colouring the vertices of a graph with prescribed colours* (in Russian), Metody Diskretnogo Analiza Teorii Kodov i Skhem **29** (1976), 3–10.

[9] X. Zhu: *Circular chromatic number: a survey*, Discrete Math. **229** (2001), 371–410.

[10] X. Zhu: *Circular choosability of graphs*, J. Graph Theory **48** (2005), 210–218.

[11] X. Zhu: *Recent developments in circular colorings of graphs*, in M. Klazar, J. Kratochvíl, J. Matoušek, R. Thomas, P. Valtr (eds.): Topics in Discrete Mathematics, Springer, 2006, 497–550.