DUALITIES AND DUAL PAIRS IN HEYTING ALGEBRAS

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Abstract. We extract the abstract core of finite homomorphism dualities using the techniques of Heyting algebras and (combinatorial) categories.

Introduction

Finite dualities appear in [24] in the categorical context of dual characterizations of various classes of structures. It is a simple idea: characterize a given class both by forbidden substructures (associated with subobjects) and by decompositions (associated with factorobjects); this proved to be surprisingly fruitful. In retrospect, it was also a timely concept as it coincided with the introduction (in the logical and artificial intelligence contexts) of the paradigm of Constraint Satisfaction ([18, 16]).

Only later it was realized (in the context of complexity theory) that these notions are two aspects of the same general problem, the study of homomorphisms of relational structures ([5]).

Finite dualities represent an extremal case of the above mentioned Contraint Satisfaction Problem. Provided we have a finite duality, the problem in question is polynomially decidable. Furthermore, in a broad context such problems coincide with the decision problem for classes of structures that are First Order decidable ([1], [32]. For general relational structures, finite dualities were characterized in [30] and a number of interesting particular cases have been investigated since ([29, 6, 3, 11, 21]).

Here, following [25] we return to the original motivation and discuss finite dualities in the categorical context. We aim at pointing out those categories in which one can describe finite dualities using the interplay of general categorical and order theoretical concepts and techniques.

1. Background: dualities in graphs

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AND SIMILAR CATEGORIES

1.1. The categories we will work with are *finitely concrete* that is, the objects are finite sets endowed with structures, and morphisms are maps respecting the structures in a specified way.

Typically we have in mind categories such as that of (finite) symmetric graphs, or oriented graphs, with edge preserving homomorphisms (or more general relations resp. relational systems), with relation(s) preserving maps. Some of the results can be applied for other choices of morphisms (strong or full homomorphisms), though.

1.2. We will assume that our categories admit finite sums (coproducts)

$$\iota_j : A_j \to A = \coprod_{i=1}^{n} A_i$$

(characterized by the property that for each system $f_j : A_j \to B$, $j = 1, \dots, n$ there is exactly one morphism $A \to B$ such that $f\iota_j = f_j$ for all j).

This is the minimal assumption; for more involved facts we will assume also the existence of finite products

$$\pi_j : A = \prod_{i=1}^{n} A_i \to A_j$$

(characterized by the property that for each system $f_j : B \to A_j$ there is exactly one morphism $B \to A$ such that $\pi_j f = f_j$ for all j), and also more (the Heyting property, see 1.6 below).

1.3. The *Constraint Satisfaction Problem* (briefly, CSP) in a category $\mathcal{C}$ is the problem of specifying the class

$$\mathbf{CSP}(\mathcal{B}) = \{X \in \mathcal{C} \mid X \to B \text{ for some } B \in \mathcal{B}\}$$

where $X \to Y$ stands for "there exists a morphism $f : X \to Y$" and $\mathcal{B}$ is a class of objects (the constraints). Here we are concerned with the situation in which this class can be represented as

$$\mathbf{Forb}(\mathcal{A}) = \{X \in \mathcal{C} \mid A \nrightarrow X \text{ for all } A \in \mathcal{A}\}$$

where $X \nrightarrow Y$ stands for "there exists no morphism $f : X \to Y$" and $\mathcal{A}$ is a finite class of objects (with an infinite $\mathcal{A}$ this is always possible). In fact the classes $\mathcal{B}$ we are interested in are also finite. Thus, we investigate the situations of finite systems $A_1, \dots, A_n$ and $B_1, \dots, B_m$ of objects such that

$$\forall i,\ A_i \nrightarrow X \quad \text{iff} \quad \exists j,\ X \to B_j \tag{1.3.1}$$

and in this case we speak of a *finite duality*.

Note. Instead of forbidding morphisms from the A_i one is sometimes interested in forbidding subobjects isomorphic to finitely many given ones. If we have (1.3.1) it is easy to replace the A_i's by finitely many other objects providing such a "subobject forbidding characterization".

1.4. The poset $\widetilde{\mathcal{C}}$. Given a category $\mathcal{C}$ consider the set of objects ordered by

$$A \le B \quad \equiv_{\text{df}} \quad \exists\, f : A \to B$$

and denote the obtained (pre-)ordered set by

$$\widetilde{\mathcal{C}}.$$

In fact, we usually think of $\widetilde{A}$ as the poset of the obvious eqivalence classes.

Note that if we assume the existence of sums as we did in 1.2 above, $\widetilde{A}$ is a join-semilttice. If we have, moreover, also the products, $\widetilde{\mathcal{C}}$ is a lattice.

Note. In the categories of graphs with standard homomorphisms, all objects posessing at least one loop are equivalent with each other, and the constraint satisfaction problem with at least one loop in at least one $B \in \mathcal{B}$ is trivial. Therefore we may think, in the case of graphs (symmetric or oriented, or otherwise specified) as of graphs without loops, with one trivial object T, consisting of one vertex with a loop, added for convenience, This T is then the top of the poset $\widetilde{A}$. Similarly, the $X \to B$ type requirements can be typically replaced by requirements of epimorphisms.

1.5. Heyting categories. The category $\mathcal{C}$ is said to be *Heyting* if $\widetilde{\mathcal{C}}$ is a Heyting algebra (that is, a lattice with an extra operation $\Rightarrow$ satisfying

$$a \wedge b \le c \quad \text{iff} \quad a \le b \Rightarrow c.$$

Trivially, any cartesian closed category (that is, a category with exponentiation $\langle X, Y\rangle$ such that the sets of morphisms

$$A \times B \to C \quad \text{and} \quad A \to \langle B, C\rangle$$

are naturally equivalent – see e.g [17]) is Heyting. Luckily enough the cartesian closedness is not necessary, and for instance the categories of graphs without loops (both in the symmetric an the oriented case) are Heyting without being cartesian closed.

1.6. Cores. In our categories the objects can be canonically reduced to make the relation $A \to B$ antisymmetric (up to isomorphism).

1.6.1. Lemma. *Let X be a finite object in a concrete category. Then each one-one morphism $\phi : X \to X$ is an isomorphism.*

Proof. ther is a $k \ne 0$ and n such that $\phi^{n+k} = \phi^n$. Thus, $\phi^k = \text{id}$ and $\phi \cdot \phi^{k-1} = \phi^{k-1} \cdot \phi = \text{id}$ □

1.6.2. Proposition. *Let X be a finite object in a concrete category. Then the smallest subobject $Y \subseteq X$ such that there is a morphism $f : X \to Y$*

(1) *is a retract of X, and*
(2) *is uniquely determined, up to isomorphism.*

Proof. Let $j : Y \to X$ be the embedding morphism. Then, by minimality, $\phi = fj : Y \to Y$ is one-one, and by 1.6.1 it is an isomorphism, and we have the retraction $r = \phi^{-1}f$. Now if $j' : Z \to X$ is an embedding of another subobject with the property, we have mutually inverse isomorphisms $r'j$ and rj'. □

The Y from 1.6.2 is called the *core* of X; denote it by cX. An X is called *core* if it is the core of some object. Note that

- a core is a core of itself, and
- in a concrete category $\mathcal{C}$ with finite objects, A and B are equivalent in $\widetilde{\mathcal{C}}$ (that is $A \leq B$ and $B \leq A$) iff cA and cB are isomorphic.

Thus, if we restrict ourselves to cores and representatives of isomorphism classes,

- the pre-ordered set $\widetilde{\mathcal{C}}$ becomes actually a poset.

Furthermore, any duality

$$\forall i,\ A_i \nrightarrow X \quad \text{iff} \quad \exists j,\ X \to B_j$$

can be replaced by the duality in the cores

$$\forall i,\ cA_i \nrightarrow X \quad \text{iff} \quad \exists j,\ X \to cB_j$$

([9, 8]).

2. Transversals and weak right duals

2.1. In a poset $(X, \leq)$ we will use the standard notation, for a subset $M \subseteq X$,

$$\downarrow M = \{x \in X \mid \exists m \in M,\ x \leq m\} \quad \text{and} \quad \uparrow M = \{x \in X \mid \exists m \in M,\ x \geq m\}$$

and, for an element $m \in X$, $\downarrow m = \downarrow\{m\}$ and $\uparrow m = \uparrow\{m\}$.

2.2. A element a of a lattice L is said to be *connected* if

$$a \leq b \vee c \quad \Rightarrow \quad a \leq b \ \text{ or } \ a \leq c.$$

Remark. Another (and perhaps more frequently used) term is *join-prime* or $\vee$*-prime*. We use "connected" because of the interpretation in the posets $\widetilde{\mathcal{C}}$ (recall 1.4) we are mostly interested in. Note that a graph without loops is connected in this sense (in any choice of morphisms at least

> as demanding as the standard graph homomorphism) iff it is connected in the usual sense.

The set of all connected elements of a semilattice L will be denoted by

$$\mathsf{Cn}L \quad \text{or simply by} \quad \mathsf{Cn}\ .$$

2.3. The upper semilattices we will be worked with will possess *finite connected decompositions*, that is

(2.3.1) for each $a \in L$, $\ {\downarrow}a \cap \mathsf{Cn}$ is finite and $a = \bigvee({\downarrow}a \cap \mathsf{Cn})$.

A *connected component* of an element $a \in L$ is a $c \in \mathsf{Cn}L$ such that there exists a decomposition $a = b \vee c \neq b$.

> **Note** that in a (semi)lattice satisfying (2.3.1) this is the same as claiming that there is a decomposition $a = \bigvee_{i\in J} c_i$ with $c_i \in \mathsf{Cn}L$ such that $c = c_j$ for some j and that for $i \neq j$, $c \not\leq c_i$.

For a subset $A \subseteq L$ write

$$A_{\mathsf{Cn}}$$

for the set of all connected componenets of the elements from P.

2.4. A *duality pair* (l, r) in L is a pair of elements such that

$${\downarrow}r = L \setminus {\uparrow}l$$

(that is,

$$l \not\leq x \quad \text{iff} \quad x \leq r.\ )$$

The element r (obviously uniquely determined by l) is then called the *right dual* of l, and similarly l called the *left dual* of r. If the other element of the pair is not specified we speak of a *right* resp. *left dual* r resp. l. Briefly, one speaks of an *r-dual* resp. *l-dual.*

> **Note.** The elements l that are left duals are always connected. More, whenever $l \leq \bigvee x_i$ for any join $\bigvee x_i$ then necessarily $l \leq x_j$ for some j (indeed, if for all j, $l \not\leq x_j$ then for all j, $x_j \leq b$, and hence $\bigvee x_i \leq b$ and $l \not\leq \bigvee x_i$). This property is usually called *supercompactness*, for obvious reasons.

More generally, a *(finite) duality* in L is a pair (A, B) of finite subsets of L such that

(1) distinct elements in A resp. B are incomparable, and
(2) $x \in {\uparrow}A$ iff $x \notin {\downarrow}B$ (in other words, ${\downarrow}B = L \setminus {\uparrow}A$).

(Compare with 1.3.) An element l (resp. r) there is a duality (A, B) and $l \in A$ (resp. $r \in B$) is called a *weak left dual* (resp. *weak right dual*); briefly one speaks of *wl-duals* resp. *wr-duals.*

2.4.1. Fact. *The object B in a finite duality (A, B) is (up to the homomorphism equivalence $B \leftrightarrow B'$) uniquely determined by A, and vice versa.*

Proof. If $(A, B_1), (A, B_2)$ are finite dualities then $\downarrow B_1 = \downarrow B_2$, and hence for each $x \in B_1$ there is an $\alpha(x) \in B_2$ such that $x \leq \alpha(x)$, and similarly for each $x \in B_2$ there is a $\beta(x) \in B_1$ such that $x \leq \beta(x)$. Thus, $x \leq \beta\alpha(x)$ and $x \leq \alpha\beta(x)$ and by uncomparability $\alpha\beta = \mathrm{id}$ and $\beta\alpha = \mathrm{id}$, and finally $x \leq \alpha(x) \leq x$ and $\alpha(x) = x$ and siilarly $\beta(x) = v$. □

2.5. We will write

$$N \leq M \quad \text{for} \quad N \subseteq \uparrow M \tag{2.5.1}$$

(the right hand side is, of course, equivalent to $N \subseteq \uparrow M \subseteq \uparrow N$); one sometimes speaks of M as of a *refinement* of N; thus, if $N \leq M$ then N is *coarser.*

A subset $M \subseteq A_{\mathsf{Cn}}$ is said to be a *transversal* (of a duality (A, M)) if

(T1) distinct elements of M are incomparable,
(T2) $A \subseteq \uparrow M$, and
(T3) in the refinement order, M is minimal with respect to the property (T2).

The subsets satisfying just (T1) and (T2) are called *quasitransversal.*

Note that by (T2)

$$\uparrow A \subseteq \uparrow M \quad \text{and hence} \quad L \setminus \uparrow M \subseteq \downarrow B.$$

2.6. Set

$$\overline{M} = A_{\mathsf{Cn}} \setminus \uparrow M.$$

Note that if $M \leq N$ then $\overline{M} = \overline{N}$.

2.6.1. Lemma. *Let $\overline{M} = \emptyset$ for a transversal M of (A, B). Then*

1. *$A = M = A_{\mathsf{Cn}}$, and*
2. *B has only one element.*

Proof. 1. Since $A_{\mathsf{Cn}} \setminus \uparrow M = \emptyset$ we have $A_{\mathsf{Cn}} \subseteq \uparrow M$ and since $M \subseteq A_{\mathsf{Cn}}$, $A_{\mathsf{Cn}} \subseteq \uparrow M \subseteq \uparrow A_{\mathsf{Cn}}$, and hence $A_{\mathsf{Cn}} \leq M$. By (T3), as A_{Cn} is a semitransversal, $A_{\mathsf{Cn}} = M$.

Suppose that there is a $c \in A_{\mathsf{Cn}}$ such that for all $l \in A$ there is a $c \in A_{\mathsf{Cn}}$, $c' \neq c$. Then $A_{\mathsf{Cn}} \setminus \{c\}$ is still a transversal and we have a contradiction with (T3) again. Hence,

$$\forall c \in A_{\mathsf{Cn}} = M \ \ \exists l_c \text{ such that } c' \leq l_c \ \& \ c' \in A_{\mathsf{Cn}} \ \Rightarrow \ c = c'.$$

But then $l_c = c$, hence $A \supseteq M = A_{\mathsf{Cn}}$, and since the elements of A are incomparable, $A \setminus M = A_{\mathsf{Cn}}$.

2. Let $r_1, r_2 \in B$ be distinct. Then, by the incomparability condition, $r_1 \vee r_2 \not\leq r$ for all $r \in B$ and hence $l \leq r_1 \vee r_2$ for some $l \in A$. Now by 1, l is connected and hence $l \leq r_1$ or $l \leq r_2$, a contradiction. □

2.6.2. Lemma. *Let M be a transversal of (M, B). Then*

1. $M = M_{\mathsf{Cn}}$, *and*
2. *there is no other transversal.*

Proof. Trivially $M \subseteq M_{\mathsf{Cn}}$. Now let $x \in M_{\mathsf{Cn}}$ an let $m \in M$ be such that $= y \vee x \neq y$. Since m is connected, $m \leq x$. Thus, $x = m \in M$.

Now if N is another transversal then $N \subseteq M_{\mathsf{Cn}} = M \subseteq {\uparrow}N$, and $N \leq M$. Thus, by (T3), $N = M$. □

Lemma 2.6.3. *Let M be a transversal of (A, B) Then there is precisely one $r \in B$ such that*

(1) $M \cap {\downarrow}r = \emptyset$, *and*
(2) $\overline{M} \subseteq {\downarrow}r$.

Proof. I. If there is $\overline{M} = \emptyset$ for a transversal M we have $B = \{r\}$ by 2.6.1, and this r satisfies the conditions (and is unique).

II. Now suppose that A is not its own transversal. Then $\overline{M} \neq \emptyset$. Set $s = \bigvee \overline{M}$. We have $s \notin {\uparrow}M$ (if $x \in M$ and $x \leq s$ then by connectedness $x \leq y \in \overline{M}$), hence $s \notin {\uparrow}A$, and consequently $s \in {\downarrow}B$ and we have a $r \in B$ such that $s \leq r$ so that $\overline{M} \subseteq {\downarrow}s \subseteq {\downarrow}r$.

Now suppose $c \in M \cap {\downarrow}r$. By (T3), $A \not\subseteq {\uparrow}(M \setminus \{c\})$. Choose a $l \notin {\uparrow}(M \setminus \{c\})$ and let $l = \bigvee c_i$ be a connected decomposition of l. If $c_i \geq b \in M$ then $b = c$ and hence $c_i = c$; thus, either $c_i = c$ or $c_i \in \overline{M}$, and $l \leq r$ contradicting the duality.

Finally let distinct r_1, r_2 have the property. Then $r_1 \vee r_2 \notin {\downarrow}B$, hence $r_1 \vee r_2 \in {\uparrow}M$ and there is a (connected) $x \in M$ such that $x \leq r_1 \vee r_2$; thus, say, $x \leq r_1$ contradicting the duality. □

2.7. The uniquely determined r from 2.6.3 will be denoted by

$$r(M).$$

Note that if A is not its own transversal $r(M)$ is determined by the formula

$$\bigvee \overline{M} \leq r(M) \in B. \tag{$*$}$$

2.8.1. Lemma. *If M_1, M_2 are distinct transversals then $\overline{M}_1 \cap M_2 \neq \emptyset$.*

Proof. If $M_1 \neq M_2$ then $M_1 \not\leq M_2$ and hence there is a $c \in M_2$ such that $c \notin {\uparrow}M_1$. Then $c \in (A_{\mathsf{Cn}} \setminus {\uparrow}M_1) \cap M_2$. □

2.8.2. Lemma. *For $r \in B$ set $M = \{x \in A_{\mathsf{Cn}} \mid x \not\leq r\}$. Then M is a quasitransversal and if M_r is a transversal with $M_r \leq M$ then $r(M_r) = r$.*

Proof. Let $l \in A$. Then $l = \bigvee\{x \in A_{\mathsf{Cn}} \mid x \leq l\} \not\leq r$ and hence there is an $x \in A_{\mathsf{Cn}}$, $x \leq l$ such that $x \not\leq r$.

We have $\overline{M}_r = \overline{M}$. Suppose $\bigvee \overline{M} \not\leq r$. Then there is an $x \in \overline{M}$ such that $x \not\leq r$, that is, $x \in M$, a contradiction. □

2.8.3. Proposition. *Let (A, B) be a duality, The formulas*

$$M \mapsto r(M) \quad \textit{where} \quad \bigvee \overline{M} \leq r(M) \in B \quad (\textit{if } \overline{M} \neq \emptyset),$$
$$r \mapsto M_r \quad \textit{where} \quad M_r \leq M = \{x \in A_{\mathsf{Cn}} \mid x \not\leq r\}$$

constitute a one-one correspondence between the transversals of (A, B) and elements of B.

Proof. We already know that $r(M_r) = r$. Let M_1, M_2 be distinct transverals. By 2.8.1 there is a $c \in \overline{M}_1 \cap M_2$. By 2.6.3(1), $c \not\leq {\downarrow} r(M_1)$ and by $(*)$ from 2.7, $c \leq {\downarrow} r(M_2)$. Hence $r(M_1) \neq r(M_2)$. □

2.9. Proposition. *Each transversal M together with the element $r(M)$ constitutes a duality $(M, \{r(M)\})$.*

Proof. Set $r = r(M)$. We have $M \subseteq L \setminus {\downarrow} r(M)$ and hence ${\uparrow} M \subseteq L \setminus {\downarrow} r(M)$ by 5(1).

Now let $x \notin {\uparrow} M = \bigcup\{{\uparrow} c \mid c \in M\}$. Thus, $c \not\leq x$ for all $c \in M$, and by connectedness $c \not\leq x \vee \bigvee \overline{M}$ for all $c \in M$, Suppose $l \leq y$ for some $l \in A$. If $c \in M$ and $c \leq l$ we have $c \leq y$ and hence $c \leq x$, a contradiction. Thus, $y \notin {\uparrow} A$ and hence $y \leq {\downarrow} B$, that is, $y \leq r'$ for some $r' \in B$. But then $\bigvee \overline{M} \leq r'$ and hence $r' = r$. Thus, $x \in {\downarrow} r$. □

3. Connected components of weak left duals are left duals

In 2.3.8 we have seen that given a finite duality (A, B), each element $r \in B$ is in a duality $(M, \{r\})$. In this section we will obtain dualities in the reversed order. Instead of dualities for elements $l \in A$ we will have them for the $c \in A_{\mathsf{Cn}}$. For these, however we will prove something stronger. Namely, we will show that each such element is a left dual.

Unlike the previous section we will have to assume that the lattice L is Heyting (and hence the categorical interpretation holds for Heyting categories only).

3.1. We will need two facts from [NTP].

3.1.1. Proposition. ([NTP] 2.6.) *The gaps in a Heyting algebra L with connected decompositions are exactly the pairs (a, b) such that for some duality (l, r),*

$$l \wedge r \leq a \leq r \quad \textit{and} \quad b = a \vee l.$$

3.1.2. Proposition. ([NTP] 3.3.) *Let L be a Heyting algebra L with connected decompositions, let $A = \{l_i \mid i \in J\}$ be a subset of L and let $r \in L$. Let either J be finite or L admit infima of sets of the size of the J. Then the pair $(A, \{r\}$ is a duality if and only if there are dualities (l_i, r_i), $i \in J$, such that*

$$r = \bigwedge_{i \in J} r_i.$$

3.2. Lemma. *Every element of a transversal is a left dual.*

Proof. Let M be a transversal and let $c \in M$. We have the duality $(M, r(M)$, by 2.9. Thus, by 3.1.2 there are dualities (m, r_m), in particular (c, r_c). □

3.3. Proposition. *In a Heyting algebra with connected decompositions, a connected component of a weak left dual is a left dual.*

Proof. Let (A, B) be a duality and let $c \in A_{\mathsf{Cn}}$. Suppose it is not a left dual; then in particular, by 3.2, it is contained in no transversal.

Set

$$a = \bigvee \{c' \in A_{\mathsf{Cn}} \mid c' < c\} \vee \bigvee \{c \wedge c' \mid c' \in A_{\mathsf{Cn}},\ c, c' \text{ incomparable}\}.$$

We have $a < c$ since else by the connectedness of c some of the summands would be equal to c which they are not. Now the couple (a, c) is not a gap: else we would have, by 3.1.1 a duality (l, r) such that $c = a \vee l$ – and hence $c = l$.

Thus there exists an x such that

$$a < x < c.$$

Claim. *If $c' \in A_{\mathsf{Cn}}$ and $c' \neq c$ then*

$$c' \leq x \quad \textit{iff} \quad c' < c, \quad \textit{and}$$
$$c' \geq x \quad \textit{iff} \quad c' > c.$$

Proof of Claim. In the first case: if $c' \leq x$ then $c' \leq x < c$, and if $c' < c$ then $c' \leq a < x$.

Now consider the second case. Trivially if $c < c'$ then $x \leq c'$. Now suppose $x \leq c'$. Then if $c' < c$ we have $x = c'$ by the first equivalence, hence $c' = x < c$ and $c' \leq a$ and we have a contradiction $x \leq a$. If c and c' are incomparable then $x \leq c \wedge c' \leq a$, a contradiction again. Thus, $c < c$' is the only alternative left. □□

Proof continued. Let $l \in A$ be such that c is one of its connected components and let $l = b \vee c \neq b$ be a decomposition witnessing the fact. Set $q = b \vee x$. We cannot have $l \leq q$ since else $c \leq b$ and $b \leq l \leq b$ contradicting the choice of the decomposition. Consequenly, we also have $l' \not\leq q$ for any other $l' \in A$ since otherwise $q \leq l$. Thus, $\forall r \in A,\ r \not\leq q$ and hence

$$\exists r \in B, \quad q \leq r.$$

Let M be such that $r = r(M)$ so that in particular

$$\forall m \in M,\ m \not\leq q.$$

We have $c \notin M$ since c is in no transversal, and hence $m \not\leq x$ for all $m \in M$. By Claim, $m \not\leq c$ for all $m \in M$, and hence $c \leq r$.

Now, since $q \leq r$, $l = q \vee c \leq r$ contradicting the duality (C, B). □

3.3.1. Corollary. *If a Heyting algebra with connected decomposition has no non-trivial duality pair then it admits no finite duality.*

3.4. Note. Compare the following two facts (the first obtained combining 2.8.3 and 3.1.2, the second is an immediate consequence of 3.3) holding in Heyting algebras with connected decompositions:

- each weak right dual is a meet of right duals, and
- each weak left dual is a join of left duals.

(The fact from which these statements follow are, of course, stronger.)

4. The transversal construction reversed: from dual pairs to finite dualities

In previous sections the notion of a transversal helped to analyze finite dualities (A, B) in terms of the individual elements of A and B. The elements $r \in B$ have been shown to be naturally associated with transversals of (A, B) (in 2.8.3), and then the elements of A have been shown to be joins of left duals (see 3.3). In this section we will use the procedure reversedly: namely, for a finite set A of sums of left duals we will construct a finite duality.

4.1. Observation. *In any lattice, if (l_i, r_i), $i = 1, \ldots, n$, are dual pairs then $(\{l_1, \ldots, l_n\}, \{\bigwedge_{1=1}^{n} r_i\})$ is a duality.*

(Indeed, $\forall i, l_i \not\leq x$ iff $\forall i, x \leq r_i$ iff $x \leq \bigwedge r_i$.)

4.2. Convention. In the definition of a transversal and semitransversal associated with a finite duality (A, B) the set B had not play any role (before one started to prove facts about the dualities). Now let us adopt the A_{Cn}, the order from (2.5.1) and the concept of a transversal and semitransversal to start with as related to a general finite subset $A \subseteq L$.

4.3. Lemma. *In a distributive lattice let c be a connected component of an $a \in L$ and let $a = \bigvee_{i=1}^{n} a_i$. Then c is a connected component of some of the a_i.*

Proof. Let $a = x \vee c \neq x$. By the connectedness, $c \leq a_i$ for some i. Then $a_i = (x \vee c) \wedge a_i = (x \wedge a_i) \vee c \neq x \vee a_i$ since otherwise $c \leq x$ and $a = x$. □

4.4. Proposition. *Let L be a Heyting algebra with finite connected decompositions. Let A be a finite set such that each $a \in A$ is a finite join of left duals. Then there exists a finite duality (A, B).*

Proof. Let $a = \bigvee_{i=1}^{n_a} c_i(a)$ be connected decompositions of the $a \in A$. Then

$$A_{\mathsf{Cn}} \subseteq \{c_i(a) \mid a \in A,\ i = 1, \dots n_a\}.$$

Now if $a = \bigvee_{j=1}^{k} a_j$ with a_j duals then each comnnected component of a is, by 4.3, a connected component of some of the a_j, and hence each $c_i(a) \in A_{\mathsf{Cn}}$ is, by 3.3, a left dual. Denote by $r_i(a)$ the corresponding right dual.

Let $\mathcal{M}$ be the set of transversals of A, hence $M \subseteq A_{\mathsf{Cn}}$. Trivially, it is finite. For $M \in \mathcal{M}$ set

$$r_M = \bigwedge \{r_i(q) \mid c_i(q) \in M\}$$

and consider

$$B = \{r_M \mid M \in \mathcal{M}\}.$$

Let $x \leq r_M$ for some $M \in \mathcal{M}$. Then for all $c_i(q) \in M$, $x \leq r_i(q)$ and hence $c_i(q) \nleq x$. For an arbitrary $a \in A$ there is a $c_i(q) \leq a$ and hence $a \nleq x$.

On the other hand let $a \nleq x$ for all $a \in A$. Thus, for each $a \in A$ we have a connected component $x_{i_a}(a)$ such that $x_{i_a}(a) \nleq x$. Set

$$M' = \{x_{i_a}(a) \mid a \in A\}$$

and consider M'' the system of all minimal elements of M' (to satisfy (T1)). Now M'' is a semitransversal and we have a transversal $M \leq M''$. Then for each $c_i(g) \in M$ we have $c_i(q) \nleq x$, hence $x \leq r_i(q)$, and finally $x \leq r$. □

From 3.3 and 4.4 we now immediately obtain

4.4.1. Corollary. *Let L be a Heyting algebra with finite connected decompositions. Then the following statements on an element $a \in L$ are equivalent:*

(1) *a is a weak left dual,*
(2) *a is a finite join of left duals.*

4.5. Define

$$\mathfrak{wld}(L)$$

as the set of all the weak left duals in L (this is the obvious abbreviation, but, by coincidence it also alludes to the german word “Wald” for forest;

it so happens that in case of binary relations the weak left duals are precisely the disjoint unions of trees, the forests). Then, by 4.4 (and 2.4.1) we have

4.5.1. Corollary. *For each subset $A \subseteq \mathfrak{wld}(L)$ there is precisely one duality (A, B).*

4.6. Note. By 4.1 and the definition of r_M we have the dualities $(M, \{r_M\})$ and hence if $c_i(q) \in \overline{M}$, that is, $c_i(q) \notin \uparrow M$, then $c_i(q) \leq r_M$. Thus,

$$\bigvee \overline{M} \leq r_M$$

and $r_M = r(M)$ as in 2.8.3.

5. Sparse incompatibility and antichains

5.1 In [26] one can find the following fact.

Sparse Incompatibility Lemma. *Let m, k be positive integers and let H be a directed graph which is not an orientation of a forest. Then there exists a directed graph H' such that*

(1) *the girth of H' is finite and greater than k,*
(2) *for each directed graph G with fewer than m vertices, we have $H' \to G$ iff $H \to G$, and*
(3) *$H \not\to H'$ and $H' \to H$.*

It should be now clear why the following assumption will be made make in the Heyting context.

Sparse incompatibility axiom – briefly, **SIA**.

This is the assumption that for any $x \in L$, any M, U finite subsets of L such that $(\{x\} \cup \uparrow U) \cap \mathfrak{wld}(L) = \emptyset$, there is a $y \in L \setminus \mathfrak{wld}(L)$ such that

$$\text{(SIA)} \quad y \in \uparrow\{x\}, \quad y \notin \uparrow(\{x\} \cup U) \quad \text{and} \quad \forall m \in M, \ y \leq m \text{ iff } x \leq m.$$

5.2. Observation. *If (A, B) is a finite duality in a lattice L, then*

$$A \cup (B \setminus \downarrow A)$$

is a finite maximal antichain in L.

(Indeed, it is an antichain because $a \not\leq b$ for any $a \in A$, $b \in B$. It is maximal because each $x \in L$ is either in $\uparrow A$ or there is a $b \in B$ with $x \leq B$; in the latter case, if $b \leq a$ for an $a \in A$ we have $x \leq a$.)

5.3.1. Lemma. *In a Heyting algebra with SIA and finite connected decompositions let C be a finite maximal antichain. Set $A = C \cap \downarrow\mathfrak{wld}(L)$. Then*

$$\uparrow C \setminus C = \uparrow A \setminus C.$$

Proof. The inclusion $\supseteq$ is trivial. Thus, let $x \in \uparrow C \setminus C$ and set $U = C \setminus A$.

If $x \notin \uparrow U$ then $x \in \uparrow A$ and hence $x \in \uparrow A \setminus C$.

If $x \in \uparrow U$ then $x \notin \mathfrak{wld}(L)$. We have $\uparrow U \cap \mathfrak{wld}(L) = \emptyset$ and hence we can apply SIA to obtain a $y \notin \mathfrak{wld}(L)$ such that

$$(*) \qquad y \notin \uparrow(\{x\} \cup U) \quad \text{and} \quad \forall m \in C \cup \{x\},\ y \leq m \text{ iff } x \leq m.$$

Now $x \in \uparrow C \setminus C$ and hence if $m \in C \cup \{x\}$ then $x \leq m$ only if $x = m$ and consequently $y \notin \downarrow C$. Since C is a maximal antichain, $y \in \uparrow C \setminus C$. By $(*)$, $y \notin U = C \setminus A$, hence $y \in \uparrow A$ and since $y \leq x$ we have, by $(*)$ again, $x \in \uparrow A$. $\square$

5.3.2. Proposition. *In a Heyting algebra with SIA and finite connected decompositions let C be a finite maximal antichain. Set $A = C \cap \downarrow \mathfrak{wld}(L)$ and consider the unique finite duality (A, B). Then*

$$C = A \cup (B \setminus \downarrow A).$$

Proof. If $b \in B$ then $b \notin \uparrow A$ and hence, by 5.3.1, $b \notin \uparrow C \setminus C$. Consequently, since C is a maximal antichain, $b \in \downarrow C$.

Now suppose that, moreover, $b \notin \downarrow A$. We want to prove that $b \in C$. If not, $b < c$ for some $c \in C$ and this means, by our assumption, that $c \in C \setminus A$. Then $c \not\leq b'$ for all $b' \in B$ and hence, by duality, $a \leq c$ for some $a \in A$ contradicting the antichain property. Thus, $b \in C$ and we have $A \cup (B \setminus \downarrow A) \subseteq C$, and since by 5.2 $A \cup (B \setminus \downarrow A)$ is a maximal antichain, $A \cup (B \setminus \downarrow A) = C$. $\square$

5.4. Remarks. Sparse Incomparability Lemma has a long and interesting history. While it seems to have been formulated specifically in this form first in [26] for $G = K_n$ and then in [31] for general G, it was preceeded in the seminal work on sparse graphs with high chromatic number by Erdös and others ([3, 7, 15, 4, 12, 19, 20, 2, 13, 14, 33]. This useful lemma is related to an important result in descriptive complexity by Kun ([13]), and to the more recent result on limits in graph sequemces ([22, 23]).

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