On characteristics of homomorphism and embedding universal graphs

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Abstract

We relate the existence problem of universal objects to the properties of corresponding enriched categories (lifts and expansions). Particularly, extending earlier results, we prove that for every countable set $\mathcal{F}$ of finite structures there exists a (countably) universal structure $\mathbf{U}$ for the class $Forb_h(\mathcal{F})$ (of all countable structures omitting a homomorphism from all members of $\mathcal{F}$). In fact $\mathbf{U}$ is the shadow (reduct) of an ultrahomogeneous structure $\mathbf{U}'$ (which however, as we will show, cannot be expressed as $Forb_h(\mathcal{F}')$ for a countable set $\mathcal{F}$; this is in a sharp contrast to the case when $\mathcal{F}$ is finite). We also put the results of this paper, perhaps for the first time, in the context of homomorphism dualities and Constraint Satisfaction Problems.

1 Introduction

A *structure* $\mathbf{A}$ is a pair $(A, (R^i_{\mathbf{A}}; i \in I))$ where $R^i_{\mathbf{A}} \subseteq A^{\delta_i}$ (i.e. $R^i_{\mathbf{A}}$ is a δ_i-ary relation on A). The family $(\delta_i; i \in I)$ is called the *type* Δ. The type

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is usually fixed and understood from the context. (Note that we consider relational structures only, and no function symbols.) If set A is finite we call $\mathbf{A}$ *finite structure*. A *homomorphism* $f : \mathbf{A} \to \mathbf{B} = (B, (R^i_{\mathbf{B}}; i \in I))$ is a mapping $f : A \to B$ satisfying for every $(x_1, x_2, \ldots, x_{\delta_i}) \in R^i_{\mathbf{A}} \implies (f(x_1), f(x_2), \ldots, f(x_{\delta_i})) \in R^i_{\mathbf{B}}$, $i \in I$. If f is 1-1, then f is called an *embedding*. The class of all (countable) relational structures of type Δ will be denoted by $Rel(\Delta)$.

The class $Rel(\Delta), \Delta = (\delta_i; i \in I)$, I finite, is fixed throughout this paper. Unless otherwise stated all structures $\mathbf{A}, \mathbf{B}, \ldots$ belong to $Rel(\Delta)$. Now let $\Delta' = (\delta'_i; i \in I')$ be a type containing type Δ. (By this we mean $I \subseteq I'$ and $\delta'_i = \delta_i$ for $i \in I$.) Then every structure $\mathbf{X} \in Rel(\Delta')$ may be viewed as structure $\mathbf{A} = (A, (R^i_{\mathbf{A}}; i \in I)) \in Rel(\Delta)$ together with some additional relations $R^i_{\mathbf{X}}$ for $i \in I' \setminus I$. To make this more explicite these additional relations will be denoted by $X^i_{\mathbf{X}}, i \in I' \setminus I$. Thus a structure $\mathbf{X} \in Rel(\Delta')$ will be written as

$$\mathbf{X} = (A, (R^i_{\mathbf{A}}; i \in I), (X^i_{\mathbf{X}}; i \in I' \setminus I))$$

and, by abuse of notation, briefly as:

$$\mathbf{X} = (\mathbf{A}, X^1_{\mathbf{X}}, X^2_{\mathbf{X}}, \ldots, X^N_{\mathbf{X}})$$

We call $\mathbf{X}$ a *lift* of $\mathbf{A}$ and $\mathbf{A}$ is called the *shadow* (or *projection*) of $\mathbf{X}$. In this sense the class $Rel(\Delta')$ is the class of all lifts of $Rel(\Delta)$. Conversely, $Rel(\Delta)$ is the class of all shadows of $Rel(\Delta')$. In this paper we will always consider types of shadows to be finite, however we allow countable types for the lifts (so I is finite and I' countable). Note that a lift is also in the model theoretic setting called an *expansion* and a shadow a *reduct*. (Our terminology is motivated by a computer science context, see [16].) We will use letters $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$ for shadows (in $Rel(\Delta)$) and letters $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ for lifts (in $Rel(\Delta')$).

For lift $\mathbf{X} = (\mathbf{A}, X^1_{\mathbf{X}}, \ldots, X^N_{\mathbf{X}})$ we denote by $\psi(\mathbf{X})$ the relational structure $\mathbf{A}$ i.e. its shadow. (ψ is called the *forgetful functor*.) Similarly, for a class $\mathcal{K}'$ of lifted objects we denote by $\psi(\mathcal{K}')$ the class of all shadows of structures in $\mathcal{K}'$.

Given a class $\mathcal{K}$ of countable structures, an object $\mathbf{U} \in \mathcal{K}$ is called *hom-universal* (or *universal*) for $\mathcal{K}$ (or shortly $\mathcal{K}$-hom-universal or $\mathcal{K}$-universal) if for every object $\mathbf{A} \in \mathcal{K}$ there exists a homomorphism (or an embedding $\mathbf{A} \to \mathbf{U}$).

For a structure $\mathbf{A} = (A, (R^i_{\mathbf{A}}, i \in I))$ the *Gaifman graph* (in combinatorics often called *2-section*) is the graph G with vertices A and all those edges which are a subset of a tuple of a relation of $\mathbf{A}$:

$$G = (V, E)$$

where $x, y \in E$ iff $x \neq y$ and there exists tuple $\vec{v} \in R^i_{\mathbf{A}}, i \in I$ such that $x, y \in \vec{v}$.

A *cut* in $\mathbf{A}$ is a subset C of A such that the Gaifman graph $G_{\mathbf{A}}$ is disconnected by removing set C (i.e. if C is graph theoretic cut of G_A). By a *minimal cut* we always mean an inclusion minimal cut.

If C is a set of vertices then $\overrightarrow{C}$ will denote a tuple (of length $|C|$) from all elements of R. Alternatively, $\overrightarrow{R}$ is arbitrary linear ordering of R.

By $Forb_h(\mathbf{F}_1, \ldots, \mathbf{F}_t)$ we denote the class of all graphs (and more generally relational structures) $\mathbf{A}$ for which there is no homomorphism $\mathbf{F}_i \to \mathbf{A}$ for every $i = 1, \ldots, t$. Formally $Forb_h(\mathbf{F}_1, \ldots, \mathbf{F}_t) = \{\mathbf{A}; \mathbf{F}_i \nrightarrow \mathbf{A} \text{ for } i = 1, 2, \ldots, t\}$.

This paper is structured as follows:

In Section 2 we prove, extending our earlier paper [13], that for every countable set $\mathcal{F}$ of finite connected structures the class $Forb_h(\mathcal{F})$ is the shadow of an amalgamation class $\mathcal{L}$ and, consequently, the Fraïssé limit $\lim \mathcal{L}$ has as its shadow structure $\mathbf{U}$ which is universal for $Forb_h(\mathcal{F})$. In [13] we proved more, for any finite set $\mathcal{F}$, there is a finite set $\mathcal{F}'$ of lifts such that actually $\lim \mathcal{L} = Forb_e(\mathcal{F}')$. As remarked at the end of Section 2 this is much to ask for the countable set $\mathcal{F}$. In Section 3 we review results related to characterizations of the existence of universal objects. We address the following questions: Given class $\mathcal{C}$ decide whether there exists $\mathcal{C}$ universal object $\mathbf{U}$. Given a countable object $\mathbf{U}$ decide whether $\mathbf{U}$ is universal for a class $\mathcal{C}$ (with further properties). While the embedding universal structures are necessarily countable the finiteness of homomorphism universal structures is a very interesting problem which leads to homomorphism dualities.

2 Classes omitting countable families of structures

Let $\mathcal{F}$ be a fixed countable set of finite relational structures of finite type Δ. For construction of universal structure of $Forb_h(\mathcal{F})$ we use special lifts, called $\mathcal{F}$-lifts. The definition of $\mathcal{F}$-lift is easy and resembles decomposition techniques standard in graph theory and thus we adopted similar terminology. The following is the basic notion:

Definition 2.1 *For a relational structure* $\mathbf{A}$ *and minimal cut* R *in* $\mathbf{A}$*, a* piece *of relational structure* $\mathbf{A}$ *is pair* $\mathcal{P} = (\mathbf{P}, \overrightarrow{R})$. *Here* $\mathbf{P}$ *is the structure*

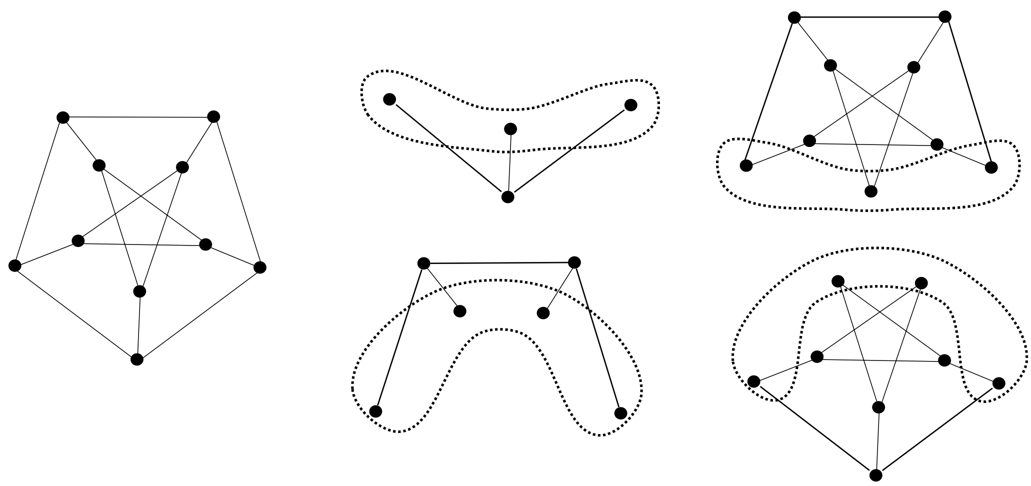

Figure 1: Pieces of the Petersen graph up to isomorphisms (and a permutations of roots).

induced on $\mathbf{A}$ *by union of* R *and vertices of some connected component of* $\mathbf{A} \setminus R$. *Tuple* $\overrightarrow{R}$ *consist of the vertices of cut* R *in a (fixed) linear order.*

Note that from inclusion-minimality of the cut R follows, that pieces of a connected structure are always connected structures.

All pieces are thought of as rooted structures: a piece $\mathcal{P}$ is a structure $\mathbf{P}$ rooted at $\overrightarrow{R}$. Accordingly, we say that pieces $\mathcal{P}_1 = (\mathbf{P}_1, \overrightarrow{R}_1)$ and $\mathcal{P}_2 = (\mathbf{P}_2, \overrightarrow{R}_2)$ are *isomorphic* if there is function $\varphi : P_1 \to P_2$ that is isomorphism of structures $\mathbf{P}_1$ and $\mathbf{P}_2$ and φ restricted to $\overrightarrow{R}_1$ is the monotone bijection between $\overrightarrow{R}_1$ and $\overrightarrow{R}_2$ (we denote this $\varphi(\overrightarrow{R}_1) = \overrightarrow{R}_2$).

Observe that for relational trees, pieces are equivalent to rooted branches. Pieces of the Petersen graph are shown at Fig 2.

Lemma 2.1 *Let* $\mathcal{P}_1 = (\mathbf{P}_1, \overrightarrow{R}_1)$ *be a piece of structure* $\mathbf{A}$ *and* $\mathcal{P}_2 = (\mathbf{P}_2, \overrightarrow{R}_2)$ *a piece of* $\mathbf{P}_1$. *If* $R_1 \cap P_2 \subseteq R_2$, *then* $\mathcal{P}_2$ *is also a piece of* $\mathbf{A}$.

Proof. Denote by $\mathbf{C}_1$ connected component of $\mathbf{A} \setminus R_1$ that produces $\mathcal{P}_1$. Denote by $\mathbf{C}_2$ component of $\mathbf{P}_1 \setminus R_2$ that produces $\mathcal{P}_2$. As $R_1 \cap P_2 \subseteq R_2$ one can check that then $\mathbf{C}_2$ is contained in $\mathbf{C}_1$ and every vertex of $\mathbf{A}$ connected by tuple to any vertex of $\mathbf{C}_2$ is contained in $\mathbf{P}_1$. Thus $\mathbf{C}_2$ is also connected component of $\mathbf{A}$ created after removing vertices of R_2. □

Fix index set I' and let $\mathcal{P}_i, i \in I'$, be all pieces of all relational structures $\mathbf{F} \in \mathcal{F}$. Notice that there are only countably many pieces.

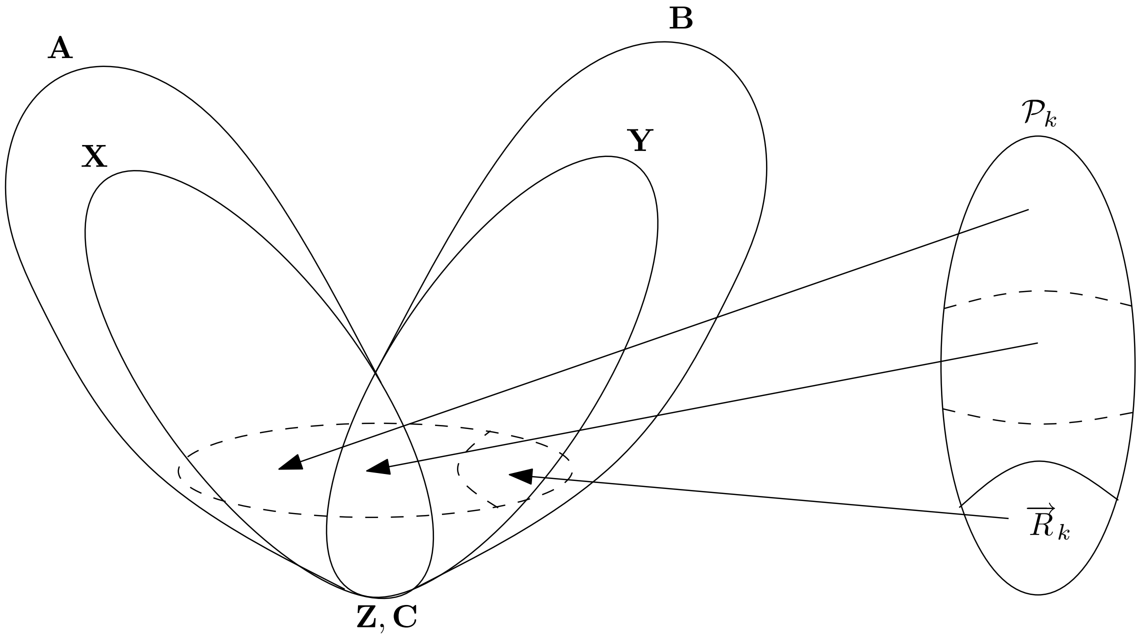

Figure 2: Construction of amalgam.

Relational structure $\mathbf{X} = (\mathbf{A}, (X^i_{\mathbf{X}}, i \in I'))$ is called $\mathcal{F}$-*lift* of relational structure $\mathbf{A}$ when the arities of relations $X^i_{\mathbf{X}}, i \in I'$ correspond to $|\overrightarrow{R}_i|$.

For relational structure $\mathbf{A}$ we define *canonical lift* $\mathbf{X} = L(\mathbf{A})$ by putting $(v_1, v_2, \ldots, v_l) \in X^i_{\mathbf{X}}$ iff there is homomorphism φ from $\mathbf{P}_i$ to A such that $\varphi(\overrightarrow{R}_i) = (v_1, v_2, \ldots, v_l)$

Theorem 2.2 *Let $\mathcal{F}$ be a countable set of finite connected relational structures. The class $\mathcal{L}$ of all induced (embedding) substructures of lifts $L(\mathbf{A})$, $\mathbf{A} \in Forb_h(\mathcal{F})$, is a Fraïssé class.*

Consequently, there is a generic structure $\mathbf{U}$ in $\mathcal{L}$ and its shadow $\psi(\mathbf{U})$ is universal structure for class $Forb_h(\mathcal{F})$.

For $\mathbf{X} \in \mathcal{L}$ we denote by $W(\mathbf{X})$ one of structures $\mathbf{A} \in Forb_h(\mathcal{F})$ such that structure $\mathbf{X}$ is induced on X by $L(\mathbf{A})$. $W(\mathbf{X})$ is called a *witness* of the fact that $\mathbf{X}$ belongs to $\mathcal{L}$.

Proof. By definition the class $\mathcal{L}$ is hereditary, isomorphism closed, has a joint embedding property. $\mathcal{L}$ is countable, because there are only countably many structures in $Forb_h(\mathcal{F})$ (because type Δ is finite) and thus also countably many lifts. To show that $\mathcal{L}$ is Fraïssé class it remains to verify that $\mathcal{L}$ has the amalgamation property.

Consider $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathcal{L}$. Assume that structure $\mathbf{Z}$ is substructure induced by both $\mathbf{X}$ and $\mathbf{Y}$ on Z and without loss of generality assume that $X \cup Y = Z$.

Put

$$\mathbf{A} = W(\mathbf{X})$$

$$\mathbf{B} = W(\mathbf{Y})$$

$$\mathbf{C} = \psi(\mathbf{Z})$$

Because $\mathcal{L}$ is closed under isomorphism, we can still assume that $\mathbf{A}$ and $\mathbf{B}$ are vertex disjoint with exception of vertices of $\mathbf{C}$.

Let $\mathbf{D}$ be free amalgam of $\mathbf{A}$ and $\mathbf{B}$ over vertices of $\mathbf{C}$: vertices of $\mathbf{D}$ are $A \cup B$ and there is $\vec{v} \in R^i_{\mathbf{D}}$ iff $\vec{v} \in R^i_{\mathbf{A}}$ or $\vec{v} \in R^i_{\mathbf{B}}$.

We claim that the structure

$$\mathbf{V} = L(\mathbf{D})$$

is (not necessarily free) amalgam of $L(\mathbf{A}), L(\mathbf{B})$ over $\mathbf{Z}$ and thus also amalgam of $\mathbf{X}, \mathbf{Y}$ over $\mathbf{Z}$.

First we show that the substructure induced by $\mathbf{V}$ on A is $L(\mathbf{A})$ and that the substructure induced by $\mathbf{V}$ on B is $L(\mathbf{B})$. In the other words no new tuples to $L(\mathbf{A}), L(\mathbf{B})$ (and thus also $\mathbf{X}$ and $\mathbf{Y}$) was introduced.

Assume the contrary that there is a new tuple $(v_1, \ldots, v_t) \in X^k_{\mathbf{V}}$ and among all tuples and possible choices of k choose one with the minimal number of vertices of the corresponding piece $\mathcal{P}_k$. By symmetry we can assume that $v_i \in A, i = 1, \ldots, t$. Explicitely, we assume

that there is homomorphism φ from $\mathbf{P}_k$ to $\mathbf{D}$ such that

$$\varphi(\overrightarrow{R_k}) = (v_1, v_2, \ldots v_t) \notin X^k_{L(\mathbf{A})}.$$

The set of vertices of $\mathbf{P}_k$ mapped to $L(\mathbf{A})$, $\varphi^{-1}(A)$, is nonempty, because it contains all vertices of $\overrightarrow{R_k}$. $\varphi^{-1}(B)$ is nonempty because there is no homomorphism φ' from $\mathbf{P}_k$ to $\mathbf{A}$ such that $\varphi'(\overrightarrow{R_k}) = (v_1, v_2, \ldots v_t)$ (otherwise we would have $(v_1, v_2, \ldots v_t) \in X^k_{L(\mathbf{A})}$).

Because there are no edges from vertices $A \setminus C$ to vertices $B \setminus C$ in $\mathbf{D}$ and because pieces are connected we also have $\varphi^{-1}(C)$ nonempty. Additionally the vertices of $\varphi^{-1}(C)$ form a cut of $\mathbf{P}_k$.

Denote by $\mathbf{K}_1, \mathbf{K}_2, \ldots \mathbf{K}_l$ all connected components of substructure induced on $P_k \setminus \varphi^{-1}(A)$ by $\mathbf{P}_k$. For each component $\mathbf{K}_i$, $1 \leq i \leq l$ there is vertex cut K'_i of $\mathbf{P}_k$ constructed by all vertices of $\varphi^{-1}(A)$ connected to K_i. This cut is always contained in $\varphi^{-1}(C)$.

Because $\mathcal{P}_k$ is piece of some $\mathbf{F} \in \mathcal{F}$ and because $(\mathbf{K}_i, \overrightarrow{K'_i})$ are pieces of $\mathbf{P}_k$, by Lemma 2.1, they are also pieces of $\mathbf{F}$. We can denote by $\mathcal{P}_{k_1}, \mathcal{P}_{k_2}, \ldots \mathcal{P}_{k_l}$ the pieces isomorphic to pieces $(\mathbf{K}_1, \overrightarrow{K'_1}), (\mathbf{K}_2, \overrightarrow{K'_2}), \ldots (\mathbf{K}_l, \overrightarrow{K'_l})$ via isomorphism $\varphi_1, \varphi_2, \ldots \varphi_l$.

Now we use the minimality of the piece $\mathcal{P}_k$. All the pieces $\mathcal{P}_k, i = 1, \ldots, l$ have smaller size than $\mathcal{P}_k$ (as $\varphi^{-1}(C)$ is a cut of $\mathcal{P}_k$). Thus we have that tuple

$\varphi(K_i)$ of $L(\mathbf{D})$ is as well a tuple of $L(\mathbf{A})$. Thus there exists homomorphism φ'_i from $\mathbf{K}_i$ to $\mathbf{D}$ such that $\varphi'_i(\overrightarrow{K'_i}) = \varphi(\overrightarrow{K'_i})$ for every $i = 1, 2, \ldots l$.

In this situation we define $\varphi'(x) : P_k \to A$ as follows:

1. $\varphi'(x) = \varphi'_i(x)$ when $x \in K_i$ for some $i = 1, 2, \ldots l$.

2. $\varphi'(x) = \varphi(x)$.

It is easy to see that $\varphi'(x)$ is homomorphism from $\mathbf{P}_k$ to $L(\mathbf{A})$. This is a contradiction.

It remains to verify that $\mathbf{D} \in Forb_h(\mathcal{F})$. We proceed analogously. Assume that φ is homomorphism of a $\mathbf{F} \in \mathcal{F}$ to $\mathbf{D}$. Because $\mathbf{A}, \mathbf{B} \in Forb_h(\mathcal{F})$, φ must use vertices of $\mathbf{C}$ and $\varphi^{-1}(C)$ forms cut of $\mathbf{F}$. Denote by E a minimal cut contained in $\varphi^{-1}(C)$. $\varphi(E)$ must contain tuples corresponding to all pieces of $\mathbf{F}$ having E as roots in $\mathbf{Z}$. This is a contradiction with $\mathbf{Z} \in \mathcal{L}$. □

For a family of relational structures $\mathcal{F}$ denote by $Forb_e(\mathcal{F})$ class of all structures omitting an embedding from all members of $\mathcal{F}$.

In [13] we showed that if the class $\mathcal{F}$ is finite and type Δ is finite then the class $\mathcal{L}$ can be described as class $Forb_e(\mathcal{F}')$ of some finite family $\mathcal{F}'$ that can be constructed from the family $\mathcal{F}$. This however holds only for lifts with finite types. For case of relational system with infinitely many relations this is not possible: there are even uncountably many relational systems with single vertex v: every relation may or may not contain tuple $(v, v, \ldots v)$. Only countably many of them can be forbidden in $Forb_e(\mathcal{F}')$ for $\mathcal{F}'$ countable.

3 Characterization theorems for homomorphism and embedding universal graphs

In this section we collect some informations about the existence of universal objects and their characterizations. This also puts the results of Section 2 and [13] in the new context.

3.1 Embedding universal

The basic problem (due to Lachlan [18], Cherlin et al. [3]) is the characterization of those families $\mathcal{F}$ of relational structures for which there exists a universal structure $\mathbf{U}$ in class $Forb_e(\mathcal{F})$ (of all countable relational structures omitting embedding from all members of $\mathbf{F} \in \mathcal{F}$). This is an open problem and [5] suggest the possibility that this problem may be undecidable. However if the universal graph $\mathbf{U}$ is demanded to be ω-categorical then [3] proves that this corresponds exactly to the case when $Forb_e(\mathcal{F})$ has finite *algebraic closure*. This model theoretic condition leads to a (model theoretic) proof which was, for the special case of forbidden graph homomorphism greatly simplified by combinatorial techniques in [13] and this paper. This was in fact our main motivation for [13].

Let us state the Lachlan—Cherlin problem more explicitly.

Problem 3.1 (Characterization of embedding universal) *For which finite families of finite structures $\mathcal{F}$ there exists structure $\mathbf{U} \in Forb_e(\mathcal{F})$ such that there is embedding from every structure $\mathbf{A} \in Forb_e(\mathcal{F})$ to $\mathbf{U}$ ($\mathbf{U}$ is embedding universal for class $Forb_e(\mathcal{F})$)?.*

This problem is still open even for finite families of finite connected structures. One should note that apart from classes $Forb_h(\mathcal{F})$ only handfull positive examples are known (see [5]). See [9] and [5] for the strongest negative result. Note also that [3] reduces this problem to the case of monadic lifts (2-colored) graphs.

3.2 Homomorphism universal structure

We say that class $\mathcal{F}$ of structures is *homomorphism monotone* iff $\mathbf{A} \in \mathcal{F}$ providing that there is structure $\mathbf{B} \in \mathcal{F}$ and homomorphism $\mathbf{B} \to \mathbf{A}$.

The existence of countable universal structures in class $Forb_h(\mathcal{F})$, or equivalently existence of countable universal structures in class $Forb_e(\mathcal{K})$ for homomorphism monotone classes $\mathcal{K}$ was settled positively:

Theorem 3.1 ([3]) *For every homomorphism monotone family $\mathcal{F}$ of finite connected relational structures the class $Forb_e(\mathcal{F})$ contains universal graph.*

In this paper we proved Theorem 3.1 in context of relational structures and possibly infinite families $\mathcal{F}$ of countable structures by an explicite construction of lifts.

For countable structures homomorphism, monomorphism and embedding universal structures mostly coincide. This has been proved by [3] for monomorphism and embedding universal objects. On the other hand the notions homomorphism universal and embedding universal are clearly different. Consider as an example the class of all planar graphs. In this case the finite homomorphism universal graph exists (the graph K_4 is hom-universal by virtue of 4-color problem) while neither an embedding and also monomorphism universal graph exists (see [11]). However in many cases we can prove that not only embedding universal do not exist but even homomorphism universal do not exist. This is the case e.g. with forbidding C_4 – cycle of length 4.

3.3 Finite hom-universal

Question of the existence of finite homomorphism universal object was studied independently in the context of finite dualities [23].

A *finite duality* (for structures of given type) is any equation

$$Forb_h(\mathcal{F}) = \{\mathbf{A}; \mathbf{A} \to \mathbf{D}\}$$

where $\mathbf{D}$ is finite relational structure and $\mathcal{F}$ is s finite set of finite relational structures. $\mathbf{D}$ is called dual of $\mathcal{F}$, the pair $(\mathcal{F}, \mathbf{D})$ is *dual pair*. For this case we also say that the class $Forb_h(\mathcal{F})$ has finite duality. We have the following:

Theorem 3.2 *Following conditions are equivalent:*

1. *$\mathcal{K}$ has finite homomorphism universal* $\mathbf{D}$,
2. *$\mathcal{K}$ has a finite monadic lift $\mathcal{K}'$ which has a finite duality,*

Existence of a finite homomorphism universal for $Forb_h(\mathcal{F})$ was characterized by Nešetřil and Tardif.

Theorem 3.3 ([23]) *For given finite class $\mathcal{F}$ of finite relational structures the class $Forb_h(\mathcal{F})$ contains a finite homomorphism universal iff $\mathcal{F}$ is a set of relational trees.*

Finite dualities also corresponds to the only first order definable Constraint Satisfaction Problems (A. Atserias [1], B. Rossman [24], see e.g. [12]).

In the opposite direction, we can ask when given finite $\mathbf{D}$ is universal for some $Forb_h(\mathcal{F})$, $\mathcal{F}$ finite. Or equivalently if $\mathbf{D}$ is dual of some finite set $\mathcal{F}$. Characterization of all structures that are duals was given by Larose, Loten, Tardif in [19]. Feder and Vardi [8] provided characterization of all structures $\mathbf{D}$ that are universal for $Forb_h(\mathcal{F})$, where $\mathcal{F}$ is infinite family of trees.

As the finite dualities are characterized by these results we define more general restricted version. A *$\mathcal{C}$-restricted duality* is the following statement:

$$\forall G \in \mathcal{C} : \mathbf{F} \not\to \mathbf{G} \iff \mathbf{G} \to \mathbf{D}$$

In the other words, $\mathbf{D}$ is an upper bound of set $Forb_h(\mathcal{F}) \cap \mathcal{C}$ in the homomorphism order.

As the extremal case we define:

Definition 3.1 *We say that class* $\mathcal{C}$ has all restricted dualities *if for every finite set* $\mathcal{F}$ *of connected structures we have*

$$\mathbf{G} \in Forb_h(\mathcal{F}) \iff \mathbf{G} \to \mathbf{D} \textit{ for every } \mathbf{G} \in \mathcal{C}$$

(or briefly $Forb_h(\mathcal{F}) \cap \mathcal{C} = CSP(\mathbf{D}) \cap \mathcal{C}$*)*

Examples of classes with all restricted dualities include: $\mathcal{C}$ include planar graphs, proper minor closed, bounded expansions [20]. The classes was recently characterized by Nešetřil and Ossona de Mendez [20] using limit objects.

3.4 Monadic lifts

The bound on arity of new relations of the lifts appears in several applications. Lifts with unary relations lead to homomorphism dualities (Theorem 3.2 while lifts with binary relations to structures similar to Urysohn spaces [13]. In this section we give characterization of all families $\mathcal{F}$, such that there exists lift $\mathcal{K}$ of class $Forb_h(\mathcal{F})$ that $\mathcal{K}$ is amalgamation class and in addition arity of all new relations is 1. Such lifts are called *monadic* lifts.

Finite relational structure $\mathbf{A}$ is called *core* iff every homomorphism $\mathbf{S} \to \mathbf{S}$ is surjective. Finite family of finite relational structures is called *minimal* iff all structures in $\mathcal{F}$ are cores and there is no homomorphism in between two structures in $\mathcal{F}$.

Observe that for every finite family $\mathcal{F}'$ of structures that is minimal family $\mathcal{F}'$ such that $Forb_h(\mathcal{F}) = Forb_h(\mathcal{F}')$.

We take time out for a Ramsey-type lemma:

Lemma 3.4 *For every* n *and* k *there is relational structure* $\mathbf{S} = (S, R_{\mathbf{S}})$, *with vertices* $S = S_1 \cup S_2 \cup \dots S_n$ *(sets* S_i *are mutually disjoint) and single relation* $R_{\mathbf{S}}$ *of arity* $2n$ *with the following properties:*

1. *For every* $(v_1, u_1, v_2, u_2, \dots v_n, u_n) \in R_{\mathbf{S}}$, $v_1, u_1 \in S_1$, $v_2, u_2 \in S_2$, $\dots v_n, u_n \in S_n$ *and every vertex appears in this tuple at most once.*

2. *For every two tuples* $\vec{v}, \vec{u} \in \mathbf{S}$, $\vec{v} \neq \vec{u}$, $\vec{v}$ *and* $\vec{u}$ *has at most one common vertex.*

3. *For every vertex coloring of* S *using* 2^k *colors, there is at least one tuple* $(v_1, u_1, v_2, u_2, \dots v_n, u_n) \in R_{\mathbf{S}}$ *such that colors of* v_i *and* u_i *are equivalent for every* $1 \leq i \leq n$.

4. *Every vertex of* $\mathbf{S}$ *is contained in at least one tuple of* $R_{\mathbf{S}}$.

Proof. We first construct set system $\mathcal{X} = (X, \mathcal{M})$ with following properties:

(a) Every set $A \in \mathcal{M}$ has precisely $2n$ elements.

(b) Every two sets $A, B \in \mathcal{M}, A \neq B$ intersect in at most 1 vertex.

(c) For every vertex coloring of $\mathcal{X}$ using at most 2^{k+n} colors there is set $A \in \mathcal{M}$ such that all vertices have same color.

Such set system can be constructed easily (see e.g. [22]): Put $\mathcal{X} = (\binom{N}{n-1}, \mathcal{M})$ with $\{A_1, A_2, \dots A_n\} \in \mathcal{M}$ iff the of union of sets A_i has size n. It is easy to see that for large enough N the set system $\mathcal{X}$ has properties (a) and (b). Property (c) follows from Ramsey Theorem.

Now assume arbitrary linear ordering $\leq_X$ on X. Put $S = X \times \{1, 2, \dots, n\}$, $S_i = \{(v, i), v \in X\}$ for $1 \leq i \leq n$. Finally put $(v_1, u_1, v_2, u_2, \dots v_n, u_n) \in R_{\mathbf{S}}$ iff $v_i = (v'_i, i), u_i = (u'_i, i)$ for every $1 \leq i \leq n$, $u_1, v_1, u_2, v_2, \dots u_n, v_n$ is increasing sequence in $(X, \leq_X)$ and set $\{v'_1, u'_1, v'_2, u'_2, \dots v'_n, u'_n\}$ is in $\mathcal{M}$.

We prove that set system $\mathbf{S} = (S, R_{\mathbf{S}})$ satisfies 1., 2. and 3. Property 1. follows from construction of $\mathbf{S}$. Because there is 1-1 correspondence in between tuples in $\mathbf{S}$ and sets of $\mathcal{X}$, we have immediately property 2. To verify property 3. observe that every coloring c of $\mathbf{S}$ using 2^k colors imply coloring c' of $\mathcal{X}$ using 2^{k+n} colors: the color of vertex $v \in X$ is the sequence of colors of vertices $(v, 1), (v, 2), \dots, (v, n)$ in S. By (c) there exists a set $A = \{a_1, a'_1, a_2, a'_2, \dots a_n, a'_n\}$ such that all its vertices get colour $(i_1, \dots i_n)$. But this in turn means that (a_i, i),(a'_i, i) have the same color in c.

Property 4. is not satisfied by the construction above, it is however obvious it is possible to remove isolated vertices from structure $\mathbf{S}$. ☐

Given a relational structure $\mathbf{S} = (S, R_{\mathbf{S}})$ with relation $R_{\mathbf{S}}$ of arity $2n$ and rooted relational structure $(\mathbf{A}, \overrightarrow{R})$ of type Δ with $\overrightarrow{R} = (r_1, r'_1, r_2, r'_2, \dots r_n, r'_n)$, we denote by $\mathbf{S} * (\mathbf{A}, \overrightarrow{R})$ the following relational structure $\mathbf{B}$ of type Δ:

$$B = (R_{\mathbf{S}} \times A)/\sim .$$

Thus the vertices of B are equivalence classes of equivalence $\sim$ generated by the following pairs:

$$(\vec{v}, r_i) \sim (\vec{u}, r_i) \text{ iff } \vec{v}_{2i} = \vec{u}_{2i}$$
$$(\vec{v}, r'_i) \sim (\vec{u}, r'_i) \text{ iff } \vec{v}_{2i+1} = \vec{u}_{2i+1}$$
$$(\vec{v}, r_i) \sim (\vec{u}, r'_i) \text{ iff } \vec{v}_{2i} = \vec{u}_{2i+1}$$

Denote by $[\vec{v}, r_i]$ the equivalence class of $\sim$ containing $(\vec{v}, r_i)$. We put $\vec{v} \in R^j_{\mathbf{B}}$ iff $\vec{v} = ([\vec{u}, v_1], [\vec{u}, v_2], \dots [\vec{u}, v_t])$ for some $\vec{u} \in R_{\mathbf{S}}$ and $(v_1, v_2, \dots v_t) \in R^j_{\mathbf{A}}$.

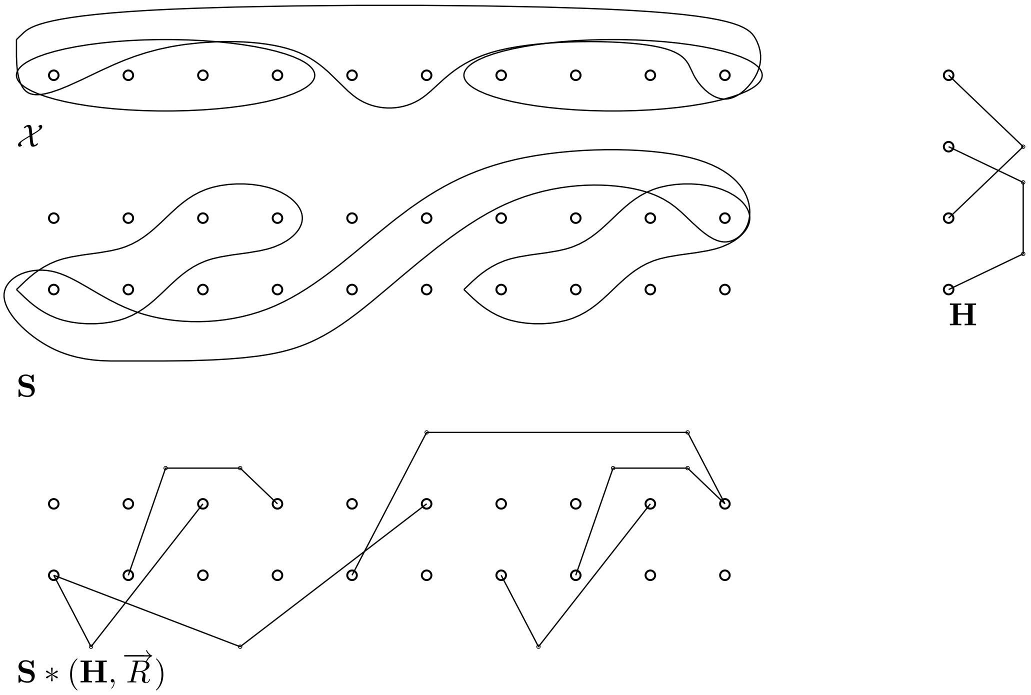

Figure 3: Construction of $\mathbf{S}$ and $\mathbf{S} * (\mathbf{H}, \vec{R})$.

This construction is commonly used in graph homomorphism context as *indicator construction.* It essentially means replacing every tuple of $R_{\mathbf{S}}$ by disjoint copy of $\mathbf{A}$ with roots $\overrightarrow{R}$ identified with vertices of the tuple.

For given vertex v of $\mathbf{S} * (\mathbf{A}, \overrightarrow{R})$ such that $v = [\vec{u}, r_i]$ (or $v = [\vec{u}, r'_i]$) we will call vertex $v' = \vec{u}_{2i}$ (or $v' = \vec{u}_{2i+1}$ respectively) *the vertex corresponding to v in* $\mathbf{S}$. Note that this give 1–1 correspondence in between vertices of $\mathbf{S}$ and $\mathbf{S} * (\mathbf{A}, \overrightarrow{R})$ restricted to vertices $[\vec{v}, r_i]$ and $[\vec{v}, r'_i]$.

Theorem 3.5 *For a minimal family $\mathcal{F}$ of relational structures, there is monadic lift of class $Forb_h(\mathcal{F})$ with just finitely many additional relations that is amalgamation class iff all minimal cuts of $\mathbf{F} \in \mathcal{F}$ consist of 1 vertex.*

Proof. Construction of lifted class $\mathcal{L}$ in proof of Theorem 2.2 adds relations of arities corresponding to the sizes of minimal cuts of $\mathbf{F} \in \mathcal{F}$ so in one direction Theorem proceede directly from the proof of Theorem 2.2.

In the opposite direction fix class $\mathcal{F}$, a relational structure $\mathbf{F} \in \mathcal{F}$ and minimal cut $C = \{1, 2, \ldots n\}$ of structure $\mathbf{F}$ of size $n > 1$. Assume, for contrary, existence of class of lifts $\mathcal{K}$ such that $\mathcal{K}$ is an amalgamation class, all new relations are monadic and shadow of $\mathcal{K}$ is $Forb_h(\mathcal{F})$. Denote by k the number of new relations.

For brevity, assume that $\mathbf{F} \setminus C$ has two connected components. Denote by $\mathcal{P}_1 = (\mathbf{P}_1, \overrightarrow{R}_1)$ and $\mathcal{P}_2 = (\mathbf{P}_2, \overrightarrow{R}_2)$ the pieces generated by C such that $\overrightarrow{R}_1 = \overrightarrow{R}_2 = \{r_1, r_2, \ldots r_n\}$. For 3 and more pieces we can proceed analogously.

Now we construct relational structure $\mathbf{H}$ as follows:

$$H = (P_1 \times \{1\}) \cup (P_2 \times \{2\})$$

and put

$$((v_1, 1), \ldots (v_t, 1)) \in R^i_{\mathbf{H}} \text{ iff } (v_1, \ldots v_t) \in R^i_{\mathbf{P_1}}$$
$$((v_1, 2), \ldots (v_t, 2)) \in R^i_{\mathbf{H}} \text{ iff } (v_1, \ldots v_t) \in R^i_{\mathbf{P_2}}$$

with no other tuples. In other words, $\mathbf{H}$ is disjoint union of $\mathbf{P}_1$ and $\mathbf{P}_2$. We will consider $\mathbf{H}$ rooted by tuple

$$\overrightarrow{R} = ((r_1, 1), (r_1, 2), (r_2, 1), (r_2, 2), \ldots, (r_n, 1), (r_n, 2)).$$

Take relational structure $\mathbf{S}$ from Lemma 3.4 and put $\mathbf{D} = \mathbf{S} * (\mathbf{H}, \overrightarrow{R})$. Construction for pieces of 5-cycle is shown at Figure 3.4. For vertex $v \in S$ denote by $m(v)$ the vertex of $\mathbf{D}$ corresponding to v (if it exists) or arbitrary vertex of $\mathbf{D}$ otherwise.

Relational structure $\mathbf{D}$ consists of disjoint copies of structures $\mathbf{P}_i$ (pieces of $\mathbf{F}$) with vertices from cut C identified. Denote by f homomorphism $\mathbf{D} \to \mathbf{F}$ mapping every vertex $(\vec{v}, (a, t)) \in D$ to $a \in F$.

First, we prove that $\mathbf{D} \in Forb_h(\mathcal{F})$. Assume, for contrary, that there is homomorphism $\varphi : \mathbf{F} \to \mathbf{D}$. By composition we have that $\varphi \circ f$ is homomorphism $\mathbf{F} \to \mathbf{F}$. Because $\mathbf{F}$ is core, we also know that $\varphi \circ f$ is surjective and thus also automorphism.

It follows that φ is injective and thus that

$\varphi(\mathbf{F})$ must use vertices from both copy of $\mathcal{P}_1$ and $\mathcal{P}_2$ (otherwise $\varphi \circ f(F)$ would be proper subset of F). Because there is at least one (relational) cycle of $\mathbf{F}$ using 2 vertices of cut C and because all copies of pieces P_i in $\mathbf{D}$ are overlapping in at most one vertex of C, we have fewer cycles in the image $\varphi(\mathbf{F})$ in $\mathbf{D}$ than in $\mathbf{F}$ itself, an contradiction with existence of homomorphism φ.

There is also no homomorphism $\mathbf{F}' \to \mathbf{D}$ for any $\mathbf{F}' \in \mathcal{F}, \mathbf{F}' \neq \mathbf{F}$ because composing such homomorphism with f would lead to homomorphism $\mathbf{F}' \to \mathbf{F}$ that does not exist.

Take generic lift $\mathbf{U} \in \mathcal{K}$ (i.e. $\mathbf{U}$ is Fraïssé limit of $\mathcal{K}$). Let φ'' be an embedding $\varphi'' : \mathbf{D} \to \psi(\mathbf{U})$ ($\psi(\mathbf{U})$ is shadow of $\mathbf{U}$). Denote by $c(v)$ the set of all new monadic relations of lift $\mathbf{U}$ associated with vertex $\varphi''(v)$. Obviously c is 2^k coloring of $\mathbf{D}$. Every vertex of S corresponds to unique vertex of $\mathbf{D}$ and thus also we get 2^k coloring of $\mathbf{S}$.

Subsequently there is monochromatic tuple $\vec{v} = \{u_1, v_1, u_2, v_2, \ldots u_n, v_n\} \in R_{\mathbf{S}}$ and thus also tuple $m(\vec{v})$ of vertices of $\mathbf{D}$ such that the relations added by lift $\mathcal{K}$ are equivalent on $\varphi''(m(u_i))$ and $\varphi''(m(v_i)), i = 1, \ldots n$. Lift $\mathbf{U}$ induce on both $\{\varphi''(m(u_1)), \varphi''(m(u_2)), \ldots, \varphi''(m(u_n))\}$ and $\{\varphi''(m(v_1)), \varphi''(m(v_2)), \ldots, \varphi''(m(v_n))\}$ same lift $\mathbf{X}$. ($\mathbf{X}$ is lift of relational structure induced by $\mathbf{F}$ on C.) Subsequently there is partial isomorphism of $\mathbf{U}$ mapping $\varphi''(m(u_i)) \rightarrow \varphi''(m(v_i))$. From genericity of lift $\mathbf{U}$ this partial isomorphism extends to automorphism of lift $\mathbf{U}$ and from construction of relational system $\mathbf{D}$ it sends roots of image of piece $\mathcal{P}_1$ to roots of image of piece $\mathcal{P}_2$ and thus shadow of $\mathbf{U}$ contains copy of $\mathbf{F} \in \mathcal{F}$, a contradiction.

□

Note that for infinite type of lift the question is trivial: take universal structure $\mathbf{U} \in Forb_h(\mathcal{F})$ and construct lift $\mathbf{X}$ with ω extended unary relations. Then chose arbitrary order of vertices $v_1, v_2, \ldots$ of $\mathbf{U}$ and put $(v_i) \in X^j_{\mathbf{X}}$ iff $i = j$. Shadow of $\mathbf{X}$ is universal for class $Forb_h(\mathcal{F})$ and because $\mathbf{X}$ has no partial isomorphisms expect for identity, trivially $Age(\mathbf{X})$ is amalgamation class.

More complicated application of Ramsey theory leads to the following strenghtening:

Theorem 3.6 ([13]) *For a minimal family $\mathcal{F}$ of relational structures, there is lift of class $Forb_h(\mathcal{F})$ with just finitely many additional relations of arity at most n that is amalgamation class iff all minimal cuts of $\mathbf{F} \in \mathcal{F}$ consist of at most n vertices.*

3.5 Lifted classes with free amalgamation

Explicit construction of the lifts provided by Theorem 2.2 allows more insight into their structure. In this section we give an answer to problem of Asterias (private communication) whether there always exists lift of class $Forb_h(\mathcal{F})$ with free amalgamation property. The answer is negative in general, we can however precisely characterize families $\mathcal{F}$ with this property.

We say that structure is *irreducible* if it does not have a cut (alternatively, any two distrinct vertices are contained in a tuple of $\mathbf{A}$).

Theorem 3.7 *Let $\mathcal{F}$ by a minimal family of finite connected relational structures. Then the following staements are euivalent:*

1. *There exists class $\mathcal{K}'$ such that:*
    (a) *$\mathcal{K}'$ is amalgamation class,*
    (b) *$\mathcal{K}'$ is closed for free amalgamation,*
    (c) *shadow of $\mathcal{K}'$ is $Forb_h(\mathcal{F})$.*

2. Every minimal cut in $\mathbf{F} \in \mathcal{F}$ *induces an irreducible subsystem.*

Proof. For 2. $\implies$ 1. it suffices to verify that for such classes $\mathcal{F}$ the amalgam $\mathbf{V}$ constructed in proof of Theorem 2.2 is free amalgam of $\mathbf{X}$ and $\mathbf{Y}$ over $\mathbf{Z}$. The amalgam is constructed as $L(\mathbf{D})$, where $\mathbf{D}$ is free amalgam of shadows of $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$. Now for every tuple $\vec{v} \in X^i_{\mathbf{V}}$ we have homomorphism $\varphi : P_i \to \mathbf{D}$. Because $\mathbf{P_i}$ induce on vertices $\overrightarrow{R}_i$ an irreducible relational structure, the map must correspond to shadow of $\mathbf{A}$ or $\mathbf{B}$ and thus there are no new edges in $\mathbf{V}$.

In the opposite direction, assume that $\mathcal{F}$ and class $\mathcal{K}'$ satisfying (a), (b) and (c) are given.

Define class $\overline{\mathcal{K}}'$ as the class of all $\mathbf{A} \in \mathcal{K}'$ and where for each tuple $\vec{v} \in X^i_{\mathbf{A}}$ the relational structure induced by $\psi(\mathbf{A})$ on $\vec{v}$ is irreducible.

We claim that $\overline{\mathcal{K}}'$ also satisfy (a), (b) and (c). Assume the contrary. Then, for some i, there is $\mathbf{A} \in \mathcal{K}'$ and $\vec{v} \in X^i_{\mathbf{A}}$ such that structure induced by $\psi(\mathbf{A})$ on $\vec{v}$ is reducible and there is no $\mathbf{B} \in \overline{\mathcal{K}}'$ such that shadow of $\mathbf{A}$ is same as shadow of $\mathbf{B}$. Without loss of generality we can assume that $\mathbf{A}$ is counterexample with minimal amount of tuples. Denote by v_1, v_2 subsets of vertices of $\vec{v}$ such that free amalgamation of structures induced on v_1 and v_2 by structure $\psi(\mathbf{A})$ over vertices $v_1 \cup v_2$ is equivalent to structure induced on $\vec{v}$ by structure $\psi(\mathbf{A})$.

Now construct $\mathbf{B}$ as free amalgam of structure induced on $(A \setminus v) \cup v_1$ and on $(A \setminus v) \cup v_2$ by $\mathbf{A}$ over vertices $v_1 \cap v_2$. Because $\mathcal{K}'$ is amalgamation class, we have $\mathbf{B} \in \mathcal{K}'$. Shadow of $\mathbf{B}$ is equivalent to shadow of $\mathbf{A}$ and either $\mathbf{B} \in \overline{\mathcal{K}}'$ or $\mathbf{B}$ is smaller counterexample, a contradiction with minimality of $\mathbf{A}$.

Now take $\mathbf{F} \in \mathcal{F}$ such that there is vertex minimal cut C and structure $\mathbf{C}$ induced on C by $\mathbf{F}$ is not irreducible. By Theorem 3.6 we know that arity of lift $\overline{\mathcal{K}}'$ must be at least $|C|$. While the lift $\overline{\mathcal{K}}'$ can have unbounded arity, from the fact that images of C are reducible, the arity of lift $\overline{\mathcal{K}}'$ on images of $\mathbf{C}$ strictly smaller than C. The proof of Theorem 3.6 (which is similar to proof of Theorem 3.5) only deals with extended tuples on images of cuts C and thus we have a contradiction. □

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