Facial parity edge colouring

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Abstract: The *facial parity edge colouring* of a connected bridgeless plane graph is such an edge colouring in which no two face-adjacent edges receive the same colour; in addition, for each face α and each colour c, no edge or an odd number of edges incident with α are coloured by c. From the Vizing's theorem it follows that every simple 3-connected plane graph has a such colouring with at most $\Delta^* + 1$ colours, where Δ^* is the size of the largest face. In this paper we prove that any connected bridgeless plane graph has a facial parity edge colouring with at most 92 colours.

Keywords: plane graph, facial walk, edge colouring

2010 Mathematics Subject Classification: 05C10, 05C15

1 Introduction

One of the motivations for this paper has come from recent papers of Bunde at al. [1, 2] who introduced parity edge colourings of graphs. Studying the parity of the usage of colours along walks suggested two edge colouring parameters that have interesting properties and applications. A *parity walk* in an edge colouring of a graph is a walk along which each colour is used an even number of times. Bunde et al. [2] introduced two parameters. Let $p(G)$ be the minimum number of colours in an edge colouring of G having no parity path (a *parity edge colouring*). Let $\widehat{p}(G)$ be the minimum number of colours in an edge colouring of G in which every parity walk is closed (a *strong parity edge colouring*). Since incident edges of the same colour would form a parity path of length 2, every parity edge colouring is a proper edge colouring, and hence $p(G) \geq \chi'(G)$, where $\chi'(G)$ is the edge chromatic index of G. Since a path is an open walk, any strong parity edge colouring has no parity path. Hence, every strong parity edge colouring is a parity edge colouring and $\widehat{p}(G) \geq p(G)$ for every graph G. Although there are graphs G with $\widehat{p}(G) > p(G)$ [2], it remains unknown how large $\widehat{p}(G)$ can be when $p(G) = k$. Elementary results on these parameters appear in [2]. In [1] it is proved that $\widehat{p}(K_n) = 2^{\lceil log(n) \rceil} - 1$ for all n. Moreover, the optimal strong parity edge colouring of the complete n-vertex graph K_n is unique when n is a power of 2. The authors of [2] mentioned that computing $p(G)$ or $\widehat{p}(G)$ is NP-hard even when G is a tree. Clearly, the parity edge colouring is such a colouring that each path uses at least one colour an odd number of times.

The vertex version of this problem (*strong parity vertex colouring*) with some restrictions was introduced in [3]. The authors of [3] conjectured that there is a constant K such that the vertices of any 2-connected plane graph can be coloured with at most K colours in such a way that for each face α and each colour c, no vertex or an odd number of vertices incident with α is coloured by c. This conjecture was proved by Kaiser et al. [4], they showed that $K \leq 300$, moreover, their colouring is proper.

Another motivation for this work has came from the papers of Pyber [6] and Mátrai [5]. A graph is called *odd* if the degree of its vertices is odd or zero. Pyber raises the problem of edge covering with odd subgraphs in [6] as the counterpart of even subgraph covering problems. He proved that the edges of every finite simple graph can be covered by at most 4 disjoint odd subgraphs; moreover, if the number of vertices is even then 3 colours are sufficient. For not necessarily disjoint coverings we have the following question: Is it true that every graph can be covered by at most 3 odd subgraphs? Mátrai in [5] showed that every finite simple graph can be covered by 3 odd subgraphs and he found an infinite sequence of finite simple connected graphs not coverable by three edge disjoint odd subgraphs.

Pyber's result implies the following: The edges of any 3-connected plane G graph can be coloured by at most 4 colours in such a way that for each face α and each colour c, no edge or an odd number of edges incident with α is coloured by c. It is sufficient to consider the dual G^* of G and its edge cover with at most 4 disjoint odd subgraphs. This cover induces the required colouring of G.

If we add a requirement that such a colouring must be proper then it is not clear whether there exits such a colouring with K colours, where K is an absolute constant. From Vizing's theorem [7] it follows that every simple 3-connected plane graph G has such a colouring which uses at most $\Delta^* + 1$ colours, where Δ^* is the size of the largest face. Consider the dual graph G^* and its proper edge colouring. This colouring induces a colouring of G in the natural way. It is such a colouring in which, for each face α of G, all the edges in the boundary of α have distinct colours.

From the result of Kaiser et al. [4] follows that each connected bridgeless plane graph has an edge colouring with at most 300 colours such that any two consecutive edges of a facial walk of any face α receive distinct colours and if a colour c appears on a α then it appears an odd number of times. It is sufficient to consider the medial graph $M(G)$ of G and its proper strong parity vertex colouring. The vertices of $M(G)$ correspond to the edges of G; two vertices $M(G)$ are adjacent in $M(G)$ if and only if the corresponding edges are face-adjacent in G. For any plane G the medial graph $M(G)$ is plane as well. Hence, every strong parity vertex colouring of $M(G)$ corresponds to an edge colouring of G.

In this paper we show that each connected bridgeless plane graph has such a colouring with at most 92 colours.

2 Notation

Let us introduce the notation used in this paper. A graph which can be embedded in the plane is called *planar graph*; a fixed embedding of a planar graph is called *plane graph*.

A *bridge* is an edge whose removal increases the number of components. A graph which contains no bridge is said to be *bridgeless*. In this paper we consider connected bridgeless plane graphs, multiple edges and loops are allowed.

Let $G = (V, E, F)$ be a connected plane graph with the vertex set V, the edge set E, and the face set F. The *degree* of a vertex v, denoted by $deg(v)$, is the number of edges incident with v, each loop counting as two edges. For a face α, the *size* of α, $deg(\alpha)$, is defined to be the length of its *facial walk*, i.e. the shortest closed walk containing all edges from the boundary of α. We often say k-face for a face of size k.

Given a graph G and one of its edge $e = uv$, (the vertices u and v do not have be different), the *contraction* of e, denoted by $G\%e$, consist of replacing

u and v by a new vertex adjacent to all the former neighbours of u and v, and removing the loop corresponding to the edge e. (We keep multiple edges if arisen). Analogously we define the contraction of the set of edges $H = \{e_1, \ldots, e_k\}$ and we denote it by $G\%\{e_1, \ldots e_k\}$ or $G\%H$.

Let H be a subgraph of G. Then the graph $G \setminus H$ is defined as a graph obtained from G by deleting the vertices in $V(H)$ together their incident edges.

Two faces are *adjacent* if they share an edge. Two (distinct) edges are *face-adjacent* if they are consecutive edges of a facial walk of some face α.

An *edge k-colouring* of a graph G is a mapping $\varphi : E(G) \to \{1, \ldots, k\}$. The *facial parity edge* (FPE) *colouring* of a connected bridgeless plane graph is such an edge colouring that no two face-adjacent edges receive the same colour; for each face α and each colour c, no edge or an odd number of edges incident with α are coloured by c.

Question 1 *What is the minimum number of colours $\chi'_{fp}(G)$ that a connected bridgeless plane graph G has a facial parity edge colouring with at most $\chi'_{fp}(G)$ colours?*

The number $\chi'_{fp}(G)$ is called the *facial parity chromatic index* of G.

3 Results

Theorem 3.1 *Let G be a connected bridgeless plane graph. Then*

$$\chi'_{fp}(G) \leq 92.$$

The proof uses the method of discharging. Let G be the counterexample with minimal number of edges, then minimal number of 1-faces, then minimal number of 2-faces. If G is a single cycle of length d, $d \leq 5$, we use exactly d colours. We consider this to be the first step of induction and call this case trivial.

First, we prove several structural properties of G.

We say that a face α is *small* if $1 \leq deg(\alpha) \leq 44$ and a face β is *big* if $deg(\beta) \geq 45$.

3.1 Reducible configurations

We find such (forbidden) subgraphs H of G that the facial parity edge colouring of $G \setminus H$ or $G\%H$ using at most 92 colours can be extended to a required colouring of G using at most 92 colours, which is a contradiction to G being a counterexample. In the sequel, whenever we speak about a FPE colouring, we always mean a FPE colouring using at most 92 colours.

3.1.1 1-faces

Claim 1 *Each vertex of G is incident with at most one 1-face.*

Proof
Let v be a vertex incident with at least two 1-faces α_1 and α_2. If we split v into two vertices v_1 and v_2 in such a way that α_1 and α_2 become a 2-face α, we obtain a graph G', see Figure 1. It has the same number of edges than G, but less 1-faces. Thus, it is not a counterexample and we can find a FPE colouring φ' of G'. It induces an edge colouring φ of G in a natural way. It is easy to see that φ is a FPE colouring. □

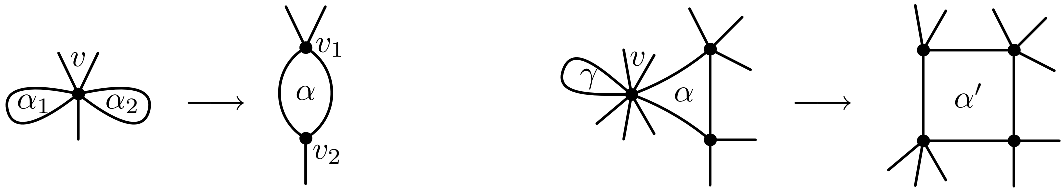

Figure 1: A vertex incident with at least two 1-faces can be split into two vertices, reducing the number of 1-faces. Similarly, one can reduce a 1-face and a d-face ($2 \leq d \leq 4$) incident with the same vertex.

Claim 2 *Each vertex of G incident with a 1-face is not incident with any d-face for $2 \leq d \leq 4$.*

Proof
We use the same reduction as in the proof of Claim 1. We split the vertex v incident with a 1-face γ and a d-face α ($2 \leq d \leq 4$) in such a way that the faces α and γ become a $(d+1)$-face α', see Figure 1 for illustration. Let the reduced graph be G'. It has less 1-faces than G, therefore, it has a FPE colouring φ'. The face α' has size at most five, therefore, its edges are coloured using $d+1$ different colours. Thus, the colouring φ of G induced by the colouring φ' of G' is a FPE colouring, too. □

Claim 3 *Each vertex of G incident with a 2-face is not incident with any d-face for $2 \leq d \leq 3$.*

Proof
We use the same reduction as in the proof of Claims 1 and 2. We omit the details. □

3.1.2 Small faces

Claim 4 *There are no two small faces adjacent to each other in G.*

Proof
Let α_1 and α_2 be two small faces adjacent to each other in G.

If both α_1 and α_2 are 1-faces, the graph consists of a single vertex and a loop; it has a FPE colouring using 1 colour. We disregard this trivial case.

Let α_1 be a 1-face and α_2 be a d-face, $d \geq 2$; let e be the loop they share, see Figure 2. Then the graph $G' = G\%e$ has less edges than G, therefore, it has a FPE colouring φ'. Let α' be a face in G' corresponding to α_1 and α_2 in G. Since α_2 is a small face, at most 43 colours occur on the edges incident with α'. To extend the colouring φ' of G' to a FPE colouring of G, it suffices to colour the edge e with any colour that does not occur on α'.

Let α_1 and α_2 be two small faces of size at least 2 and let e be the edge they share, see Figure 2. The graph $G' = G\%e$ has less edges than G, therefore, it has a FPE colouring φ'. Let α_1' and α_2' be the faces of G' corresponding to the faces α_1 and α_2 in G. (Since α_1 and α_2 are small faces of size at least 2, the size of α_1' and α_2' is at least 1.)

Consider the set of colours different from the colours occurring on the edges of α_1' and α_2' (the colours admissible for the edge e). Since α_1 and α_2 are small, at most $2 \cdot (44 - 1) = 86$ colours occur on their edges. Hence, there is an admissible colour c. We can extend φ' to a FPE colouring φ of G by setting $\varphi(e) = c$. $\square$

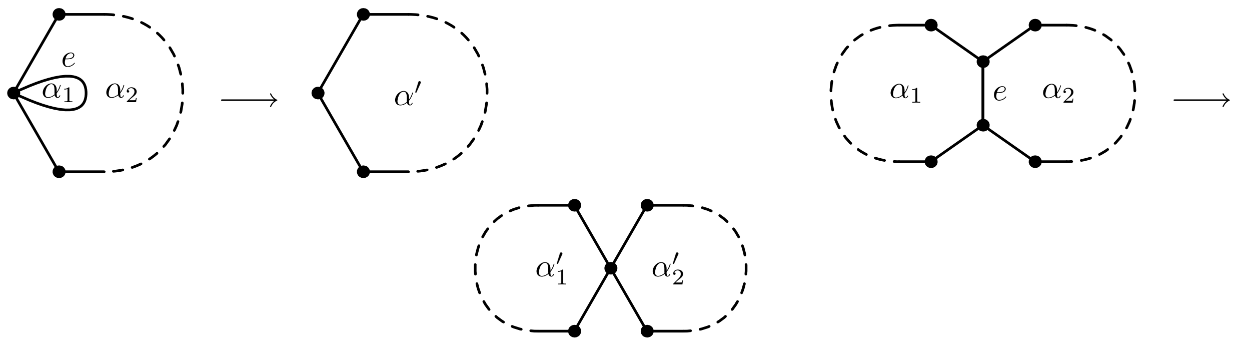

Figure 2: Two adjacent small faces form a reducible configuration: one can contract the edge they share.

Claim 5 *Let β be a big face adjacent to two small faces α_1 and α_2. Let e_i be an edge incident with β and α_i, $i = 1, 2$. Then e_1 and e_2 are face-adjacent.*

Proof

Let e_1 and e_2 not be face-adjacent. See Figure 3 for illustration. The graph $G' = G\%\{e_1, e_2\}$ has less edges than G, therefore, it has a FPE colouring φ'. Let α'_1, α'_2, β' be the faces of G' corresponding to the faces α_1, α_2, β in G, respectively.

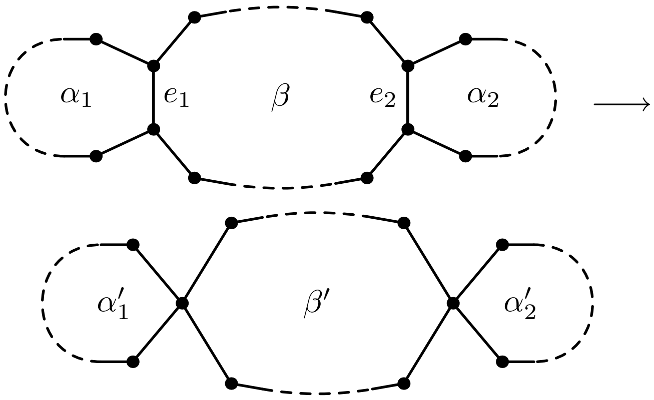

Figure 3: A big face β adjacent to two different small faces α_1 and α_2 forms a reducible configuration unless the edges e_1 and e_2 are face-adjacent.

We extend the colouring φ' of G' to a FPE colouring of G in the following way: Consider the set of colours different from the colours occurring on the edges of α'_1 and α'_2; also different from the colours occurring on the edges of G' corresponding to the edges of G face-adjacent to e_1 and e_2. There are at least $92 - 2 \cdot (44 - 1) - 4 = 2$ such colours, say c_1 and c_2. If at least one of them, say c_i, already occurs on β', we set $\varphi(e_1) = \varphi(e_2) = c_i$. If none of them occurs on β', we set $\varphi(e_i) = c_i$, $i = 1, 2$. □

Claim 6 *Each big face is adjacent to at most one* 1*-face.*

Proof

It follows from Claims 1 and 5. □

Claim 7 *Each big face is adjacent to at most two small faces.*

Proof

Let a big face β be adjacent to small faces α_1, α_2, and α_3. Consider the edges that β shares with α_i, $i = 1, 2, 3$. It is easy to see that there must be a pair of edges e_i and e_j, incident to α_i and α_j, respectively $(i \neq j)$, which are not face-adjacent. It is a contradiction with Claim 5. □

3.1.3 Chains of 2-vertices

Claim 8 *There is no chain consisting of at least* 5 *consecutive* 2*-vertices in* G*.*

Proof
Let $v_0e_0v_1e_1\dots v_pe_pv_{p+1}$ be a chain consisting of p vertices of degree 2, $(v_1,\dots,v_p)$, where $p\geq 5$. The graph $G'=G\%\{e_1,e_2,e_3,e_4\}$ has a FPE colouring φ'. Let e_0' and e_5' be the edges in G' corresponding to the edges e_0 and e_5 in G. Let $\varphi'(e_0')=c_1$ and $\varphi'(e_5')=c_2$. The FPE colouring φ' of G' can be extended to a FPE colouring φ of G by setting $\varphi(e_2)=\varphi(e_4)=c_1$ and $\varphi(e_1)=\varphi(e_3)=c_2$, see Figure 4. □

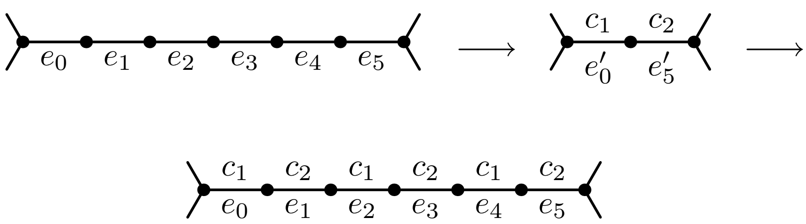

Figure 4: A chain of (at least) five 2-vertices is a reducible configuration.

A d-face α is *hanging* on a vertex v, if all vertices incident with α are 2-vertices except for the vertex v. By Claim 8 we have $d\leq 5$. (If $\deg(v)=2$, the graph G consists of a single cycle of length at most 5, which is the trivial case).

We colour the vertices of G with black, blue and white colour in the following way:

Let all 2-vertices be black, all 3-vertices be blue and all k-vertices for $k\geq 6$ be white.

A 4-vertex v is black if there is a face hanging on it, else it is white. See Figure 5 for illustration of all types of black vertices.

A 5-vertex v is blue, if there is a face hanging on it, else it is white.

Observe that any black 2-vertex v is incident with two faces. We say v is *bad* for both faces it is incident with. Any black 4-vertex v is incident with a small face α of size at most 5 and two other faces. The face β adjacent to α must be big (see Claim 4); the vertex v occurs twice on the facial walk of β. The other face γ can be big or small. We say v is *bad* for the face γ.

Claim 9 *For any face* α*, there is no chain of at least* 5 *bad consecutive vertices. Each chain of bad vertices contains at most one bad* 4*-vertex.*

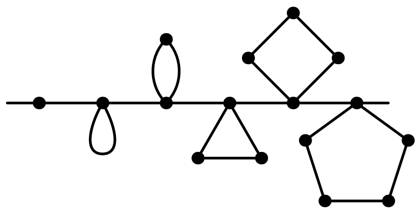

Figure 5: A vertex is black, if it is a 2-vertex, or a 4-vertex with a hanging face.

Proof
Let $v_0e_0v_1e_1v_2e_2v_3e_3v_4e_4v_5e_5v_6$ be a subpath of the facial walk of α containing 5 vertices bad for α, $(v_1,\dots,v_5)$. It is easy to see that all the edges $e_0,\dots,e_5$ are incident with the same face β. If at least two of the vertices the vertices $v_1,\dots,v_5$ are bad 4-vertices, we come to a contradiction with Claim 4 or with Claim 5. If none of them is a 4-vertex, we are in the case of Claim 8. If precisely one of them is a 4-vertex, say v_i, it is incident with a small face γ of size $d \le 5$.

The graph $G' = G\%\{e_1,e_2,e_3,e_4\}$ has a FPE colouring φ'. Let e_0' and e_5' be the edges in G' corresponding to the edges e_0 and e_5 in G; let α', β', γ' be the faces in G' corresponding to the faces α, β, γ in G; let v' be the vertex in G' corresponding to $v_1,\dots,v_5$ in G. Let $\varphi'(e_0') = c_1$ and $\varphi'(e_5') = c_2$. Since e_0' and e_5' are face-adjacent in G', $c_1 \ne c_2$. To extend φ' to a FPE colouring of G, we proceed in the following way:

If $i = 1$, 3, or 5, we simply set $\varphi(e_2) = \varphi(e_4) = c_1$ and $\varphi(e_1) = \varphi(e_3) = c_2$.

If $i = 2$ or $i = 4$, we set $\varphi(e_2) = \varphi(e_4) = c_1$ and $\varphi(e_1) = \varphi(e_3) = c_2$ and switch the order of colours of edges incident with γ', see Figure 6 for illustration (here the small face γ is a bigon.).

□

Claim 10 *Let v be a black vertex bad for a small face α. If v is a 4-vertex, then the face hanging on it is a 1-face.*

Proof
Let v be a black 4-vertex, let γ be the face hanging on v of size at least 2 and let e_1 and e_2 be the edges of α incident with v. It is easy to see that the edges e_1 and e_2 are incident with the same big face, say β.

There is an edge e_γ incident with γ and β, which is not face-adjacent to $e_\alpha \in \{e_1,e_2\}$, what is a contradiction with Claim 5. □

Claim 11 *Let α be a small face sharing at least two bad black vertices with a big face β. Then all the bad black vertices incident with α and β are 2-vertices.*

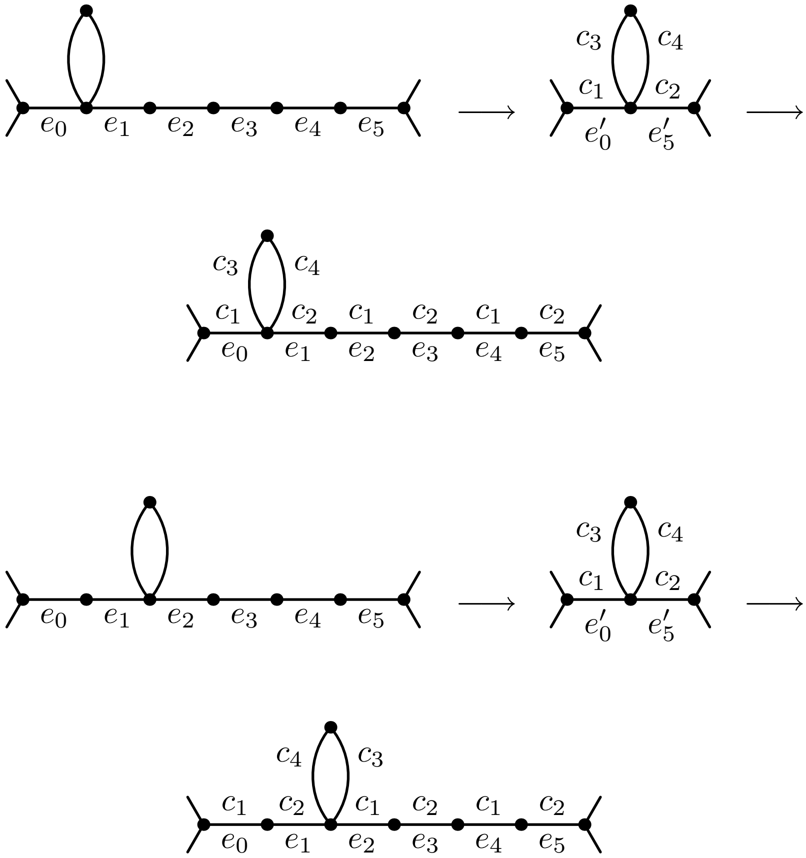

Figure 6: A chain of (at least) five bad black vertices is a reducible configuration as well.

Proof
Let v_1 and v_2 be black vertices incident with α and β. If v_1 is a 4-vertex, it is bad for α, therefore, there is a small face γ hanging on v_1, adjacent to β. By Claim 10, the face γ is a 1-face. Thus, we can find an edge (a loop) e_γ incident with γ and β and an edge e_α incident with α and β, which are not face-adjacent. It is a contradiction with Claim 5. □

Claim 12 *Let v be a black vertex bad for a small d-face α, $d \in \{2, 3, 4\}$. Then v is a 2-vertex.*

Proof
Let v be a black 4-vertex, let γ be the face hanging on v. By Claim 10, the face γ is a 1-face. On the other hand, by Claim 2, the face γ cannot be a 1-face, what is a contradiction. □

Claim 13 *If a big face β shares a 2-vertex v with a small face α, then β is not adjacent to any other small face.*

Figure 7: Reducible pairs of edges (e_γ and e_α). For details see Claims 10 and 11.

Proof
It follows immediately from Claim 5. □

Claim 14 *Let γ be a d-face, $d \in \{2,3,4,5\}$, hanging on a vertex v, adjacent to a big face β. Then β is not adjacent to any other small face.*

Proof
It follows immediately from Claim 13. □

3.2 Discharging rules

Let G be a counterexample. It contains no reducible configuration. Let the initial charge of each vertex be $\psi(v) = \deg(v) - 6$ and the initial charge of each face be $\psi(\alpha) = 2\deg(\alpha) - 6$. From Euler's formula we can easily derive that

$$\sum_{\alpha \in F} (2\deg(\alpha) - 6) + \sum_{v \in V} (\deg(v) - 6) = -12.$$

It is obvious that all the negative charge is in the vertices of degree 2, 3, 4, and 5 and in the faces of size 1 and 2.

Rule 1: Let β be a big face.

- If β is adjacent to a single small face α, it sends 3 units of charge to α.
- If β is adjacent to two small faces α_1 and α_2, such that $\deg(\alpha_1) \leq \deg(\alpha_2)$, it sends 2 units of charge to α_1 and 1 unit of charge to α_2. (If $\deg(\alpha_1) = \deg(\alpha_2)$, we do it arbitrarily.)

Rule 2: Let β be a big face.

- It sends 2 units of charge to any black vertex bad for β.
- It sends 1 unit of charge to any other black, blue or white vertex incident with β. (Multiply incident vertices are considered as different).

Rule 3: Let α be a small face.

- It sends 2 units of charge to any black vertex bad for α.
- It sends 1 unit of charge to any other black or blue vertex incident with α.

Rule 4: Let v be a black 4-vertex.

- It sends 2 units of charge to the incident small hanging face γ.

Rule 5: Let v be a blue 5-vertex.

- It sends 2 units of charge to the incident small hanging face γ.

Rule 6: Let v be a k-vertex, $k \geq 6$.

- It sends 2 units of charge to any incident small hanging face γ.

3.3 Analysis of the graph

3.3.1 Vertices

Every 2-vertex is bad black for both faces incident with it, hence it receives 2 units of charge from both incident faces (Rules 2 and 3). Its charge is $-4 + 2 + 2 = 0$.

Every 3-vertex is blue, hence it receives 1 unit of charge from all the three incident faces (Rules 2 and 3). Its charge is $-3 + 3 \cdot 1 = 0$.

Every black 4-vertex v receives 2 units of charge from the face α it is bad for (Rules 2 and 3) and $2 \cdot 1$ units of charge from the doubly-incident big face β (Rule 2). It sends 2 units of charge to the hanging face γ (Rule 4). The charge of v is $-2 + 2 + 2 \cdot 1 - 2 = 0$.

Every white 4-vertex is incident with at least 2 big faces (see Claim 4), therefore, its charge is at least $-2 + 2 \cdot 1 = 0$.

Every blue 5-vertex v is incident with a hanging face γ, doubly-incident with a big face β and incident with two more faces α_1 and α_2. Therefore, v receives $4 \cdot 1$ units of charge from the incident faces (Rules 2 and 3). It sends 2 units of charge to γ (Rule 5). Therefore, the charge of v is $-1+4\cdot 1-2 = 1$.

Every white 5-vertex is incident with at least 3 big faces (see Claim 4), therefore, its charge is at least $-1 + 3 \cdot 1 = 2$.

Every (white) k-vertex v, $k \geq 6$, has non-negative initial charge. It receives charge from big faces (Rule 2) and sends charge to the hanging faces $\gamma_1, \ldots, \gamma_r$ (Rule 6). For each hanging face γ_i, the adjacent big face β_i is doubly-incident to v. The faces β_i and β_j are different for different γ_i and γ_j (see Claims 5 and 6). Therefore, the charge of v is at least $r \cdot (2 \cdot 1 - 2) = 0$.

3.3.2 1-faces

Let γ be a 1-face. It is adjacent to a big face β and it is hanging on a vertex v. The face β is adjacent to at most one 1-face (Claim 6). Therefore, it sends at least 2 units of charge to the face γ (Rule 1).

If the vertex v is a 2-vertex, we get the trivial graph, which was omitted before. If the vertex v is a 3-vertex, the third edge incident with v is a bridge in G, which is not allowed. Therefore, $\deg(v) \geq 4$ and the vertex v sends 2 units of charge to the face γ (Rules 4 – 6). The charge of γ is at least $-4 + 2 + 2 = 0$.

3.3.3 2-faces

Let α be a 2-face. Its initial charge is -2. Let v_1 and v_2 be the vertices incident with α. The face α is adjacent to at most 2 faces, which must be big (see Claim 4). Consider the number of black vertices bad for α. Note that by Claim 12 each such vertex is a 2-vertex.

1. Let both v_1 and v_2 be bad black. Then the graph G consists of a single cycle on 2 vertices, which is not a counterexample.

2. Let v_1 be a black 2-vertex. Then α is adjacent to a single big face β. The big face β is not adjacent to any other small face (see Claim 13). Therefore, β sends 3 units of charge to α (Rule 1). The face α sends 2 units of charge to v_1 and at most 1 unit of charge to v_2 (Rule 3). On the other hand, α is hanging on v_2, therefore, it receives 2 units of charge from v_2 (Rules 4 – 6). Note that v_2 cannot be a 3-vertex, otherwise there would be a bridge in G. The charge of α is at least $-2 - 2 - 1 + 2 + 3 = 0$.

3. Let none of v_1 and v_2 be bad black. Consider the number of faces adjacent to α. If α is adjacent to a single big face β, then β is not adjacent to any other small face. Therefore, β sends 3 units of charge to α. Moreover, in this case none of v_1 and v_2 can be black nor blue. The charge of α is $-2 + 3 = 1$.

   If α is adjacent to two big faces β_1 and β_2, the face α sends at most $2 \cdot 1$ unit of charge to v_1 and v_2. The big face β_i, $i = 1, 2$, is not adjacent to any other small face of size at most 2 (see Claims 5, 2, and 3). Therefore, by Rule 1, the face β_i, $i = 1, 2$, sends at least 2 units of charge to α. The charge of α is at least $-2 - 2 \cdot 1 + 2 \cdot 2 = 0$.

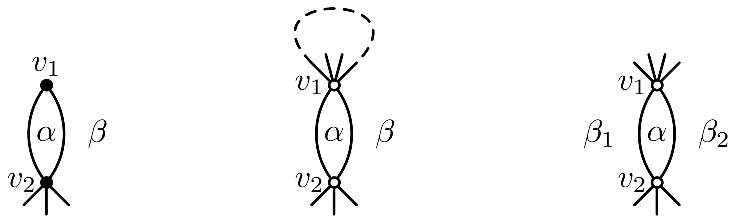

Figure 8: Different possible neighbourhoods of a 2-face α.

3.3.4 3-faces

Let α be a 3-face. Its initial charge is 0. Let v_1, v_2, v_3 be the vertices incident with α. (They do not have to be pairwise different.) Consider the number of black vertices bad for α. Note that by Claim 12 each such vertex is a 2-vertex.

1. Let all the three vertices v_1, v_2, v_3 be bad black. Then the graph G consists of a single cycle on 3 vertices, which is not a counterexample.

2. Let v_2 and v_3 be bad black. Then all the three edges of α are incident with the same big face β and α is hanging on v_1. Then by Claim 14 the face β sends 3 units of charge to α (Rule 1). The face α then sends $2 \cdot 2$ units of charge to the 2-vertices v_2, v_3 and at most 1 unit of charge to the vertex v_1 (Rule 3). If the vertex v_1 is a 3-vertex, the third edge incident with it is a bridge in G. Therefore $\deg(v_1) \geq 4$ and v_1 sends 2 units of charge to α (Rules 4 – 6). The charge of α is $3-2\cdot 2-1+2=0$.

3. Let v_3 be bad black. Let β_1 be the big face incident with the edge v_1v_2 and β_2 be the big face incident with the edges v_2v_3 and v_3v_1.

   The face β_2 sends 3 units of charge to α, β_1 sends at least 1 unit of charge to α. The face α then sends 2 units of charge to v_3 and at most 1 unit of charge to v_1 and v_2. The charge of α is at least $3+1-2-2\cdot 1=0$.

4. Let none of the vertices v_1, v_2, v_3 be bad black. Consider the number of faces adjacent to α.

   If there are three different faces adjacent to α, (they must be big, see Claim 4), the face α receives at least 1 unit of charge from each of them, and sends at most 1 unit of charge to each incident vertex. Hence, the charge of α is at least 0.

   If there is a big face β sharing at least two edges with α, these edges are not face-adjacent in β. Therefore, β is not adjacent to any other small face, thus, it sends 3 units of charge to α. The charge of α is non-negative again.

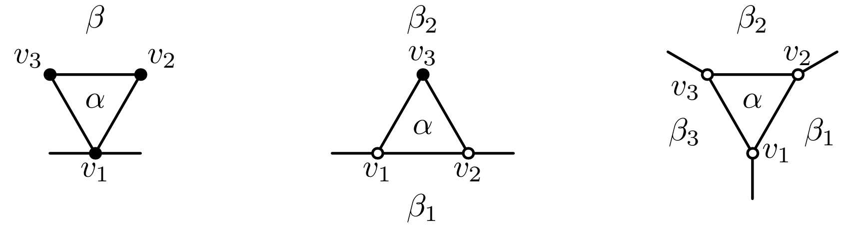

Figure 9: Different possible neighbourhoods of a 3-face α.

3.3.5 4-faces

Let α be a 4-face. Its initial charge is 2. Let v_1, v_2, v_3, v_4 be the vertices incident with α. (They do not have to be pairwise different.) Consider the number of black vertices bad for α. Note that by Claim 12 each such vertex is a 2-vertex.

1. Let all the four vertices v_1, v_2, v_3, v_4 be bad black. Then the graph G consists of a single cycle on 4 vertices, which is not a counterexample.

2. Let v_1, v_2, and v_3 be bad black. Then all the four edges of α are incident with the same big face β and α is hanging on v_4. Hence, β sends 3 units of charge to α (Rule 1). The face α then sends $3 \cdot 2$ units of charge to the 2-vertices v_1, v_2, v_3 and at most one unit of charge to v_4 (Rule 3). If the vertex v_4 is a 3-vertex, the third edge incident with it is a bridge in G. Therefore, $\deg(v_4) \geq 4$ and v_4 sends 2 units of charge to α (Rules $4-6$). The charge of α is $2+3-3\cdot 2-1+2=0$.

3. Let v_1 and v_3 be bad black. Let β_1 be the big face incident with v_1, let β_2 be the big face incident with v_3.

   If $\beta_1 \neq \beta_2$, both β_1 and β_2 send 3 units of charge to α. The face α then sends 2 units of charge to the vertices v_1 and v_3 and at most 1 unit of charge to the vertices v_2 and v_4. The charge of α is at least $2+2\cdot 3-2\cdot 2-2\cdot 1=2$. If $\beta_1=\beta_2$, then v_2 and v_4 are not blue, and the charge of α is at least $2+3-2\cdot 2=1$.

4. Let v_1 and v_2 be bad black. Let β_1 be the big face incident with the edge v_3v_4 and β_2 be the big face incident with the vertices v_1 and v_2. If $\beta_1 \neq \beta_2$, β_2 sends 3 units of charge to α and β_1 sends at least 1 unit of

charge to α. The face α then sends 2 units of charge both to v_1 and v_2 and at most 1 unit of charge to v_3 and v_4. The charge of α is at least $2+3+1-2\cdot 2-2\cdot 1=0$. If $\beta_1=\beta_2$, then v_3 and v_4 are not blue, and the charge of α is at least $2+3-2\cdot 2=1$.

5. Let v_1 be bad black. The big face β_1 incident with v_1 sends 3 units of charge to α. The face α then sends 2 units of charge to v_1 and at most 1 unit of charge to v_2, v_3, and v_4. The charge of α is at least $2+3-2-3\cdot 1=0$.

6. Let no bad black vertex be incident with α. Then the big faces adjacent to α send together at least 2 units of charge (if there was only one big face, it would send 3 units of charge), and α sends at most 1 unit of charge to each incident vertex. The charge of α is at least $2+2-4=0$.

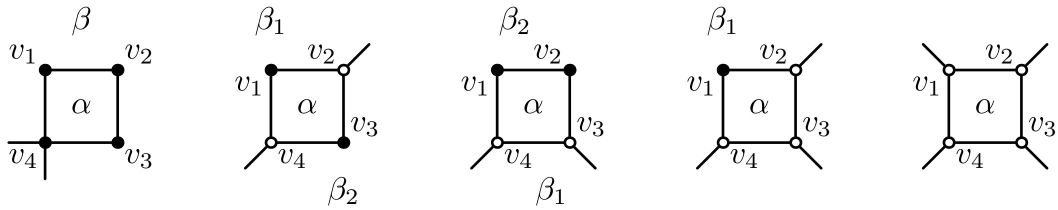

Figure 10: Different possible neighbourhoods of a 4-face α.

3.3.6 5-faces

Let α be a 5-face. Its initial charge is 4. Let v_1, v_2, v_3, v_4, v_5 be the vertices incident with α. (They do not have to be pairwise different.) Consider the number of black vertices bad for α. Note that by Claim 10 each such vertex is either a 2-vertex or a 4-vertex with a hanging 1-face.

1. Let all the five vertices v_1, v_2, v_3, v_4, v_5 be bad black. Then the graph G contains only 5 vertices, hence, it is not a counterexample.

2. Let v_1, v_2, v_3, and v_4 be bad black. From Claim 5 follows that none of them is incident with a 1-face. Then all the five edges of α are incident with the same big face β and α is hanging on v_5. Hence, β sends 3 units of charge to α (Rule 1). The face α then sends $4\cdot 2$ units of charge to the 2-vertices v_1, v_2, v_3, and v_4 (Rule 3) and at most 1 unit of charge to the vertex v_5 (Rule 3). If the vertex v_5 is a 3-vertex, the third edge incident with it is a bridge in G. Therefore $\deg(v_5)\geq 4$

and v_5 sends 2 units of charge to α (Rules 4 – 6). The charge of α is $4 + 3 - 4 \cdot 2 - 1 + 2 = 0$.

3. Let v_1, v_2, and v_3 be bad black. Let β_1 be the big face incident with the vertices v_1, v_2, and v_3 and β_2 be the big face incident with the edge v_4v_5. By Claim 11 v_1, v_2, and v_3 are 2-vertices. If $\beta_1 \neq \beta_2$, β_1 sends 3 units of charge to α and β_2 sends at least 1 unit of charge to α. The face α then sends 2 units of charge to v_1, v_2, and v_3 and at most 1 unit of charge to v_4 and v_5. The charge of α is at least $4 + 3 + 1 - 3 \cdot 2 - 2 \cdot 1 = 0$. If $\beta_1 = \beta_2$, then v_4 and v_5 are not blue, and the charge of α is at least $4 + 3 - 3 \cdot 2 = 1$.

4. Let v_1, v_2, and v_4 be bad black. Let β_1 be the big face incident with v_1 and v_2, let β_2 be the big face incident with v_4. By Claim 11 v_1 and v_2 are 2-vertices, thus β_1 sends 3 units of charge to α.

   If $\beta_1 \neq \beta_2$, the face α sends 2 units of charge to the vertices v_1, v_2, and v_4 and at most 1 unit of charge to the vertices v_3 and v_5. The charge of α is at least $4 + 3 + 1 - 3 \cdot 2 - 2 \cdot 1 = 0$. If $\beta_1 = \beta_2$, then v_3 and v_5 are not blue, and the charge of α is $4 + 3 - 3 \cdot 2 = 1$.

5. Let v_1 and v_2 be bad black. Let β_1 be the big face incident with the vertices v_1 and v_2. The face β_1 sends 3 units of charge to α. The face α then sends 2 units of charge both to v_1 and v_2 and at most 1 unit of charge to v_3, v_4, and v_5. The charge of α is at least $4+3-2\cdot2-3\cdot1 = 0$.

6. Let v_1 and v_3 be bad black. Let β_1 be the big face incident with v_1, β_2 be the big face incident with v_3, and β_3 be the big face incident with the edge v_4v_5. If β_1, β_2, and β_3 are three different faces, the charge of α is at least $4 + 3 - 2 \cdot 2 - 3 \cdot 1 = 0$. If two of them coincide, then at least one of the vertices v_2, v_4, and v_5 is not blue and the charge of α is at least $4 + 2 - 2 \cdot 2 - 2 \cdot 1 = 0$. If $\beta_1 = \beta_2 = \beta_3$, then v_2, v_4, and v_5 are not blue and the charge of α is at least $4 + 1 - 2 \cdot 2 = 1$.

7. Let v_1 be black. Then the big faces adjacent to α send together at least 2 units of charge (if there was only one big face, it would send 3 units of charge). The charge of α is at least $4 + 2 - 2 - 4 \cdot 1 = 0$.

8. Let no black vertex be incident with α. Then the big faces adjacent to α send together at least 1 unit of charge. The charge of α is at least $4 + 1 - 5 = 0$.

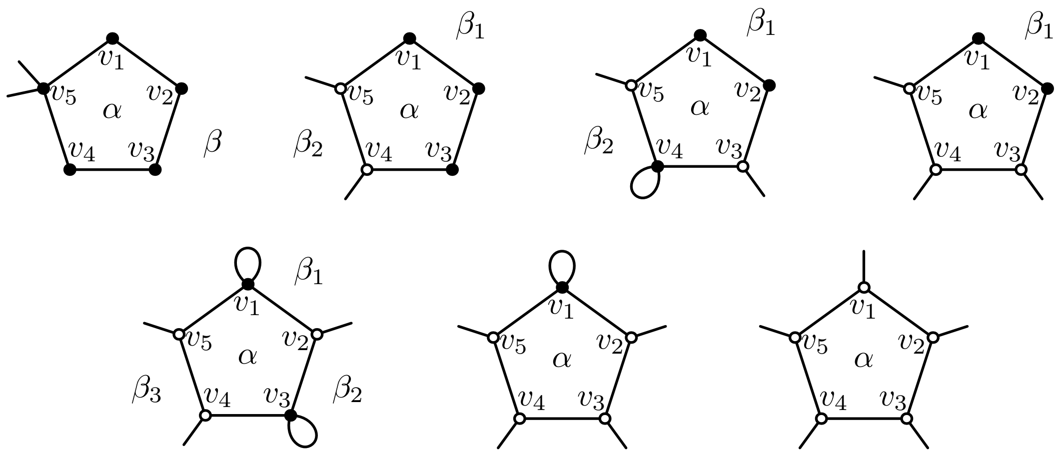

Figure 11: Different possible neighbourhoods of a 5-face α.

3.3.7 Small faces of size at least 6

Let α be a d-face, $6 \leq d \leq 44$. Its initial charge is $2d-6$. Let $v_1, \ldots, v_d$ be the vertices incident with α. (They do not have to be pairwise different.) Consider the black vertices incident with α. Let v_i be a black 4-vertex. It cannot be good for α, since no two small faces are adjacent (see Claim 4). Therefore, each black 4-vertex is bad for α. By Claim 9 at most $d-2$ vertices incident with α are bad. Let $k \leq d-2$ be the number of black vertices incident with α. We can divide the facial walk of α into $d-k \geq 2$ parts, each beginning and ending in a blue or white vertex, each incident with α and a big face β_i, $i \in \{1, \ldots, d-k\}$. Each of these big faces sends at least 1 unit of charge to α. (If $\beta_i = \beta_j$ for some $1 \leq i < j \leq d-k$, the face β_i cannot be adjacent to other small face than α, therefore, it sends 3 units to α, which is even more than what two different big faces would send.)

The face α then sends 2 units of charge to each of the k incident black vertices, and at most 1 unit of charge to each of the other incident vertices. Together, the charge of α is at least

$$2d - 6 + (d-k) \cdot 1 - k \cdot 2 - (d-k) \cdot 1 = 2(d-k) - 6.$$

If $d - k \geq 3$, the charge of α is non-negative.

Let $d-k=2$. It means there are only two vertices which are not black. Since $d \geq 6$, at least one big face β shares at least 2 black vertices with α, say v_1 and v_2. By Claim 9 at least one from v_1 and v_2 is a 2-vertex, hence, by Claim 13 the face β sends 3 units of charge to α. The charge of α is therefore at least

$$2d - 6 + 3 + 1 - (d-2) \cdot 2 - 2 \cdot 1 = 0.$$

3.3.8 Big faces

Let β be a d-face, $d \geq 45$. Its initial charge is $2d - 6$. It sends 3 units of charge to the small faces it is adjacent to (Rule 1). It sends 2 units of charge to all bad black vertices; 1 unit of charge to all other vertices. Let k be the number of black vertices bad for β. By Claim 9, $k \leq \frac{4}{5} \cdot d$. The charge of β is therefore at least

$$2d - 6 - 3 - k \cdot 2 - (d - k) \cdot 1 = d - k - 9 \geq d - \frac{4d}{5} - 9 = \frac{d}{5} - 9 = \frac{d - 45}{5} \geq 0.$$

The charge of all elements of the graph is non-negative, but the sum of all the charge is -12, what is a contradiction.

Acknowledgments: This work was supported by the Slovak Science and Technology Assistance Agency under the contract No. APVV-0007-07. Institute for Theoretical Computer Science (ITI) is supported by Ministry of Education of the Czech Republic as project 1M0545.

References

[1] D. P. Bunde, K. Milans, D. B. West, and H. Wu, Optimal strong parity edge-coloring of complete graphs, *Combinatorica* **28 (6)** (2008), 625–632.

[2] D. P. Bunde, K. Milans, D. B. West, and H. Wu, Parity and strong parity edge-coloring of graphs, *Congressus Numerantium* **187** (2007), 193–213.

[3] J. Czap and S. Jendroľ, Colouring vertices of plane graphs under restrictions given by faces, *Discussiones Math. Graph Theory* (in press).

[4] T. Kaiser, O. Rucký, R. Škrekovski, and M. Stehlík, Strong parity colorings of planar graphs, in preparation.

[5] T. Mátrai, Covering the edges of a graph by three odd subgraphs, *J. Graph Theory* **53** (2006), 75–82.

[6] L. Pyber, Covering the edges of a graphs by ..., *Sets, Graphs and Numbers, Colloquia Mathematica Societatis János Bolyai* **60** (1991), 583–610.

[7] V. G. Vizing, On an estimate of the chromatic class of a p-graph, *Diskret. Analiz* **3** (1964), 25–30.