A short proof of the tree-packing theorem

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Abstract

We give a short elementary proof of Tutte and Nash-Williams' characterization of graphs with k edge-disjoint spanning trees.

We deal with graphs that may have parallel edges and loops; the vertex and edge sets of a graph H are denoted by $V(H)$ and $E(H)$, respectively. Let G be a graph. If $\mathcal{P}$ is a partition of $V(G)$, we let $G/\mathcal{P}$ be the graph on the set $\mathcal{P}$ with an edge joining distinct vertices $P_1, P_2 \in \mathcal{P}$ for every edge of G with one end in P_1 and another in P_2. Tutte [5] and Nash-Williams [3] proved the following classical result:

Theorem 1. *A graph G contains k pairwise edge-disjoint spanning trees if and only if for every partition $\mathcal{P}$ of $V(G)$, the graph $G/\mathcal{P}$ has at least $k(|\mathcal{P}| - 1)$ edges.*

An elegant proof of Theorem 1 is based on the matroid union theorem (see, e.g., [4, Corollary 51.1a]); a relatively short elementary proof appears in [1, Theorem 2.4.1]. In this paper, we give another elementary proof which is also short and perhaps somewhat more straightforward. The argument directly translates to an efficient algorithm to find either k disjoint spanning trees, or a proof that none exist. To an extent, the method can also be applied to the packing of structures without the matroidal properties of spanning trees, as shown, e.g., in the forthcoming paper [2].

Let $k \geq 1$ and $\mathcal{T} = (T_1, \ldots, T_k)$ be an ordered partition of G into k spanning subgraphs of G. We define the sequence $\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_\infty$ of partitions

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of $V(G)$ *associated with* $\mathcal{T}$ as follows. First, $\mathcal{P}_0 = \{V(G)\}$. For $i \geq 0$, if there is $c \in \{1, \dots, k\}$ such that the induced subgraph $T_c[P]$ is disconnected for some $P \in \mathcal{P}_i$, then c_i is defined as the least such c and $\mathcal{P}_{i+1}$ consists of the vertex sets of all components of $T_{c_i}[Q]$, where Q ranges over all the classes of $\mathcal{P}_i$. Otherwise, the process ends by setting $\mathcal{P}_\infty = \mathcal{P}_i$.

We define the *level* $\ell(e)$ of an edge $e \in E(G)$ (with respect to $\mathcal{T}$) as the largest i (possibly ∞) such that both ends of e are contained in one class of $\mathcal{P}_i$. An edge $e \in E(T_k)$ is *superfluous* (for $\mathcal{T}$) if $\ell(e) < \infty$ and e is contained in some cycle of T_k. We say that an edge *leaves* $P \subset V(G)$ if it has precisely one end in P.

To keep the notation simple, the symbols $\mathcal{P}_i$ and $\ell(e)$ (as well as $\mathcal{P}_\infty$ and c_i) will relate to a partition $\mathcal{T}$, while $\mathcal{P}'_i$ and $\ell'(e)$ relate to a partition $\mathcal{T}'$. Thus, for instance, the level $\ell'(e)$ of an edge e with respect to $\mathcal{T}'$ is defined using the partitions $\mathcal{P}'_i$ associated with $\mathcal{T}'$.

Proof of Theorem 1. The necessity of the condition is clear. To prove the sufficiency, we proceed by induction on k. The claim is trivially true for $k = 0$, so assume $k \geq 1$ and choose an ordered partition $\mathcal{T} = (T_1, \dots, T_k)$ of G into spanning subgraphs of G such that $T_1, \dots, T_{k-1}$ are trees and the following holds:

(1) the number of components of T_k is as small as possible,

(2) either there is no superfluous edge for $\mathcal{T}$, or the minimum level of a superfluous edge is as small as possible subject to (1).

If T_k is connected, then we are done. For the sake of a contradiction, suppose that T_k has at least two components (i.e., $|\mathcal{P}_1| \geq 2$). We prove that there exists a superfluous edge for $\mathcal{T}$. For all $i = 1, \dots, k-1$ and $P \in \mathcal{P} := \mathcal{P}_\infty$, the graph $T_i[P]$ is connected. Hence $T_i/\mathcal{P}$ is a tree and has exactly $|\mathcal{P}| - 1$ edges. By the assumption on G, the graph $T_k/\mathcal{P}$ has at least $k(|\mathcal{P}| - 1) - (k-1)(|\mathcal{P}| - 1) = |\mathcal{P}| - 1$ edges. Since $T_k/\mathcal{P}$ has $|\mathcal{P}|$ vertices and is disconnected, it must contain a cycle. But then T_k contains a cycle as each $T_k[P]$ is connected for $P \in \mathcal{P}$. All the edges of this cycle joining different classes of $\mathcal{P}$ (there are at least two such edges) are superfluous for $\mathcal{T}$.

Let $e \in E(T_k)$ be an edge of minimum level that is superfluous for $\mathcal{T}$ and set $m = \ell(e)$. Let P be the class of $\mathcal{P}_m$ containing both ends of e. Since e joins different components of $T_{c_m}[P]$, we have $c_m \neq k$ and the unique cycle C in $T_{c_m} + e$ contains an edge leaving P. Thus, for an edge e' of C of lowest possible level we have $j := \ell(e') < \ell(e) = m$. Let Q be the class of $\mathcal{P}_j$ containing both ends of e'; note that (the vertex set of) C is contained within Q. We will exchange e for e' in the respective sets T_i to eventually obtain the desired contradiction.

Let $\mathcal{T}'$ be the partition obtained from $\mathcal{T}$ by replacing T_{c_m} with $T'_{c_m} = T_{c_m} + e - e'$ and T_k with $T'_k = T_k - e + e'$; we set $T'_i = T_i$ for $i \notin \{c_m, k\}$. If the ends of e' are in different components of T_k (i.e., $j = 0$), then $T_k - e + e'$ has fewer components than T_k, which is impossible. Thus, all of C is contained within one class $R \in \mathcal{P}_1$. Furthermore, since e is superfluous for $\mathcal{T}$, $T_k[R] - e$ is connected and so e' is superfluous for $\mathcal{T}'$.

Claim. *For all $i \le j$, $\mathcal{P}'_i = \mathcal{P}_i$.*

The proof is by induction on i, with the $i = 0$ case being trivial. We assume that $\mathcal{P}'_i = \mathcal{P}_i$ and prove that every component of $T_c[U]$, where $U \in \mathcal{P}_i$ and $c \le c_i$, has the same vertex set as some component of $T'_c[U]$. This will clearly imply the Claim.

Let K be the vertex set of a component of $T_c[U]$. First of all, T'_c contains no edge leaving K, because T_c and T'_c only differ by edges contained within $Q \in \mathcal{P}_j$, which is either disjoint from K or contained in it. Thus, it suffices to prove that $T'_c[K]$ is connected. Furthermore, we may restrict our attention to $c \in \{c_m, k\}$. Suppose that $T'_c[K]$ is disconnected; there are two cases.

Case 1: $c = k$ and $T_k[K] - e$ is disconnected. Since e is superfluous, it is contained in a cycle D of T_k. By the assumption, D is not contained within $K \in \mathcal{P}_i$, so some edge f of D leaves K. However, f is then superfluous for $\mathcal{T}$ and $\ell(f) \le i < m$, a contradiction.

Case 2: $c = c_m$ and $T_{c_m}[K] + e - e'$ is disconnected. The class $Q \in \mathcal{P}_j$ containing the cycle C is thus contained in K. But then C is a cycle in $T_{c_m}[K] + e$ containing e', so $T_{c_m}[K] + e - e'$ cannot be disconnected. This contradiction concludes the proof of the Claim.

We continue with the proof of the theorem. The above Claim implies that $\ell'(e') \ge j$. On the other hand, if $c_j \ne k$, then e' joins different components of $T'_{c_j}[Q]$ and so we actually have $\ell'(e') = j < i$, a contradiction with the choice of e since we have seen that e' is superfluous for $\mathcal{T}'$. Thus, we conclude that $c_j = k$. Since $T_k[R]$ is connected, the ends of e' are joined by a path S in $T_k[R]$. However, S is not contained within Q as e' joins different components of $T_k[Q]$. Hence, S contains an edge e'' leaving Q. Observing that $\ell(e'') < j$, we use the Claim to obtain $\ell'(e'') = \ell(e'') < j < \ell(e)$, a final contradiction with the minimality of e since e'' is superfluous for $\mathcal{T}'$. $\square$

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