Every plane graph of maximum degree 8 has an edge-face 9-colouring

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Abstract

An edge-face colouring of a plane graph with edge set E and face set F is a colouring of the elements of $E \cup F$ such that adjacent or incident elements receive different colours. Borodin proved that every plane graph of maximum degree $\Delta \geqslant 10$ can be edge-face coloured with $\Delta + 1$ colours. Borodin's bound was recently extended to the case where $\Delta = 9$. In this paper we extend it to the case $\Delta = 8$.

1 Introduction

Let G be a plane graph with vertex set V, edge set E and face set F. Given a positive integer k, an *edge-face k-colouring of G* is a mapping $\lambda : E \cup F \to \{1, 2, \ldots, k\}$ such that

(*i*) $\lambda(e) \neq \lambda(e')$ for every pair (e, e') of adjacent edges;

(*ii*) $\lambda(e) \neq \lambda(f)$ for edge e and every face f incident to e;

(*iii*) $\lambda(f) \neq \lambda(f')$ for every pair (f, f') of adjacent faces with $f \neq f'$.

The requirement in (*iii*) that f and f' be distinct is only relevant for graphs containing a cut-edge; such graphs would not have an edge-face colouring otherwise. Let $\chi_{ef}(G)$ be the value of the smallest integer k such that there exists an edge-face k-colouring of G.

Edge-face colourings were first studied by Jucovič [4] and Fiamčík [3], who considered 3- and 4-regular graphs. A conjecture of Mel'nikov [5] spurred

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research into upper bounds on $\chi_{ef}(G)$ for plane graphs G with $\Delta(G) \leqslant \Delta$. For small values of Δ, the best bounds known are $\Delta+3$ for $\Delta \in \{2, \dots, 6\}$ [1, 6, 9] and $\Delta + 2$ for $\Delta = 7$ [7]. For $\Delta \geqslant 10$, Borodin [2] proved the bound of $\Delta + 1$. This is tight, as can be seen by considering trees. Recently, the second and third authors [8] extended the $\Delta + 1$ bound to the case $\Delta = 9$ by proving that every plane graph of maximum degree 9 has an edge-face 10-colouring. Here, using a more delicate analysis, we settle the case $\Delta = 8$.

Theorem 1. *Every plane graph of maximum degree* 8 *has an edge-face* 9*-colouring.*

The problem of finding the provably optimal upper bounds on $\chi_{ef}(G)$ for plane graphs G with $\Delta(G) \leqslant \Delta$ remains open for $\Delta \in \{4, 5, 6, 7\}$.

We prove Theorem 1 by contradiction. From now on, we let $G = (V, E, F)$ be a counter-example to the statement of Theorem 1 with as few edges as possible. That is, G is a plane graph of maximum degree 8 and no edge-face 9-colouring, but every plane graph of maximum degree at most 8 with less than $|E|$ edges has an edge-face 9-colouring. In particular, for every edge $e \in E$ the plane subgraph $G - e$ of G has an edge-face 9-colouring. First, we describe various structural properties of G in Section 2; the proofs of these properties are given at the end of this paper in Section 5. In Section 3 we describe the discharging rules. In Section 4 we use the discharging rules and the structural properties of G to obtain a contradiction, and thus a proof of Theorem 1.

In the sequel, a vertex of degree d is called a *d-vertex*. A vertex is an *$(\leqslant d)$-vertex* if its degree is at most d; it is an *$(\geqslant d)$-vertex* if its degree is at least d. The notions of *d-face*, *$(\leqslant d)$-face* and *$(\geqslant d)$-face* are defined analogously as for the vertices, where the *degree* of a face is the number of edges incident to it. A face of length 3 is called a *triangle*. For integers a, b, c, an *$(\leqslant a, \leqslant b, \leqslant c)$-triangle* is a triangle xyz of G with $\deg(x) \leqslant a$, $\deg(y) \leqslant b$ and $\deg(z) \leqslant c$. The notions of $(a, \leqslant b, \leqslant c)$-triangles, $(a, b, \geqslant c)$-triangles, $(a, \leqslant b, c, d)$-faces, and so on, are defined analogously. A vertex is *triangulated* if all its incident faces are triangles.

As mentioned above, our proof uses discharging techniques. It was developed through several rounds, with corrective adjustments and optimisations included in each, starting from a naïve scheme in which only the $(\geqslant 7)$-vertices compensated for the deficit of charge on triangles. A breakthrough in the design of our strategy was the realisation that the reducible configuration A3 (defined below) could allow us to conserve considerable charge at $(\geqslant 7)$-vertices incident to $(\geqslant 5)$-faces of a particular type (cf. the rule R1g mentioned at the end of Section 3). We could then balance these savings against the loss of charge to incident triangles with the development of further reducible configurations. As will become apparent, the analysis of the final charge of ($\geqslant$7)-vertices is particularly difficult.

2 Reducible configurations

In this section, we present a catalogue of the structural properties of the graph G that are necessary for our proof of Theorem 1. In particular, we identify that some plane graphs are *reducible configurations*, i.e. that they cannot be part of the chosen embedding of G. Their reducibility follows from Lemmas 2–10 presented in Section 5.

For convenience, we depict these configurations in Figure 1. We use the following notational conventions for vertices: 2-, 3- and 4-vertices are depicted by black bullets, black triangles and black squares, respectively; a white bullet containing a number represents a vertex of degree that quantity; an empty white bullet represents a vertex of arbitrary degree (but at least that shown in the figure). For faces, we use the following conventions: a straight line indicates a single edge; a curved line indicates a portion of the face with an unspecified number of edges; a curved face that is shaded grey represents an ($\leqslant$4)-face.

The following configurations are reducible. Note that, for any of the below, if an edge can be removed without affecting the specific incidence or facial structure, then the configuration remains reducible.

A0 A 1-vertex.

Configurations with faces incident to a 2-vertex

A1 A triangle incident to a 2-vertex.

A2 A 4-face incident to a 2-vertex and an ($\leqslant$3)-vertex.

A3 A face incident to an edge uv such that $\deg(u) = 2$ and $\deg(v) = 6$.

Configurations with an edge incident to a ($\leqslant$4)-face

B1 An edge uv that is incident to an ($\leqslant$4)-face, with $\deg(u) + \deg(v) \leqslant 9$.

B2 A triangle uvw with $\deg(u) + \deg(v) \leqslant 10$ and $\deg(w) = 6$.

B3 A triangle uvw with uw incident to two ($\leqslant$4)-faces, and $\deg(u) + \deg(v) \leqslant 10$ and $\deg(w) = 7$.

B4 A triangle uvw with uw adjacent to two ($\leqslant$4)-faces, vw incident to two ($\leqslant$4)-faces, and $\deg(u) + \deg(v) \leqslant 10$.

Configurations with an edge incident to two ($\leqslant$4)-faces

C1 An edge uv that is incident to two ($\leqslant$4)-faces, with $\deg(u) + \deg(v) \leqslant 10$.

C2 A triangle uvw with uv incident to two ($\leqslant$4)-faces, and $\deg(u) + \deg(v) \leqslant 11$ and $\deg(w) = 6$.

C3 A triangle uvw with uv and uw each incident to two ($\leqslant$4)-faces, and $\deg(u) + \deg(v) \leqslant 11$ and $\deg(w) = 7$.

C4 A triangle uvw with vw incident to the triangle vwx and wx incident to two ($\leqslant$4)-faces, and $\deg(u) = \deg(x) = 3$.

C5 A triangle uvw with vw incident to the triangle vwx and wx incident to two ($\leqslant$4)-faces, and $\deg(u) + \deg(v) \leqslant 10$ and $\deg(v) + \deg(x) \leqslant 11$.

Configurations along a 2-path

D1 A 2-path uvw such that vwx is a triangle, with uv incident to an ($\leqslant$4)-face, vw and vx each incident to two ($\leqslant$4)-faces, and $\deg(u) + \deg(v) \leqslant 10$ and $\deg(v) + \deg(w) \leqslant 11$.

D2 A 2-path uvw such that vwx is a triangle, with uv, vw and vx each incident to two ($\leqslant$4)-faces, and $\deg(u) + \deg(v) \leqslant 11$ and $\deg(v) + \deg(w) \leqslant 11$.

D3 A 2-path uvw such that vwx is a triangle, with vx incident to two ($\leqslant$4)-faces, and $\deg(u) = 2$, $\deg(v) = 7$ and $\deg(u) = 3$.

D4 A 2-path uvw such that vwx is a triangle, with vw and vx each incident to two ($\leqslant$4)-faces, and $\deg(u) = 2$, $\deg(v) = 7$ and $\deg(u) = 4$.

Note on configurations D1 and D2. An ($\leqslant$4)-face incident to uv is not ruled out from also being an ($\leqslant$4)-face (distinct from vwx) incident to vw or vx. In this sense, the figures representing D1 and D2 in Figure 1 belie the configurations' fuller forms.

Special configurations

E1 A 4-path $uvwxy$, such that uvz, vwz, wxz and xyz are triangles, with yz incident to two ($\leqslant$4)-faces, and $\deg(v) = 3$, $\deg(x) = 4$.

E2 A 4-path $uvwxy$, such that uvz, vwz, wxz and xyz are triangles, and $\deg(v) = 3$, $\deg(x) = 4$ and $\deg(y) = 6$.

E3 A triangulated 8-vertex that is adjacent to both a 3-vertex and a 4-vertex.

3 Discharging rules

Recall that $G = (V, E, F)$ is a plane graph that is a minimum counter-example to the statement of Theorem 1, in the sense that $|E|$ is minimum. (In particular, a planar embedding of G is fixed.) We obtain a contradiction by using the Discharging Method. Each vertex and face of G is assigned an initial charge; the total sum of the charge is negative by Euler's Formula. Then vertices and faces send or receive charge according to certain redistribution rules. The total sum of the charge remains unchanged, but ultimately (by using all of the reducible configurations in Section 2) we deduce that the charge of each face and vertex is non-negative, a contradiction.

Initial charge

We assign a charge to each vertex and face. For every vertex $v \in V$, we define the initial charge $\mathrm{ch}(v)$ to be $2 \cdot \deg(v) - 6$, while for every face $f \in F$, we define the initial charge $\mathrm{ch}(f)$ to be $\deg(f) - 6$. The total sum is

$$\sum_{v \in V} \mathrm{ch}(v) + \sum_{f \in F} \mathrm{ch}(f) = -12\,.$$

Indeed, by Euler's formula $|E| - |V| - |F| = -2$. Thus, $6|E| - 6|V| - 6|F| = -12$. Since $\sum_{v \in V} \deg(v) = 2|E| = \sum_{f \in F} \deg(f)$, it follows that

$$\begin{aligned} -12 &= 4 \cdot |E| - 6 \cdot |V| + \sum_{f \in F} (\deg(f) - 6) \\ &= \sum_{v \in V} (2\deg(v) - 6) + \sum_{f \in F} (\deg(f) - 6)\,. \end{aligned}$$

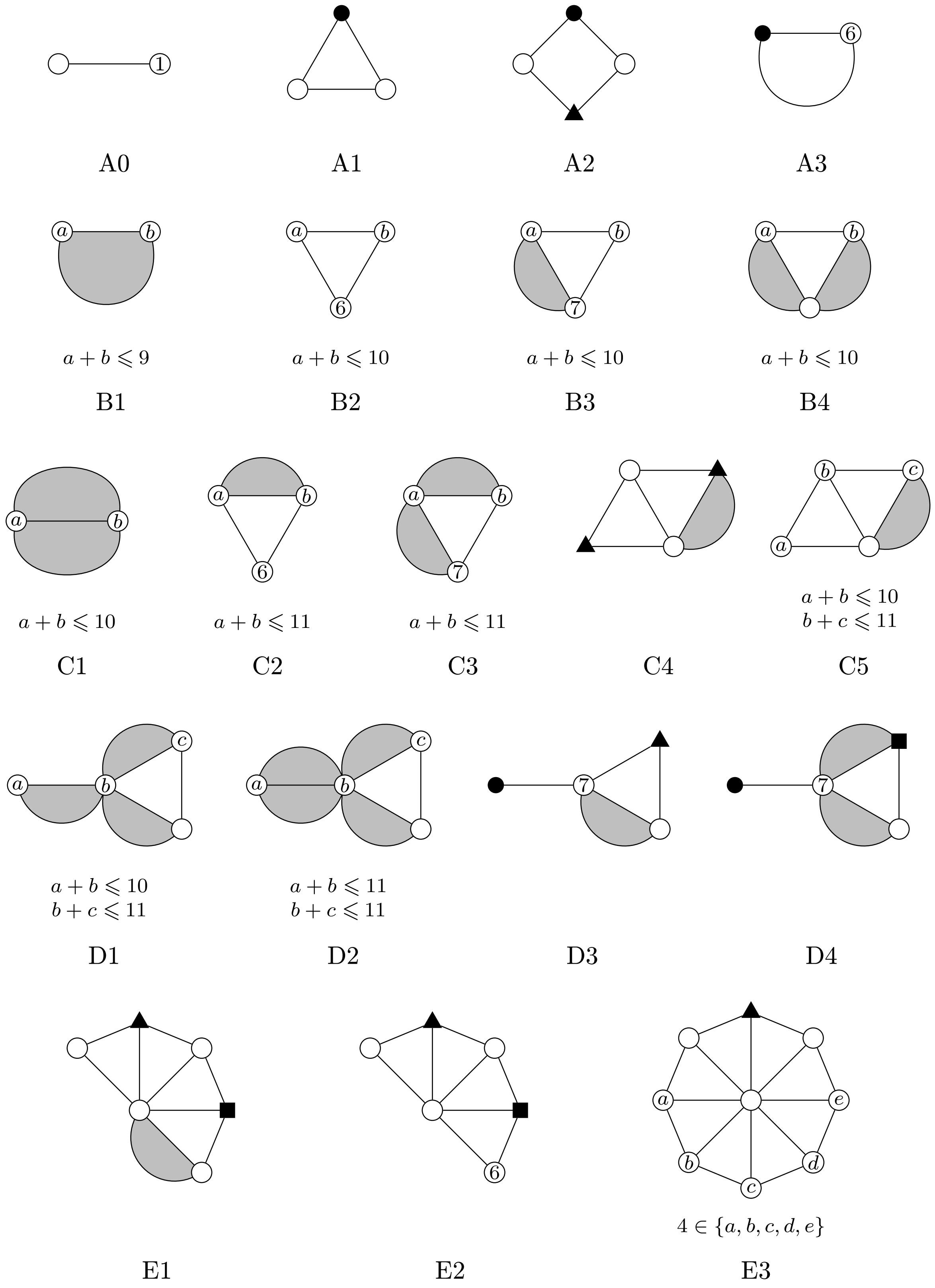

Figure 1: The reducible configurations.

Rules

We need the following definition to state the discharging rules. Given a vertex v, a face is *special* (for v) if it is an ($\geqslant$5)-face that is incident to a degree 2 neighbour of v (and so, in particular, such a face is incident to v).

Since G may have cut-vertices (of a type not forbidden by Lemma 2), some vertices may be incident to the same face several times. Thus, in the rules below, when we say that a vertex or a face sends charge to an incident face or vertex, we mean that the charge is sent as many times as these elements are incident to each other.

The following describe how the charge is redistributed among the edges and faces in G.

R0 An ($\geqslant$4)-face sends 1 to each incident 2-vertex.

R1 An ($\geqslant$7)-vertex sends

- **R1a** 3/2 to incident $(3, \geqslant 7, \geqslant 7)$-triangles and $(4, 6, \geqslant 7)$-triangles;
- **R1b** 7/5 to incident $(5, 5, \geqslant 7)$-triangles;
- **R1c** 5/4 to incident $(4, \geqslant 7, \geqslant 7)$-triangles and $(2, 8, 4, 8)$-faces;
- **R1d** 6/5 to incident $(5, 6, 8)$-triangles;
- **R1e** 11/10 to incident $(5, 6, 7)$- and $(5, \geqslant 7, \geqslant 7)$-triangles, and incident $(2, 8, 5, 8)$-faces;
- **R1f** 1 to incident $(\geqslant 6, \geqslant 6, \geqslant 6)$-triangles, to every other incident 4-face, and to special faces.

R2 A 6-vertex sends

- **R2a** 11/10 to incident $(5, 6, 6)$- and $(5, 6, 7)$-triangles;
- **R2b** 1 to every other incident triangle and to each incident 4-face.

R3 A 5-vertex sends 4/5 to each incident face.

R4 A 4-vertex sends 1/2 to each incident face.

Note on rules R1 and R2. Since the configurations A1, B1 and B2 are reducible, it follows from rule R1 that an ($\geqslant$7)-vertex sends positive charge to every incident triangle. We conclude that an ($\geqslant$7)-vertex sends zero charge only to incident ($\geqslant$5)-faces that are not special; we refer to this as rule R1g. Similarly, R2c is the "rule" that a 6-vertex sends zero charge to each incident ($\geqslant$5)-face.

4 Proof of Theorem 1

In this section, we prove that the final charge $\mathrm{ch}^*(x)$ of every $x \in V \cup F$ is non-negative. Hence, we obtain

$$-12 = \sum_{x \in V \cup F} \mathrm{ch}(x) = \sum_{x \in V \cup F} \mathrm{ch}^*(x) \geqslant 0\,,$$

a contradiction. This contradiction establishes Theorem 1.

Final charge of faces

Let f be a d-face. Our goal is to show that $\mathrm{ch}^*(f) \geqslant 0$. Recall that the initial charge of f is $\mathrm{ch}(f) = d - 6$.

First suppose that $d \geqslant 5$. Let p be the number of ($\geqslant$7)-vertices for which f is special, and q the number of 2-vertices incident to f. Since the configuration A3 is reducible, $p \geqslant q+1$ if d is odd and $p \geqslant q$ otherwise. Thus, by rules R0 and R1f, the final charge of f satisfies $\mathrm{ch}^*(f) \geqslant d - 6 + p - q$. Hence, $\mathrm{ch}^*(f) \geqslant 0$ if $d \geqslant 6$, and if $d = 5$ then $d-6+p-q \geqslant 5-6+q+1-q = 0$.

Next suppose that $d = 4$. Let the four vertices incident to f be $v_0, \dots, v_3$ in clockwise order and suppose without loss of generality that v_0 has the least degree among $v_0, \dots, v_3$. First, if $\deg(v_0) \geqslant 4$, then by rules R1f, R2b, R3 and R4, the charge sent to f by each incident vertex is at least 1/2, so that $\mathrm{ch}^*(f) \geqslant -2 + 4 \cdot 1/2 = 0$. If $\deg(v_0) = 3$, then since the configuration B1 is reducible $\deg(v_1) \geqslant 7$ and $\deg(v_3) \geqslant 7$. Thus, by rule R1f, $\mathrm{ch}^*(f) \geqslant -2+2 = 0$. Last, assume that $\deg(v_0) = 2$. Since the configuration B1 is reducible, $\deg(v_1) = \deg(v_3) = 8$, and since the configuration A2 is reducible, $\deg(v_2) \geqslant 4$. By rule R0, f sends charge 1 to v_0. But f receives charge 3: by rules R1c and R4 if f is a $(2, 8, 4, 8)$-face; by rules R1e and R3 if f is a $(2, 8, 5, 8)$-face; and by rules R1f an R2b if f is a $(2, 8, \geqslant 6, 8)$-face. Thus, $\mathrm{ch}^*(f) \geqslant 0$.

Finally suppose that $d = 3$. Let the three vertices incident to f be v_0, v_1 and v_2, and let us assume without loss of generality that $\deg(v_0) \leqslant \deg(v_1) \leqslant \deg(v_2)$. Since the configuration A1 is reducible, $\deg(v_0) \geqslant 3$. Thus f sends no charge, but needs to make up for an initial charge of -3. We analyze several cases according to the value of $\deg(v_0)$.

$\deg(v_0) = 3$. Since the configuration B1 is reducible, $\deg(v_1) \geqslant 7$. By rule R1a, f receives charge $2 \cdot 3/2 = 3$.

$\deg(v_0) = 4$. Since the configuration B1 is reducible, $\deg(v_1) \geqslant 6$. If $\deg(v_1) \geqslant 7$, then f receives charge $2{\cdot}5/4{+}1/2 = 3$ by rules R1c and R4. Otherwise, $\deg(v_1) = 6$ and hence $\deg(v_2) \geqslant 7$ since the configuration

B2 is reducible, but then f receives charge $3/2+1+1/2=3$ by rules R1a, R2b and R4.

$\deg(v_0)=5$. If $\deg(v_1)=5$, then $\deg(v_2)\geqslant 7$ since the configuration B2 is reducible, but then f receives charge $7/5+2\cdot 4/5=3$ by rules R1b and R3. If $\deg(v_1)=6$, then we separately consider the cases of $\deg(v_2)\in\{6,7,8\}$. If $\deg(v_2)\in\{6,7\}$, then f receives charge $2\cdot 11/10+4/5=3$ by rules R1e, R2a and R3; if $\deg(v_2)=8$, then f receives charge $6/5+1+4/5=3$ by rules R1d, R2b and R3. Last, if $\deg(v_1)\geqslant 7$, then f receives charge $2\cdot 11/10+4/5=3$ by rules R1e and R3.

$\deg(v_0)\geqslant 6$. f receives charge at least 3 by rules R1f and R2b.

This concludes our analysis of the final charge of f, verifying that $\mathrm{ch}^*(f)\geqslant 0$.

Final charge of ($\leqslant$6)-vertices

Let v be an arbitrary vertex of G. Our goal is to show that $\mathrm{ch}^*(v)\geqslant 0$. Recall that the initial charge of v is $\mathrm{ch}(v)=2\cdot\deg(v)-6$. Moreover, $\deg(v)\geqslant 2$ since the configuration A0 is reducible.

If $\deg(v)=2$, then v is incident to two ($\geqslant$4)-faces since the configuration A1 is reducible; thus, v receives charge 1 from both incident faces by rule R0 and the final charge of v is $\mathrm{ch}^*(v)=-2+2=0$.

If $\deg(v)=3$, then v neither sends nor receives any charge; hence, the final charge of v is $\mathrm{ch}^*(v)=\mathrm{ch}(v)=0$.

If $\deg(v)\in\{4,5\}$, then v sends charge $\mathrm{ch}(v)/\deg(v)$ to each incident face by rules R3 and R4; the final charge of v is $\mathrm{ch}^*(v)=0$.

Suppose now that $\deg(v)=6$. The initial charge of v is $\mathrm{ch}(v)=6$. Since the configuration A3 is reducible, all adjacent vertices have degree at least 3. Thus, no ($\geqslant$5)-face is special for v and by rule R2c any incident ($\geqslant$5)-face is sent no charge. If there is an incident ($\geqslant$5)-face, then by rule R2 the total charge sent by v is at most $5\cdot 11/10<6$. We conclude that v is only incident to ($\leqslant$4)-faces. Then, since the configuration C2 is reducible, v has no incident $(5,6,6)$-face; furthermore, since the configuration C3 is reducible, v has no incident $(5,6,7)$-face. Therefore, the charge sent by v is at most 6 and the final charge of v satisfies $\mathrm{ch}^*(v)\geqslant 0$.

Final charge of 7-vertices

Next, suppose that $\deg(v)=7$. For convenience, let $v_0,v_1,\ldots,v_6$ be the neighbours of v in clockwise order, and let f_i be the face vv_iv_{i+1} for $i\in\{0,1,\ldots,6\}$, where the index is modulo 7. The initial charge of v is $\mathrm{ch}(v)=8$.

We partition our analysis based on the number of incident special ($\geqslant$5)-faces. Note that since the configuration B1 is reducible, if v is adjacent to a 2-vertex then both of the 2-vertex's incident faces are special for v.

We first treat the cases in which v is adjacent to some 2-vertex. In these cases, we may assume that there is no incident ($\geqslant$5)-face that is not special (and sent no charge from v by rule R1g), for otherwise the total charge sent by v is at most $4 \cdot 3/2 + 2 = 8$ (due to rule R1f). Furthermore, by rules R1a and R1b, any face that is sent charge more than $5/4$ must be a $(3, 7, \geqslant 7)$-, $(4, 6, 7)$- or $(5, 5, 7)$-triangle. And so we assert that if f_i is such a triangle, then both f_{i-1} and f_{i+1} are ($\geqslant$5)-faces. The assertion holds if f_i is a $(3, 7, \geqslant 7)$-triangle since the configurations C1 and D3 are reducible, and the fact that the configuration B3 is reducible implies the assertion for the two other cases.

If v has at least five ($\geqslant$5)-faces, then the charge sent is at most $2\cdot 3/2+5 = 8$ due to rule R1f.

If v is incident to exactly four ($\geqslant$5)-faces, all of which are special, then there must be two incident ($\leqslant$4)-faces that are adjacent. (Recall that each special face is adjacent to another special face.) By the assertion in the second paragraph of the 7-vertex analysis, both of these are sent charge at most $5/4$. Therefore, the total charge sent by v in this case is at most $3/2 + 2 \cdot 5/4 + 4 = 8$.

If v is incident to exactly three ($\geqslant$5)-faces, all of which are special, then these faces are sequentially adjacent around v. Hence, by the assertion in the second paragraph of the 7-vertex analysis, no face is sent charge more than $5/4$ and the total charge sent is at most $4 \cdot 5/4 + 3 = 8$

Suppose that v is incident to exactly two ($\geqslant$5)-faces, say f_0 and f_1, both special (so v_1 is a 2-vertex). Recall that all other incident faces have size at most 4. Let us analyse which incident faces can be sent charge $5/4$. By rule R1c, such a face must be a $(4, 7, \geqslant 7)$-triangle. Since the configuration D4 is reducible, such a face must be adjacent to a special face for v. Thus, there are at most two such faces, namely f_2 and f_6. Consequently, the total charge sent by v is at most $2 \cdot 5/4 + 3 \cdot 11/10 + 2 < 8$ by rules R1c, R1e and R1f.

Now we may assume that v is not adjacent to a 2-vertex and thus any ($\geqslant$5)-face incident to v is sent no charge. Thus, v is incident to at most one ($\geqslant$5)-face, for otherwise the total charge sent by v is at most $5 \cdot 3/2 < 8$. Since the configuration B3 is reducible, v is incident to no $(3, 7, 7)$-, $(4, 6, 7)$- or $(5, 5, 7)$-triangles. Furthermore, since the configuration C1 is reducible, if f_i is a $(3, 7, 8)$-triangle then f_{i-1} or f_{i+1} is an ($\geqslant$5)-face. In particular, v is incident to at most two such triangles; if it is incident to at least one then it is also incident to an ($\geqslant$5)-face, in which case the total charge sent is at most $2\cdot 3/2+4\cdot 5/4 = 8$. Therefore, we may assume that v does not send more than

5/4 charge to any incident face. As a result, v has no incident ($\geqslant$5)-faces, for otherwise the total charge sent is $6 \cdot 5/4 < 8$. Since the configuration D2 is reducible, v has at most one degree 4 neighbour. Therefore, at most two faces are sent charge 5/4 by rule R1c, and all other faces are sent charge at most 11/10 by rule R1e. Hence, the total charge sent by v is at most $2 \cdot 5/4 + 5 \cdot 11/10 = 8$.

Final charge of 8-vertices

Last, suppose that $\deg(v) = 8$. For convenience, let $v_0, \ldots, v_7$ be the neighbours of v in clockwise order, and for $i \in \{0, \ldots, 7\}$, let f_i be the face of G incident with vv_i and vv_{i+1}, where the index is modulo 8. The initial charge of v is $\mathrm{ch}(v) = 10$. We partition our analysis based on the number of incident special faces. Note that since the configuration C1 is reducible, if v is adjacent to a 2-vertex then at least one of the 2-vertex's incident faces is special for v. Furthermore, since the configurations A2 and B1 are reducible, if one of the 2-vertex's incident faces is an ($\leqslant$4)-face, then it must be a $(2, 8, \geqslant 4, 8)$-face.

We first treat the cases in which v is adjacent to some 2-vertex. In these cases, we may assume that there is no incident ($\geqslant$5)-face that is not special (and sent no charge from v by rule R1g), for otherwise the total charge sent by v is at most $6 \cdot 3/2 + 1 = 10$ (due to rule R1f).

If v has at least four incident special faces, then the charge sent is at most $4 \cdot 3/2 + 4 = 10$.

Suppose that v is incident to exactly three special faces. If v is incident to at least two $(2, 8, \geqslant 4, 8)$-faces, each sent charge at most 5/4 by rule R1c, then the total charge sent by v is at most $3 \cdot 3/2 + 2 \cdot 5/4 + 3 = 10$. If v is incident to exactly one $(2, 8, \geqslant 4, 8)$-face, then it must be that v is incident to three sequentially adjacent ($\leqslant$4)-faces, say f_0, f_1 and f_2. Since the configuration B4 is reducible, f_1 is not a $(3, 7, 8)$-, $(4, 6, 8)$- or $(5, 5, 8)$-face; since v is incident to a $(2, 8, \geqslant 4, 8)$-face and D1 is reducible, f_1 is not a $(3, 8, 8)$-face; hence, f_1 receives charge at most 5/4. Consequently, the total charge sent by v is at most $3 \cdot 3/2 + 2 \cdot 5/4 + 3 = 10$. If v is not incident to a $(2, 8, \geqslant 4, 8)$-face, then the three incident special faces are sequentially adjacent around v. In the following we shall assume that f_5, f_6 and f_7 are the three special faces. The remaining five faces are are all triangles, otherwise (by rule R1f) the total charge sent by v is at most $4 \cdot 3/2 + 4 = 10$. Note that there is no $i \in \{2, 3, 4\}$ such that all of f_{i-1}, f_i, f_{i+1} are $(3, \geqslant 7, 8)$-triangles since the configuration C4 is reducible. Since the configuration B4 is reducible, none of f_1, f_2, f_3 is a $(3, 7, 8)$-, $(4, 6, 8)$- or $(5, 5, 8)$-triangle. Furthermore, since the configuration D2 is reducible, v is incident to at most one pair of adjacent

$(3, \geqslant 7, 8)$-triangles. Thus, at most one of $v_1, \dots, v_4$ is a 3-vertex.

If none of $v_1, \dots, v_4$ is a 3-vertex, then the only faces that can be sent charge more than $5/4$ are f_0 and f_4. Therefore, the total charge sent by v is at most $2 \cdot 3/2 + 3 \cdot 5/4 + 3 < 10$.

Suppose that v_2 or v_3 is a 3-vertex, say v_2, by symmetry. Then v_1 and v_3 are 8-vertices. Since the configuration C4 is reducible, v_0 and v_4 are $(\geqslant 4)$-vertices and hence f_0 and f_3 are each sent charge at most $5/4$. Thus, the total charge sent by v is at most $3 \cdot 3/2 + 2 \cdot 5/4 + 3 = 10$.

Suppose that v_1 or v_4 is a 3-vertex, say v_1, by symmetry. Then f_4 is the only face other than f_0 and f_1 that can be sent charge more than $5/4$. In this case, the total charge sent by v is at most $3 \cdot 3/2 + 2 \cdot 5/4 + 3 = 10$.

Suppose that v is incident to exactly two special faces (and hence is incident to at most two $(2, 8, \geqslant 4, 8)$-faces). First, assume that v is incident to a $(2, 8, \geqslant 4, 8)$-face. Since the configuration B4 is reducible, if f_i is a $(3, 7, 8)$-, $(4, 6, 8)$- or $(5, 5, 8)$-triangle, then f_{i-1} or f_{i+1} is a $(\geqslant 5)$-face; also, since the configuration D1 is reducible (and v is incident to a $(2, 8, \geqslant 4, 8)$-face), the same conclusion holds if f_i is a $(3, 8, 8)$-triangle. Since each incident special face is sequentially adjacent either to an incident $(2, 8, \geqslant 4, 8)$-face or to the other special face, we deduce that at most two faces are sent charge more than $5/4$. Thus, the total charge sent by v is at most $2 \cdot 3/2 + 4 \cdot 5/4 + 2 = 10$.

Now we deal with the case where v is not incident to a $(2, 8, \geqslant 4, 8)$-face. Suppose f_6 and f_7 are the two special faces, with v_7 being a 2-vertex. Recall that none of $f_0, \dots, f_5$ is an $(\geqslant 5)$-face. Note also that at most one of $f_0, \dots, f_5$ is a 4-face, for otherwise the charge sent by v is at most $4 \cdot 3/2 + 4 = 10$. We analyse possible $(3, \geqslant 7, 8)$-triangles among these six faces. First, note that there is no $i \in \{2, 3, 4, 5\}$ such that all of f_{i-1}, f_i, f_{i+1} are $(3, \geqslant 7, 8)$-triangles since the configuration C4 is reducible. Since the configuration B4 is reducible, none of $f_1, \dots, f_4$ is a $(3, 7, 8)$-, $(4, 6, 8)$- or $(5, 5, 8)$-triangle. Furthermore, since the configuration D2 is reducible, v is incident to at most one pair of adjacent $(3, \geqslant 7, 8)$-triangles. First, we suppose that all of $f_0, \dots, f_5$ are triangles. It follows that at most one of $v_1, \dots, v_5$ is a 3-vertex. We consider several cases regarding which neighbours of v are $(\leqslant 4)$-vertices.

If none of $v_1, \dots, v_5$ is a 3-vertex, then the only faces that can be sent charge more than $5/4$ are f_0 and f_5. Therefore, the total charge sent by v is at most $2 \cdot 3/2 + 4 \cdot 5/4 + 2 = 10$.

Suppose that v_3 is a 3-vertex. Hence, v_2 and v_4 are 8-vertices. We show that f_0 and f_1 are sent charge at most $5/2$ altogether by v. Indeed, if v_1 is a 4-vertex, then $\deg(v_0) \geqslant 7$ because the configuration E2 is reducible. Hence, v sends charge $5/4$ to each of f_0 and f_1 by rule R1c. If v_1 is a 5-vertex, then $\deg(v_0) \geqslant 5$ and hence v sends charge $11/10$ to f_1 and at most $7/5$ to f_0 by rules R1b and R1e. Last, if $\deg(v_1) \geqslant 6$ then v sends charge 1 to f_1 and at most $3/2$ to f_0 by rules R1a and R1f. Similarly, we deduce that v sends charge at most $5/2$ to f_4 and f_5 altogether. Therefore, the total charge sent by v is at most $2 \cdot 3/2 + 2 \cdot 5/2 + 2 = 10$.

Suppose that v_2 or v_4 is a 3-vertex, say v_2, by symmetry. Then, v_1 and v_3 are 8-vertices. We have $\deg(v_0) \geqslant 4$ since the configuration C4 is reducible, so that f_0 receives charge at most $5/4$ from v. Thus, it suffices to show that v sends to f_3, f_4 and f_5 charge at most $15/4$ altogether: the total charge sent by v would then be at most $2 \cdot 3/2 + 5/4 + 15/4 + 2 = 10$. First, $\deg(v_4) \geqslant 5$ since the configuration E1 is reducible. Recall that $\deg(v_4) + \deg(v_5) \geqslant 11$. If $\deg(v_4) + \deg(v_5) \geqslant 12$, then f_3 and f_4 are sent charge at most $9/4$ altogether by v. Thus, the conclusion holds since f_6 is sent charge at most $3/2$ by v. Now, if $\deg(v_4) + \deg(v_5) = 11$, then $\deg(v_5) + \deg(v_6) \geqslant 11$ since the configuration C5 is reducible. Consequently, each of f_4 and f_5 is sent charge at most $5/4$ by v. Moreover, f_3 is sent charge at most $11/10$ by rule R1e, so that the conclusion holds.

Suppose that v_1 or v_5 is a 3-vertex, say v_1, by symmetry. Then, $\deg(v_0) = 8 = \deg(v_2)$, and $\deg(v_3) \geqslant 5$ since E1 is reducible. Further, recall that v_4 and v_5 both have degree at least 4. If $\deg(v_6) \leqslant 4$, then $\deg(v_5) \geqslant 6$. Since $\deg(v_3) + \deg(v_4) \geqslant 11$ (because B4 is reducible), at least one of f_2 and f_4 is sent charge at most 1, implying that the total charge sent by v is at most $3 \cdot 3/2 + 2 \cdot 5/4 + 3 = 10$. If $\deg(v_6) \geqslant 5$, then f_5 is sent charge at most $7/5$ and f_2 is sent charge at most $11/10$, so the total charge sent by v is at most $2 \cdot 3/2 + 7/5 + 2 \cdot 5/4 + 11/10 + 2 = 10$.

Assume now that (exactly) one of $f_0, \dots, f_5$ is a 4-face. (Such a 4-face is assumed to not have an incident 2-vertex.) Without loss of generality, we may suppose that it is one of f_0, f_1 and f_2. Recall that none of $f_1, \dots, f_4$ is a $(3,7,8)$-, $(4,6,8)$- or $(5,5,8)$-triangle since the configuration B4 is reducible. Also, at most one of $v_1, \dots, v_5$ is a 3-vertex since the configurations C4 and D2 are reducible. Moreover, if at most one of $f_1, \dots, f_4$ is sent charge $3/2$ by v (i.e. is a $(3,8,8)$-triangle), then the total charge sent by v is at most $3 \cdot 3/2 + 2 \cdot 5/4 + 3 = 10$. In particular, we assume that (exactly) one of v_2, v_3, v_4 has degree 3.

Suppose that the 4-face is f_0. At most three of $f_1, \dots, f_5$ are sent charge more than $5/4$, so the total charge sent by v is at most $3 \cdot 3/2 + 2 \cdot 5/4 + 3 = 10$.

Suppose that the 4-face is f_1. By the remark above, one of v_3 and v_4 has degree 3. If $\deg(v_4) = 3$, then $\deg(v_3) = \deg(v_5) = 8$. Further, $\deg(v_6) \geqslant 4$ since the configuration C4 is reducible. Consequently, the total charge sent by v is at most $3 \cdot 3/2 + 2 \cdot 5/4 + 3 = 10$. If $\deg(v_3) = 3$, then f_4 and f_5 are sent at most $5/2$ altogether by v. Indeed, let us check all of the subcases: if $\deg(v_6) \leqslant 4$, then $\deg(v_5) \geqslant 6$, implying that f_4 is sent charge 1 and f_5 is sent charge at most $3/2$; if $\deg(v_6) = 5$, then $\deg(v_5) \geqslant 5$, implying that f_4 is sent charge at most $11/10$ and f_5 is sent charge at most $7/5$; if $\deg(v_6) = 6$, then $\deg(v_5) \geqslant 5$ since the configuration E1 is reducible, and so each of f_4 and f_5 are sent charge at most $6/5$; if $\deg(v_6) \geqslant 7$, then each of f_4 and f_5 are sent charge at most $5/4$. Therefore, the total charge sent by v is at most $3 \cdot 3/2 + 3 + 5/2 = 10$.

Suppose that the 4-face is f_2. Then, $\deg(v_4) = 3$, for otherwise at most one face among $f_1, \ldots, f_4$ is sent charge $3/2$ by v. Then, $\deg(v_5) = 8$ and $\deg(v_6) \geqslant 4$ since the configuration C4 is reducible. Therefore, the total charge sent by v is at most $3 \cdot 3/2 + 2 \cdot 5/4 + 3 = 10$.

Suppose that v is incident to exactly one special face. Then v is incident to a $(2, 8, \geqslant 4, 8)$-face and, since the configurations B4 and D1 are reducible, v is incident to at most one $(3, \geqslant 7, 8)$-, $(4, 6, 8)$- or $(5, 5, 8)$-face; the total charge sent by v is at most $3/2 + 6 \cdot 5/4 + 1 = 10$.

Suppose that v is not incident to a special face. Any $(\geqslant 5)$-face incident to v is sent no charge. If there are two such faces, then the total charge sent by v is at most $6 \cdot 3/2 < 10$. If there is one such face, since configurations B4, C4 and D2 are reducible, we conclude that v is incident to at most four faces that are sent more than $5/4$ charge; thus, the total charge sent by v is at most $4 \cdot 3/2 + 3 \cdot 5/4 < 10$.

Finally, we are in the case that v is only incident to $(\leqslant 4)$-faces (none of which are $(2, 8, \leqslant 5, 8)$-faces since the configuration C1 is reducible). Since the configuration B4 is reducible, v is incident to no $(3, 7, 8)$-, $(4, 6, 8)$- or $(5, 5, 8)$-triangle. If v is not incident to a $(3, 8, 8)$-triangle, then no face is sent charge more than $5/4$ and hence the total charge sent by v is at most $8 \cdot 5/4 = 10$. So assume that v_7 is a 3-vertex, and that f_7 is a $(3, 8, 8)$-triangle. Since the configuration D2 is reducible, v is adjacent to no other 3-vertices. We may assume that v is incident to fewer 4-faces than the number of $(3, 8, 8)$-triangles incident to v. (Otherwise, if x is the number of $(3, 8, 8)$-triangles incident to v, then the total charge sent by v is at most $x \cdot (3/2 + 1) + (8 - 2 \cdot x) \cdot 5/4 = 10$.)

If v were incident to only one $(3,8,8)$-triangle then the other face incident to w would necessarily be a 4-face. We conclude, therefore, that v is incident to exactly two $(3,8,8)$-triangles, namely f_6 and f_7, and to at most one 4-face. Now, if v is incident to only triangles, then since the configuration E3 is reducible, every neighbour of v other than v_1 has degree at least 5, and so the total charge sent is at most $2 \cdot 3/2 + 4 \cdot 6/5 + 2 \cdot 11/10 = 10$ (where we observe that the faces adjacent around v to the $(3,8,8)$-triangles are $(\geqslant 5, 8, 8)$-faces and hence sent charge at most $11/10$).

Therefore, in addition to the two $(3,8,8)$-triangles, v must be incident to exactly one 4-face. Since the configuration E3 is reducible, we may assume, without loss of generality, by symmetry, that f_0 is an $(\geqslant 5, 8, 8)$-triangle, and f_1 and f_2 are both $(\geqslant 5, \geqslant 5, 8)$-triangles. If f_1 is a $(5,8,8)$-triangle, then (since the configuration B4 is reducible) f_2 and f_3 must both be $(\geqslant 5, \geqslant 6, 8)$-triangles; in this case, the total charge sent by v is at most $2 \cdot 3/2 + 2 \cdot 5/4 + 2 \cdot 6/5 + 11/10 + 1 = 10$. Otherwise, f_1 is an $(\geqslant 6, 8, 8)$-triangle and sent charge at most 1 by rule R1f and the total charge sent by v is at most $2 \cdot 3/2 + 4 \cdot 5/4 + 2 = 10$.

We have shown that if $\deg(v) = 8$, then $\mathrm{ch}^*(v) \geqslant 0$. This allows us to conclude our analysis of the final charge of v, having shown $\mathrm{ch}^*(v) \geqslant 0$ in all cases. This completes the proof of Theorem 1. □

5 Proofs of reducibility

In this section, we prove that the graph G cannot contain any of the configurations given in Section 2.

Let λ be a (partial) edge-face 9-colouring of G. For each element $x \in E \cup F$, we define $\mathcal{C}(x)$ to be the set of colours (with respect to λ) of the edges and faces incident or adjacent to x. If $x \in V$ we define $\mathcal{E}(x)$ to be the set of colours of the edges incident to x. Moreover, λ is *nice* if only some $(\leqslant 4)$-faces are uncoloured. Observe that every nice colouring can be greedily extended to an edge-face 9-colouring of G, since $|\mathcal{C}(f)| \leqslant 8$ for each $(\leqslant 4)$-face f, i.e. f has at most 8 forbidden colours. Therefore, in the rest of the paper, we shall always suppose that such faces are coloured at the very end. More precisely, every time we consider a partial colouring of G, we uncolour all $(\leqslant 4)$-faces, and implicitly colour them at the very end of the colouring procedure of G. We make the following observation about nice colourings, which we rely on frequently.

Observation. Let e be an edge of G incident to two faces f and f'. There exists a nice colouring λ of $G - e$, and hence a partial edge-face 9-colouring of G in which only e and f are uncoloured. Moreover, if f is an ($\leqslant$4)-face, then it suffices to properly colour the edge e with a colour from $\{1, 2, \ldots, 9\}$ to extend λ to a nice colouring of G.

The following lemma implies the reducibility of configuration A0. We require the stronger form as it is necessary for later arguments.

Lemma 2. *Let v be a vertex of G with neighbours $v_0, v_1, \ldots, v_{d-1}$ in clockwise order. If v is a cut-vertex of G, then no component C of $G - v$ is such that the neighbourhood of v in C is contained in $\{v_i, v_{i+1}\}$ for some $i \in \{0, 1, \ldots, d-1\}$, where the index i is taken modulo d.*

Proof. Suppose on the contrary that C is a component of $G - v$ such that the neighbourhood of v in C is contained in, say, $\{v_0, v_1\}$.

First, assume that the neighbourhood of v is $\{v_0, v_1\}$. Then G is the edge-disjoint union of two plane graphs $G_1 = (C \cup \{v\}, E_1)$ and $G_2 = (V \setminus C, E_2)$. The outer face f_1 of G_1 corresponds to a face f_2 of G_2. By the minimality of G, the graph G_i has an edge-face 9-colouring λ_i for $i \in \{1, 2\}$. Since both vv_0 and vv_1 are incident in G_1 to f_1, we may assume that $\lambda_1(f_1) = 1$, $\lambda_1(vv_0) = 8$ and $\lambda_1(vv_1) = 9$. Regarding λ_2, we may assume that $\lambda_2(f_2) = 1$. Furthermore, up to permuting the colours, we can also assume that the colours of the edges of G_2 incident to v are contained in $\{1, 2, \ldots, 7\}$, since there are at most 6 such edges.

We now define an edge-face 9-colouring λ of G as follows. For every edge e of G, set $\lambda(e) := \lambda_1(e)$ if $e \in E_1$ and $\lambda(e) := \lambda_2(e)$ if $e \in E_2$. To colour the faces of G, let f be the face of G incident to both vv_0 and vv_{d-1}. (Note that there is only one such face, since otherwise v would have degree 2, which would be a contradiction.) Now observe that there is a natural one-to-one correspondence between the faces of G_1 and a subset F_1 of the face set F of G that maps f_1 to f. Similarly, there is a natural one-to-one correspondence between the faces of G and a subset F_2 of F that maps f_2 to f. Note that $F_1 \cap F_2 = \{f\}$. Now, we can colour every face $f \in F_i$ using λ_i. This is well defined since $\lambda_1(f_1) = \lambda_2(f_2) = 1$.

Let us check that λ is proper. Two adjacent edges of G are assigned different colours. Indeed, if the two edges belong to E_i for some $i \in \{1, 2\}$, then it comes from the fact that λ_i is an edge-face 9-colouring of G_i. Otherwise, both edges are incident with v, and one is in G_1 and the other in G_2. The former is coloured either 8 or 9, and the latter with a colour of $\{1, 2, \ldots, 7\}$ by the choice of λ_1 and λ_2. Two adjacent faces in G necessarily correspond to two adjacent faces in G_1 or G_2, and hence are assigned different colours. Last, let g be a face of G and e an edge incident to g in G. If $g \neq f$, then g and e are incident in G_1 or G_2, and hence coloured differently. Otherwise e is incident to f_i in G_i for some $i \in \{1, 2\}$, and hence $\lambda(e) = \lambda_i(e) \neq \lambda_i(f_i) = 1 = \lambda(f)$.

The case where the neighbourhood of v is $\{v_0\}$, i.e. vv_0 is a cut-edge, is dealt with in the very same way so we omit it. □

The next lemma shows the reducibility of configurations A3, B1 and C1.

Lemma 3. *Let uv be an edge of G, and let $s \in \{0, 1, 2\}$ be the number of $(\leqslant 4)$-faces incident to uv. Then* $\deg(u) + \deg(v) \geqslant 9 + s$.

Proof. Suppose on the contrary that $\deg(u) + \deg(v) \leqslant 8 + s$. Let f and f' be the two faces incident to uv.

First assume that $s \geqslant 1$. Without loss of generality assume that f is an $(\leqslant 4)$-face. By the minimality of G, the graph $G - e$ has a nice colouring λ. Let f'' be the face of $G - e$ corresponding to the union of the two faces f and f' of G after having removed the edge e. We obtain a partial edge-face 9-colouring of G in which only e, f and the $(\leqslant 4)$-faces are uncoloured by just assigning the colour $\lambda(f'')$ to f', and keeping all the other assignments.

Consequently, $|\mathcal{C}(uv)| \leqslant \deg(u) + \deg(v) - 2 + 2 - s \leqslant 8$. Hence, we can properly colour the edge uv, thereby obtaining a nice colouring of G; a contradiction.

Now assume that $s = 0$. The graph G' obtained by contracting the edge uv is planar, simple and has maximum degree at most 8. By the minimality of G, let λ be a nice colouring of G'. Let g and g' be the faces of G' corresponding to the contracted faces f and f' of G, respectively. We obtain a partial edge-face 9-colouring of G in which only e is uncoloured by assigning the colour $\lambda(g)$ to f, $\lambda(g')$ to f', and keeping all the other assignments.

Consequently, $|\mathcal{C}(uv)| \leqslant \deg(u) + \deg(v) - 2 + 2 \leqslant 8$. Hence, we can properly colour the edge uv, thereby obtaining a nice colouring of G; a contradiction. □

In light of Lemma 3, we make the following definition and observation. An edge uv of G is called *tight* if $\deg(u) + \deg(v) - s = 9$, where $s \in \{0, 1, 2\}$ is the number of $(\leqslant 4)$-faces incident to uv.

Observation. Assume that c is an edge-face 9-colouring of G in which only uv and the $(\leqslant 4)$-faces are uncoloured. Let S be the (possibly empty) set of colours assigned by c to the $(\geqslant 5)$-faces incident to uv. If uv is tight, then the sets $\mathcal{E}(u)$, $\mathcal{E}(v)$ and S are pairwise disjoint, and $\mathcal{C}(uv) = \mathcal{E}(u) \cup \mathcal{E}(v) \cup S = \{1, \ldots, 9\}$.

The reducibility of configurations B2, B3, B4, C2 and C3 follows from the next lemma.

Lemma 4. *Let uvw be a triangle of G such that* $\deg(u) + \deg(v) = 10 + s$, *where $s \in \{0, 1\}$ is the number of* ($\leqslant$4)*-faces distinct from uvw incident to uv, and let $t \in \{0, 1, 2\}$ be the number of* ($\leqslant$4)*-faces distinct from uvw incident to uw or vw. Then* $\deg(w) \geqslant 7 + t$.

Proof. As we pointed out, there exists a partial edge-face 9-colouring c of G in which only uv and the ($\leqslant$4)-faces are left uncoloured. Let α_{uv}, α_{vw} and α_{uw} be the colours, if any, assigned to the ($\geqslant$5)-faces incident to uv, vw and uw, respectively. Since the edge uv is tight, $\mathcal{E}(u)$, $\mathcal{E}(v)$ and $\{\alpha_{uv}\}$ form a partition of $\{1, 2, \ldots, 9\}$. Thus, if there is a colour $\xi \in \mathcal{E}(u) \cup \{\alpha_{uv}\}$ that is not in $\mathcal{E}(w) \cup \{\alpha_{vw}\}$, then we can colour uv with $c(vw)$ and next recolour vw with ξ to obtain a nice colouring of G. We deduce that $\mathcal{E}(u) \cup \{\alpha_{uv}\} \subseteq \mathcal{E}(w) \cup \{\alpha_{vw}\}$. Similarly, $\mathcal{E}(v) \cup \{\alpha_{uv}\} \subseteq \mathcal{E}(w) \cup \{\alpha_{uw}\}$. Hence, $\deg(w) + 2 - t \geqslant 9$, so $\deg(w) \geqslant 7 + t$, as required. □

The following verifies that the configuration A1 is reducible.

Lemma 5. *Let u, v, w be vertices of G with* $\deg(v) = 2$*. Then uvw is not a face of G.*

Proof. Suppose on the contrary that uvw is a face of G. There exists a partial edge-face 9-colouring c of G in which only uv and the ($\leqslant$4)-faces are uncoloured. Let α be the colour, if any, assigned to the ($\geqslant$5)-face incident to both uv and vw, and let β_{uw} be the colour, if any, assigned to the ($\geqslant$5)-face incident to uw.

By Lemma 2, observe that $\alpha \notin \{\beta_{uw}, c(uw)\}$. Since uv is tight, the sets $\mathcal{E}(u)$, $\{c(vw)\}$ and $\{\alpha\}$ are pairwise disjoint. Since vw is tight, we deduce that $\alpha \notin \mathcal{E}(w)$, for otherwise we could colour uv with $c(vw)$ and next recolour vw with a colour from $\{1, \ldots, 9\} \setminus \mathcal{E}(w)$. Hence $\alpha \notin \mathcal{E}(u) \cup \mathcal{E}(w) \cup \{\beta_{uw}\}$, so then colouring uv with $c(uw)$ and next recolouring uw with α yields a nice colouring of G; a contradiction. □

Since the configuration B1 is reducible, to demonstrate that the configuration A2 is reducible, it suffices to show the following.

Lemma 6. *Let u, v, w, x be vertices of G with* $\deg(v) = 2$ *and* $\deg(x) \leqslant 3$*. Then $uvwx$ is not a face of G.*

Proof. Suppose on the contrary that $uvwx$ is a face of G. There exists a partial edge-face 9-colouring c of G in which only uv and the ($\leqslant$4)-faces are uncoloured. Let α be the colour, if any, assigned to the ($\geqslant$5)-face incident to both uv and vw, and let β_{ux} and β_{wx} be the colours, if any, assigned to the ($\geqslant$5)-faces incident to ux and wx, respectively.

By Lemma 2, observe that $\alpha \notin \{\beta_{ux}, \beta_{wx}, c(ux), c(wx)\}$. Since uv is tight, the sets $\mathcal{E}(u)$, $\{c(vw)\}$ and $\{\alpha\}$ are pairwise disjoint. Since vw is tight, we deduce that $\alpha \notin \mathcal{E}(w)$, for otherwise we could colour uv with $c(vw)$ and next recolour vw with a colour from $\{1, \ldots, 9\} \setminus \mathcal{E}(w)$. Hence $\alpha \notin \mathcal{E}(u) \cup \mathcal{E}(w) \cup \{\beta_{ux}, \beta_{wx}\}$.

Let x' be the vertex adjacent to x distinct from u and w. We must have $c(xx') = \alpha$, otherwise we could colour uv with $c(ux)$ and next recolour ux with α. Since $\beta_{ux} \neq \beta_{wx}$, at least one of β_{ux} and β_{wx} is distinct from $c(vw)$. Observing that we can colour uv with $c(vw)$ and next uncolour vw, we may assume without loss of generality that $\beta_{wx} \neq c(vw)$. As a result, colouring uv with $c(vw)$, and next swapping the colours of vw and xw yields a nice colouring of G; a contradiction. □

The following verifies that the configurations C4 and C5 are reducible.

Lemma 7. *Let uvw and vwx be triangles of G such that wx is incident to two $(\leqslant 4)$-faces.*

(i) At least one of u and x has degree at least 4.

(ii) If uv is tight, then $\deg(v) + \deg(x) \geqslant 12$.

Proof. (i). Suppose on the contrary that both u and x have degree less than 4. Then both have degree 3 by Lemma 5. Let u' (respectively x') be the neighbour of u (respectively x) distinct from v and w. Let c be a partial edge-face 9-colouring of G in which only wx and the $(\leqslant 4)$-faces are uncoloured. Let α_{uv}, α_{uw} and α_{vx} be the colours, if any, assigned to the $(\geqslant 5)$-faces incident to uv, uw and vx, respectively.

Since the edge wx is tight, the sets $\mathcal{E}(w)$ and $\mathcal{E}(x)$ are disjoint. Hence $c(xx') \in \mathcal{E}(v)$, otherwise we could colour wx with $c(vw)$ and recolour vw with $c(xx')$.

We first assert that $\alpha_{vx} \neq c(vw)$. Otherwise, $|\mathcal{C}(vx)| = |\mathcal{E}(v)| \leqslant 8$ and there exists $\xi \in \{1, 2, \ldots, 9\} \setminus \mathcal{C}(vx)$. Now, colouring wx with $c(vx)$ and recolouring vx with ξ yields a nice colouring of G; a contradiction. Consequently, we can safely swap the colours of vw and vx, if necessary.

Our next assertion is that $\{c(uu'), \alpha_{uw}\} = \{c(vw), c(vx)\}$. For, if $c(vx) \notin \{c(uu'), \alpha_{uw}\}$, we can colour wx with $c(uw)$ and recolour uw with $c(vx)$; a contradiction. The same argument after swapping the colours of vw and vx shows that $c(vw) \in \{c(uu'), \alpha_{uw}\}$. Thus, up to swapping the colours of vw and vx, we may assume that $c(vx) = \alpha_{uw}$.

Let us recolour uv with $c(vx)$, colour wx with $c(vx)$ and uncolour vx. The obtained colouring is proper, since $\alpha_{uv} \neq \alpha_{uw} = c(vx)$ and $\mathcal{E}(w) \cap \mathcal{E}(x) = \emptyset$. Now, if vx cannot be coloured greedily, then for the obtained colouring $\mathcal{E}(v) \cup$

$\mathcal{E}(x)\cup\{\alpha_{vx}\} = \{1,2,\dots,9\}$ since vx is tight. As a result, $c(xx') \notin \mathcal{E}(v)\cup\mathcal{E}(w)$ and hence we can colour vx with $c(vw)$ and colour vw with $c(xx')$ to obtain a nice colouring of G; a contradiction.

(ii). Suppose on the contrary that uv is tight and $\deg(v) + \deg(x) = 11$. Let c be a partial edge-face 9-colouring of G in which only uv and the ($\leqslant$4)-faces are uncoloured. Let α_{uv}, α_{uw} and α_{vx} be the colours, if any, assigned to the ($\geqslant$5)-faces incident to uv, uw and vx, respectively. Since the edge uv is tight, the sets $\mathcal{E}(u)$, $\mathcal{E}(v)$ and $\{\alpha_{uv}\}$ are pairwise disjoint.

Let $\xi \in \{1,\dots,9\} \setminus \mathcal{E}(w)$. Then $\xi \in \mathcal{E}(v)$, otherwise we could colour uv with $c(vw)$ and recolour vw with ξ. It follows that $\xi \notin \mathcal{E}(u)$. Therefore, $\alpha_{uw} = \xi$, otherwise we could colour uv with $c(uw)$ and recolour uw with ξ. Thus, the colours of uw and vw may be exchanged, if necessary.

Let us show that $\mathcal{E}(u) \cup \{\alpha_{uv}, c(vw)\} \subseteq \mathcal{E}(x) \cup \{\alpha_{vx}\}$. First, if there is a colour $\gamma \in \mathcal{E}(u) \cup \{\alpha_{uv}\}$ that is not in $\mathcal{E}(x) \cup \{\alpha_{vx}\}$, then we can recolour vx with γ and then colour uv with $c(vx)$ to obtain a nice colouring of G, which is a contradiction. Similarly, by exchanging the colours of uw and vw, we conclude that $c(vw) \in \mathcal{E}(x) \cup \{\alpha_{vx}\}$.

Since uv is tight and $\deg(v)+\deg(x) = 11$, we deduce that $\mathcal{E}(x)\cup\{\alpha_{vx}\} = \mathcal{E}(u)\cup\{\alpha_{uv}, c(vw), c(vx)\}$. (Indeed, $|\mathcal{E}(x)\cup\{\alpha_{vx}\}| \leqslant \deg(x)+1 = 12-\deg(v)$, and $|\mathcal{E}(u) \cup \{\alpha_{uv}\}| = 9 - (\deg(v) - 1) = 10 - \deg(v)$.) In particular, $\alpha_{vx} \neq c(wx)$ and $\xi \notin \mathcal{E}(x) \setminus \{c(vx)\}$. Now, colour uv with $c(vx)$, and then recolour vx with $c(wx)$, wx with ξ to obtain a nice colouring of G; a contradiction. □

The next lemma implies that the configurations D1–D4 are reducible.

Lemma 8. *Let vwx be a triangle of G and u a neighbour of v distinct from x and w. If vx is incident to two ($\leqslant$4)-faces, then not both uv and vw are tight.*

Proof. Suppose on the contrary that both uv and vw are tight. Let c be a partial edge-face 9-colouring of G in which only vw and the ($\leqslant$4)-faces are left uncoloured. Let α be the colour, if any, assigned to the ($\geqslant$5)-face incident to vw. Since vw is tight, we know that the sets $\mathcal{E}(v)$, $\mathcal{E}(w)$ and $\{\alpha\}$ form a partition of $\{1,2,\dots,9\}$. In particular, $c(vx) \notin \mathcal{E}(w)$ and $c(wx) \notin \mathcal{E}(v)$.

If an edge e that is adjacent to vw could be properly recoloured with a colour ξ, then colouring vw with $c(e)$ and recolouring e with ξ would yield a nice colouring of G; a contradiction. Applying this to vx yields that $\mathcal{E}(w) \cup \{\alpha\} \subseteq \mathcal{E}(x)$, since $\mathcal{C}(vx) = \mathcal{E}(x) \cup \mathcal{E}(v)$, and as we noted above $\{1,\dots,9\} \setminus \mathcal{E}(v) = \mathcal{E}(w) \cup \{\alpha\}$. Applying the same remark to wx, we obtain $\mathcal{E}(v) \cup \{\alpha\} \subseteq \mathcal{E}(x) \cup \{\beta\}$, where β is the colour, if any, assigned to the ($\geqslant$5)-face incident to wx.

Since $9 = |\mathcal{E}(v) \cup \mathcal{E}(w) \cup \{\alpha\}| \leqslant |\mathcal{E}(x) \cup \{\beta\}| \leqslant 9$, we deduce that $\beta \notin \mathcal{E}(x)$. Therefore, we can safely swap the colours of vx and wx if needed (recalling that $\mathcal{E}(v) \cap \mathcal{E}(w) = \emptyset$).

Let S be the set of colours of the ($\geqslant$5)-faces incident to uv. Thus, $|S| = 2 - s$ where s is the number of ($\leqslant$4)-faces incident to uv. Again, we apply the same arguments as above to uv: since uv cannot be recoloured, we deduce that $\mathcal{E}(u) \cup \mathcal{E}(v) \cup S = \{1, 2, \ldots, 9\}$. But $|\mathcal{E}(u) \cup \mathcal{E}(v) \cup S| \leqslant \deg(u) - 1 + \deg(v) - 1 + 2 - s = \deg(u) + \deg(v) - s \leqslant 9$ since uv is tight and vw is uncoloured. Consequently, $\mathcal{E}(u)$, $\mathcal{E}(v)$ and S are pairwise disjoint. In particular, $c(vx) \notin \mathcal{E}(u) \cup S$. As a result, colouring vw with $c(uv)$, then recolouring uv with $c(vx)$ and finally swapping the colours of vx and wx yields a nice colouring of G; a contradiction. $\square$

The next lemma implies that configurations E1 and E2 are reducible.

Lemma 9. *Let v be an 8-vertex of G with neighbours $v_0, v_1, \ldots, v_7$ in anti-clockwise order. Assume that $v_i v_{i+1}$ is an edge for $i \in \{0, 1, 2, 3\}$, and that v_1 an ($\leqslant$4)-vertex. If v_0 is an ($\leqslant$6)-vertex or vv_0 is adjacent to two ($\leqslant$4)-faces, then v_3 is an ($\geqslant$4)-vertex.*

Proof. Suppose on the contrary that v_3 is a 3-vertex. By the minimality of G, the graph $G - vv_3$ has a nice colouring and hence G has a partial edge-face 9-colouring c in which only vv_3 and the ($\leqslant$4)-faces are left uncoloured. Since vv_3 is tight, we deduce that $|\mathcal{E}(v) \cup \mathcal{E}(v_3)| = 9$ and $\mathcal{E}(v) \cap \mathcal{E}(v_3) = \emptyset$.

Let α be the colour, if any, of the ($\geqslant$5)-face incident with both v_2v_3 and v_3v_4. If v_2v_3 can be recoloured with a colour ξ, then colouring vv_3 with $c(v_2v_3)$ and then v_2v_3 with ξ would yield a nice colouring of G; a contradiction. Thus, $\mathcal{E}(v) \subseteq \mathcal{E}(v_2) \cup \{\alpha\}$.

Let $j \in \{1, 2\}$. If there exists a colour $\xi \in \mathcal{E}(v_3) \setminus \mathcal{E}(v_j)$, then colouring vv_3 with $c(vv_j)$ and then vv_j with ξ yields a nice colouring of G (recalling that $\mathcal{E}(v_3)$ and $\mathcal{E}(v)$ are disjoint). Therefore, $\mathcal{E}(v_3) \subseteq \mathcal{E}(v_j)$ for $j \in \{1, 2\}$. Letting γ be the colour, if any, of the ($\geqslant$5)-face incident to vv_0 we similarly find that $\mathcal{E}(v_3) \subseteq \mathcal{E}(v_0) \cup \{\gamma\}$.

Since $\mathcal{E}(v_2) \cup \{\alpha\} \supseteq \mathcal{E}(v) \cup \mathcal{E}(v_3) = \{1, 2, \ldots, 9\}$ and $|\mathcal{E}(v_2) \cup \{\alpha\}| \leqslant 9$, it follows that $\alpha \neq c(vv_2)$. As $\mathcal{E}(v) \cap \mathcal{E}(v_3) = \emptyset$, this implies that the colours of vv_2 and v_2v_3 can be freely swapped. By doing so, we can conclude that $\mathcal{E}(v_3) \cup \{c(vv_2)\} \subseteq \mathcal{E}(v_j)$ for $j \in \{1, 2\}$ and $\mathcal{E}(v_3) \cup \{c(vv_2)\} \subseteq \mathcal{E}(v_0) \cup \{\gamma\}$.

Since $\deg(v_1) = 4$, we find that $\mathcal{E}(v_1) = \{c(vv_1), c(vv_2)\} \cup \mathcal{E}(v_3)$. Furthermore, by swapping the colours of vv_2 and v_2v_3 if necessary, we may assume that $c(v_0v_1) \in \mathcal{E}(v_3)$. Now, if v_0v_1 could be recoloured with a colour ξ, then colouring vv_3 with $c(vv_1)$, then vv_1 with $c(v_0v_1)$ and then v_0v_1 with ξ would yield a nice colouring of G. Thus, letting β be the colour, if any, of the ($\geqslant$5)-face incident to v_0v_1 we obtain $\mathcal{E}(v_0) \cup \mathcal{E}(v_1) \cup \{\beta\} = \{1, 2, \ldots, 9\}$.

Let us partition our analysis now based on if v_0 is an ($\leqslant$6)-vertex or if vv_0 is adjacent to two ($\leqslant$4)-faces.

Suppose we are in the former case. Since $\mathcal{E}(v_3) \cup \{c(vv_2)\} \subseteq (\mathcal{E}(v_0) \cup \{\gamma\}) \cap \mathcal{E}(v_1)$, we deduce that $|\mathcal{E}(v_0) \cup \mathcal{E}(v_1)| \leqslant \deg(v_0) + \deg(v_1) - 2 \leqslant 8$. Consequently, $\beta \neq c(vv_1)$ and $c(vv_1) \notin \mathcal{E}(v_0)$. In particular, the colours of vv_1 and v_0v_1 can safely be swapped if needed. As a result, colouring vv_3 with $c(vv_1)$ and then swapping the colours of vv_1 and v_0v_1 yields a nice colouring of G; a contradiction.

Now suppose we are in the latter case. Then there is no colour γ. For $j \in \{0, 1\}$, it cannot be that $c(vv_j) \in \mathcal{E}(v_{1-j})$ (and hence $c(vv_j) \in \mathcal{E}(v_0) \cap \mathcal{E}(v_1)$). Otherwise, we would have, using $\mathcal{E}(v_3) \cup \{c(vv_2)\} \subseteq \mathcal{E}(v_0) \cap \mathcal{E}(v_1)$, that $|\mathcal{E}(v_0) \cup \mathcal{E}(v_1)| \leqslant \deg(v_0) + \deg(v_1) - 4 \leqslant 8$, in which case, recolouring as we did in the last paragraph, we would reach a contradiction. However, for some $j \in \{0, 1\}$, we must have $\beta \neq c(vv_j)$, and so the colours of vv_j and v_0v_1 can be swapped safely. Thus, colouring vv_3 with $c(vv_j)$ and then swapping the colours of vv_j and v_0v_1 yields a nice colouring of G; a contradiction. $\square$

In the following lemma, we show that the configuration E3 is reducible.

Lemma 10. *Let v be a triangulated 8-vertex of G with neighbours $v_0, v_1, \ldots, v_7$ in anti-clockwise order. If v_0 is a 3-vertex, then every vertex v_i with $i \neq 0$ has degree at least 5.*

Proof. Suppose on the contrary that v_j is an ($\leqslant$4)-vertex with $j \in \{1, \ldots, 7\}$. First, note that $j \notin \{1, 7\}$ since the configuration B2 is reducible. By the minimality of G, the graph $G - vv_0$ has a nice colouring, and hence the graph G has a partial edge-face 9-colouring in which only vv_0 and the ($\leqslant$4)-faces are left uncoloured. Since vv_0 is tight and incident to two triangles, we infer that $|\mathcal{E}(v) \cup \mathcal{E}(v_0)| = 9$ and $\mathcal{E}(v) \cap \mathcal{E}(v_0) = \emptyset$.

Note that $\mathcal{E}(v_0) \subset \mathcal{E}(v_i)$ for $i \neq 0$, for otherwise we could colour vv_0 with $c(vv_i)$ and then recolour vv_i with a colour in $\mathcal{E}(v_0) \setminus \mathcal{E}(v_i)$ to obtain a nice colouring of G (recalling that $\mathcal{E}(v) \cap \mathcal{E}(v_0) = \emptyset$). Since $\{c(vv_j)\} \cup \mathcal{E}(v_0) \subseteq \mathcal{E}(v_j)$ and $\deg(v_j) \leqslant 4$, we deduce that one of $c(vv_1)$ and $c(vv_7)$ does not belong to $\mathcal{E}(v_j)$, say $c(vv_7)$.

Let α be the colour of the face incident to both v_0v_1 and v_0v_7. We prove that $\alpha \neq c(vv_7)$. Indeed, suppose on the contrary that $\alpha = c(vv_7)$. Then, there exists a colour ξ that does not belong to $\mathcal{E}(v_7) \cup \{\alpha\} = \mathcal{E}(v_7)$, since $\deg(v_7) \leqslant 8$. As $\mathcal{E}(v_0) \subset \mathcal{E}(v_7)$, we deduce that $\xi \notin \mathcal{E}(v_0) \cup \mathcal{E}(v_7) \cup \{\alpha\}$. Therefore, colouring vv_0 with $c(v_0v_7)$ and then v_0v_7 with ξ yields a nice colouring of G; a contradiction. Hence, $\alpha \neq c(vv_7)$. Consequently, we can freely swap the colours of vv_7 and v_0v_7. Now, colouring vv_0 with $c(vv_j)$, then recolouring vv_j with $c(vv_7)$ and last swapping the colours of vv_7 and v_0v_7 yields a nice colouring of G; a contradiction. □

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