Hamiltonian Cycles in the Square of a Graph

Jan Ekstein
ekstein@kma.zcu.cz
Department of Mathematics
University of West Bohemia
Univerzitní 8
306 14 Plzeň
Czech Republic

September 2, 2010

Abstract

We show that under certain conditions the square of the graph obtained by identifying a vertex in two graphs with hamiltonian square is also hamiltonian. Using this result, we prove necessary and sufficient conditions for hamiltonicity of the square of a connected graph such that every vertex of degree at least three in a block graph corresponds to a cut vertex and any two these vertices are at distance at least four.

Keywords: hamiltonian cycle; connection of graphs; block graph; square; star

1 Introduction and notation

The graphs considered in this paper are undirected and simple. If G is a graph, we denote by $V(G)$ the vertex set of G, by $E(G)$ the edge set of G. For $x \in V(G)$, $d_G(x)$ denotes the *degree* of x and $N_G(x)$ denotes the *neighborhood* of x. For $x, y \in V(G)$, $\mathrm{dist}_G(x, y)$ denotes the *distance* between x, y. For $A \subseteq V(G)$, $\langle A \rangle$ denotes the subgraph of G induced by A.

The *k-star* is a tree on $k+1$ vertices with one vertex of degree k, called the center, and the others of degree 1, $k = 0, 1, 2, \ldots$. The graph $S(K_{1,3})$ is the graph $K_{1,3}$ in which each edge is subdivided once. Given sets A, B of vertices, we call $P = x_0, ..., x_k$ an *(A, B)-path* if $V(P) \cap A = \{x_0\}$ and $V(P) \cap B = \{x_k\}$, we write (a, B)-path rather than $(\{a\}, B)$-path. For a graph G we define $V_i(G) = \{v \in V(G) : d(v) = i\}$ and $W(G) = V(G) \setminus V_2(G)$. A *branch* in G is a nontrivial path whose ends are in $W(G)$ and whose internal vertices, if any, are of degree 2 in G.

The *square* of G, denoted G^2, is the graph with the vertex set $V(G)$ in which two vertices are adjacent if their distance in G is one or two. We say that two graphs are *homeomorphic* if they can be turned into isomorphic graphs by finite number of edge-subdivisions. Let G' be an subgraph of G. We say that G' is *maximal* with respect to a given graph property if G' itself has the property but no graph $G' + A$ does, for any nonempty subset $A \subseteq E(G) \setminus E(G')$.

A connected graph that has no cut vertices is called a *block*. A *block* of a graph is a subgraph that is a block and is maximal with respect to this property. The *degree of a block B* of a graph G, denoted by $d(B)$, is the number of cut vertices of G belonging to $V(B)$. A block of degree 1 is called an *endblock* of G, otherwise it is a *non-end block*. A block is said to be *acyclic* if it is isomorphic to one edge, otherwise we say it is *cyclic*. The *block graph* of a graph G is the graph $\mathrm{Bl}(G)$ such that the vertices of $\mathrm{Bl}(G)$ are the blocks and cut vertices of G, and two vertices are adjacent in $\mathrm{Bl}(G)$ if one of them is a block of G and the second one is its vertex.

Let G_1, G_2 be connected graphs, $x \notin V(G_1) \bigcup V(G_2)$, $V(G_1) \bigcap V(G_2) = \emptyset$, and let $x_i \in V(G_i)$, $i = 1, 2$. Then the graph G with vertex set $V(G) = (V(G_1) \setminus \{x_1\}) \bigcup (V(G_2) \setminus \{x_2\}) \bigcup \{x\}$ and with edge set $E(G) = E(G_1 - x_1) \bigcup E(G_2 - x_2) \bigcup \{ux|\ u \in V(G_1), ux_1 \in E(G_1)\} \bigcup \{vx|\ v \in V(G_2), vx_2 \in E(G_2)\}$ is called the *connection of the graphs G_1, G_2 over the vertices x_1, x_2*, denoted $G = G_1[x_1 = x_2]G_2$.

Let G be a connected graph such that G^2 is hamiltonian and let $x \in V(G)$. We say that

a) the vertex x is *of type 1* if there exists a hamiltonian cycle C of G^2 such that both edges of C incident with x are in G,

b) the vertex x is *of type 2* if x is not of type 1 and there exists a hamiltonian cycle C of G^2 such that exactly one edge of C incident with x is in G,

c) the vertex x is *of type 3* if x is not of type 1 or 2 and there exists a hamiltonian cycle C of G^2 such that for some two vertices $u, v \in$

$N_G(x)$ is $uv \in E(C)$,

d) the vertex x is *of type 4* if x is not of type 1 or 2 or 3.

We denote $V_{[i]}(G) = \{x \in V(G) \mid x \text{ is of type } i\}, i = 1, 2, 3, 4$.

2 The connection of graphs

Let us first mention the following result by Fleischner [2] that will be used many times in proofs.

Theorem 2.1 [2]. *Let y and z be arbitrarily chosen vertices of a 2-connected graph G. Then G^2 contains a hamiltonian cycle C such that the edges of C incident with y are in G and at least one of edges of C incident with z is in G. If y and z are adjacent in G, then these are three different edges.*

It is easy to see that Theorem 2.1 implies that the square of a 2-connected graph is hamiltonian.

The following result shows that, under certain conditions, the square of the connection of two graphs with hamiltonian square is also hamiltonian.

Theorem 2.2. *Let G_1, G_2 be connected graphs such that $(G_1)^2$, $(G_2)^2$ are hamiltonian, let $x_i \in V(G_i)$, $i = 1, 2$. If*

I) $G = G_1[x_1 = x_2]G_2$ and $x_i \in V_{[1]}(G_i) \cup V_{[2]}(G_i)$, $i = 1, 2$, or

II) $G = G_1[x_1 = x_2]K_2$, $x_1 \in V_{[1]}(G_1) \cup V_{[2]}(G_1) \cup V_{[3]}(G_1)$ and $V(K_2) = \{x_2, u\}$ or

III) $G = G_1[x_1 = x_2]G_2$, $x_1 \in V_{[3]}(G_1)$ and $x_2 \in V_{[1]}(G_2)$,

then G^2 is hamiltonian.

Moreover under the assumptions of I),

a) if $x_i \in V_{[1]}(G_i)$, $i = 1, 2$, then $x = x_1 = x_2 \in V_{[1]}(G)$;

b) if $x_1 \in V_{[1]}(G_1)$ and $x_2 \in V_{[2]}(G_2)$, then $x = x_1 = x_2 \in V_{[2]}(G)$;

c) if G_2 is 2-connected and $x_1 \in V_{[1]}(G_1) \cup V_{[2]}(G_1)$, then $v \in V_{[1]}(G)$ for any $v \in V(G_2)$, $v \neq x_2$;

d) if $x_i \in V_{[2]}(G_i)$, $i = 1, 2$, then $x = x_1 = x_2 \notin V_{[1]}(G) \cup V_{[2]}(G)$.

Moreover under the assumptions of II),

a) if $x_1 \in V_{[1]}(G_1)$, then $x = x_1 = x_2 \in V_{[1]}(G)$;

b) if $x_1 \in V_{[2]}(G_1)$, then $x = x_1 = x_2 \in V_{[2]}(G)$;

c) if $x_1 \in V_{[1]}(G_1) \cup V_{[2]}(G_1)$, then $u \in V_{[2]}(G)$.

Proof.

I) Let $x = x_1 = x_2$ and let C_1, C_2 be hamiltonian cycles in $(G_1)^2$, $(G_2)^2$ such that $a_1x, b_1x \in E(G)$ for $a_1 \in N_{C_1}(x)$, $b_1 \in N_{C_2}(x)$, respectively. Let $a_2 \in N_{C_1}(x)$, $a_1 \neq a_2$, and $b_2 \in N_{C_2}(x)$, $b_1 \neq b_2$, and let $P_{a_1a_2} = C_1 - x_1$ and $P_{b_2b_1} = C_2 - x_2$. Then $P_{a_1a_2}$, $P_{b_2b_1}$ are hamiltonian paths in $(G_1 - x)^2$, $(G_2 - x)^2$, respectively, and the cycle $C = a_1P_{a_1a_2}a_2xb_2P_{b_2b_1}b_1a_1$ is a hamiltonian cycle in G^2.

a) If moreover $x_i \in V_{[1]}(G_i)$, $i = 1, 2$, then we can assume that $a_2x, b_2x \in E(G)$ and therefore $x \in V_{[1]}(G)$.

b) If moreover $x_1 \in V_{[1]}(G_1)$ and $x_2 \in V_{[2]}(G_2)$, then we can assume that $a_2x \in E(G)$ and it is obvious that there is no C_2 such that $b_2x \in E(G)$, therefore $x \in V_{[2]}(G)$.

c) If moreover G_2 is 2-connected, then for any $v \in V(G_2)$, $v \neq x_2$, by Theorem 2.1 we can assume without loss of generality that $c_1v, c_2v \in E(G_2)$, $c_1, c_2 \in N_{C_2}(v)$, $c_1 \neq c_2$. If $v \neq b_1$, then $c_1v, c_2v \in E(C)$, therefore $v \in V_{[1]}(G)$. If $v = b_1$, then we have $vx_2 \in V(G_2)$ and by Theorem 2.1 $x_2b_2 \in E(G)$. Then $\widetilde{C} = a_1P_{a_1a_2}a_2xb_1P_{b_1b_2}b_2a_1$ is also a hamiltonian cycle in G^2 and moreover the edges of C_2 incident with $v = b_1$ are in $\widetilde{C}$. Therefore $v \in V_{[1]}(G)$.

d) If moreover $x_i \in V_{[2]}(G_i)$, $i = 1, 2$, then it is obvious that there are no C_1, C_2 such that $a_2x \in E(G)$ or $b_2x \in E(G)$ and therefore $x \notin V_{[1]}(G) \bigcup V_{[2]}(G)$.

II) *Case 1:* $x_1 \in V_{[3]}(G_1)$.

Let $x = x_1 = x_2$ and let C_1 be a hamiltonian cycle in $(G_1)^2$ such that $yw \in E(C_1)$ for some $y, w \in N_{G_1}(x)$. Let $P_{yw} = C_1 - yw$. Then P_{yw} is a hamiltonian path in $(G_1)^2$ and the cycle $C = yP_{yw}wuy$ is a hamiltonian cycle in G^2.

Case 2: $x_1 \in V_{[1]}(G_1) \bigcup V_{[2]}(G_1)$.

Let $x = x_1 = x_2$ and let C_1 be a hamiltonian cycle in $(G_1)^2$ such that $yx \in E(G)$ for $y \in N_{C_1}(x)$. Let $z \in N_{C_1}(x)$, $z \neq y$, and let $P_{zy} = C_1 - x_1$. Then P_{yz} is a hamiltonian path in $(G_1 - x)^2$ and the cycle $C = zP_{zy}yuxz$ is a hamiltonian cycle in G^2.

a) If moreover $x_1 \in V_{[1]}(G_1)$, then we can assume that $xz \in E(G)$ and therefore $x \in V_{[1]}(G)$.

b) If moreover $x_1 \in V_{[2]}(G_1)$, then it is obvious that there is no C_1 such that $xz \in E(G)$ and therefore $x \in V_{[2]}(G)$.

c) Since $ux \in E(G)$ and $N_G(u) = \{x\}$, it is obvious that there is no cycle $\widetilde{C}$ in G^2 such that both edges of $\widetilde{C}$ incident with u are in G and therefore $u \in V_{[2]}(G)$.

III) Let $x = x_1 = x_2$, let C_1 be a hamiltonian cycle in $(G_1)^2$ such that $yw \in E(C_1)$ for some $y, w \in N_{G_1}(x)$ and let C_2 be a hamiltonian cycle in $(G_2)^2$ such that $ax, bx \in E(G)$ for $a, b \in N_{C_2}(x)$, $a \neq b$. Let $P_{yw} = C_1 - yw$ and $P_{ab} = C_2 - x_2$. Then P_{yw}, P_{ab} are hamiltonian paths in $(G_1)^2$, $(G_2 - x)^2$, respectively, and the cycle $C = wP_{wy}yaP_{ab}bw$ is a hamiltonian cycle in G^2.

■

3 The hamiltonian square of a graph

This work is motivated by the following result due to El Kadi Abderrezzak, Flandrin and Ryjáček [1].

Theorem 3.1 [1]. *If G is a connected graph such that every induced $S(K_{1,3})$ has at least three edges in a block of degree at most 2, then G^2 is hamiltonian.*

The following result, originally by Thomassen [5], is an immediate corollary of Theorem 3.1.

Theorem 3.2 [5]. *If the block graph of G is a path, then G^2 is hamiltonian.*

We consider the graph in Figure 1. It is easy to see that the cycle $C = v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_1$ is a hamiltonian cycle in G^2 but the induced subgraph $H = \langle\{v_1, v_2, v_3, v_5, v_6, v_9, v_{10}\}\rangle$ is isomorphic to $S(K_{1,3})$ and does not have at least three edges in a block of degree at most 2. This example shows that the assumptions in Theorem 3.1 are sufficient but not necessary. We looked for other conditions implying that the square of a graph is hamiltonian.

Before the presentation of our main results we first give the following slight strengthening of Theorem 3.2 which will be needed in our proofs.

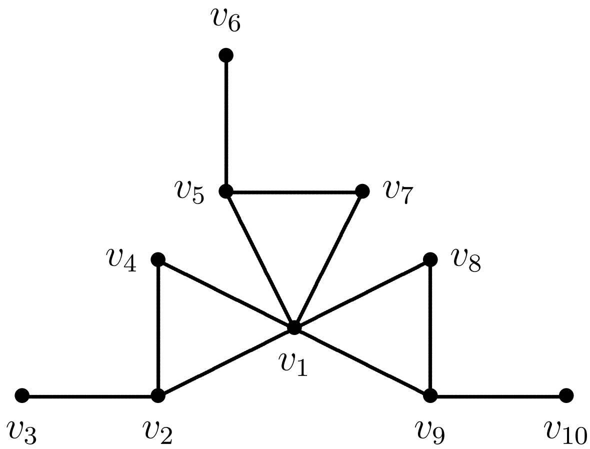

Figure 1:

Theorem 3.3. *Let G be a graph such that its block graph is a path and let u_1, u_2 be arbitrary vertices which are not cut vertices and are contained in different endblocks of G.*
Then G^2 contains a hamiltonian cycle C such that, for $i = 1, 2$,

- *if u_i is contained in a cyclic block, then both edges of C incident with u_i are in G, and*
- *if u_i is contained in an acyclic block, then exactly one edge of C incident with u_i is in G.*

Proof. If G is a path of length at least 2, then the theorem is obvious. Thus, suppose that G contains at least one cyclic block B_1 and let k denote the number of blocks of G.

We prove the theorem by induction on k.

1. Let $k = 2$, let B_2 be the second block of G, let $x = V(G)$ be the (only) cut vertex of G and let u_1, u_2 be arbitrary vertices such that $u_1 \in V(B_1)$, $u_2 \in V(B_2)$ and $u_1 \neq x$, $u_2 \neq x$. The graph $(B_1)^2$ contains a hamiltonian cycle C_1 such that the edges of C_1 incident with u_1 are in B_1 and at least one of edges of C_1 incident with x is in B_1. If u_1 and x are adjacent in G, then these are three different edges by Theorem 2.1. Then we can assume that $x \in V_{[1]}(B_1) \bigcup V_{[2]}(B_1)$ and $G = B_1[x_1 = x_2]B_2$, where x_1 and x_2 is the copy of x in B_1 and B_2, respectively.

a) If B_2 is cyclic, then the graph G^2 contains a hamiltonian cycle C such that both edges of C incident with u_2 are in G by Theorem 2.2 Ic) and it is obvious that we can find C such that also both edges of C incident with u_1 are in G.

b) If $B_2 = K_2 = x_2u_2$, then the graph G^2 contains a hamiltonian cycle C such that exactly one edge of C incident with u_2 is in G by Theorem 2.2 IIc) and it is obvious that we can find C such that both edges of C incident with u_1 are in G.

2. Suppose the assertion is true for each graph with at most k blocks, let G be a graph with $k+1$ blocks such that its block graph is a path and let u_1, u_2 be arbitrary vertices which are not cut vertices and are contained in different endblocks of G, $k \geq 2$.

Let B_{k+1} be the endblock of G containing u_2. We denote $\widetilde{G} = G - V(B_{k+1} - x)$, where $x \in V(B_{k+1})$ is a cut vertex of G. Then $G = \widetilde{G}[x_1 = x_2]B_{k+1}$, where x_1 and x_2 is the copy of x in $\widetilde{G}$ and B_{k+1}, respectively, and we can assume by the induction hypothesis that $\widetilde{G}$ contains a hamiltonian cycle C_1 such that if u_1, x_1 is contained in a cyclic block, then both edges of C_1 incident with u_1, x_1 are in $\widetilde{G}$, and if u_1, x_1 is contained in an acyclic block, then exactly one edge of C_1 incident with u_1, x_1 is in $\widetilde{G}$, respectively. Then we can assume that $x_1 \in V_{[1]}(\widetilde{G}) \bigcup V_{[2]}(\widetilde{G})$.

a) If B_{k+1} is cyclic, $u_2 \in V(B_{k+1})$ and $u_2 \neq x_2$, then the graph G^2 contains a hamiltonian cycle C such that both edges of C incident with u_2 are in G by Theorem 2.2 Ic) and it is obvious that we can find C such that if u_1 is contained in a cyclic block, then both edges of C incident with u_1 are in G, and if u_1 is contained in an acyclic block, then exactly one edge of C incident with u_1 is in G.

b) If $B_{k+1} = K_2 = x_2u_2$, then the graph G^2 contains a hamiltonian cycle C such that exactly one edge of C incident with u_2 is in G by Theorem 2.2 IIc) and it is obvious that we can find C such that if u_1 is contained in a cyclic block, then both edges of C incident with u_1 are in G, and if u_1 is contained in an acyclic block, then exactly one edge of C incident with u_1 is in G.

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4 Main result

Let $V_{\geq 3}(G) = \{x \in V(G) | d_G(x) \geq 3\}$ and, for $x \in V(G)$, $t_G(x)$ denotes the number of acyclic non-end blocks of G containing x. First of all we prove the following lemma.

Lemma 4.1. *Let G be a connected graph with exactly one vertex in $V_{\geq 3}(Bl(G))$ corresponding to a cut vertex a of G. If a is contained in at most two acyclic non-end blocks of G, then G^2 contains a hamiltonian cycle C such that if $t_G(a) = 0$, then both edges of C incident with a are in G, if $t_G(a) = 1$, then exactly one edge of C incident with a is in G, if $t_G(a) = 2$, then no edge of C incident with a is in G.*

Proof. Let $r \geq 0$, $s \geq 0$ and $t = t_G(a) \geq 0$ denote the number of cyclic blocks, acyclic endblocks and acyclic non-end blocks of G containing a, respectively, and choose the notation such that if $r > 0$, then $B_1, ..., B_r$ are all cyclic blocks, if $s > 0$, then $B_{r+1}, ..., B_{r+s}$ are all acyclic endblocks, and if $t > 0$, then $B_{r+s+1}, ..., B_{r+s+t}$ are all acyclic non-end blocks of G containing the vertex a.

By the assumption, $t \leq 2$ and $r + s + t \geq 3$, hence $r + s > 0$.

Case 1: $r = 0$.
If $t = 0$, then G is a star and the assertion is obvious. Let $t \geq 1$.

Subcase 1.1: $s = 1$. Then necessarily $t = 2$. Let $B_1 = au$ and let b_2, b_3 be the branch of $\mathrm{Bl}(G)$ containing the vertex corresponding to B_2, B_3 and denote H_2, H_3 the subgraph corresponding to b_2, b_3, respectively. For $i = 2, 3$, $\mathrm{Bl}(H_i)$ is a path and therefore $(H_i)^2$ is hamiltonian and $a \in V_{[2]}(H_i)$ by Theorem 3.3. If $G_1 = H_2[x_1 = x_2]B_1$, where x_1 and x_2 is the copy of a in H_2 and B_1, respectively, then $(G_1)^2$ is hamiltonian and $a \in V_{[2]}(G_1)$ by Theorem 2.2 IIb). Moreover $G = G_1[y_1 = y_2]H_3$, where y_1 and y_2 is the copy of a in G_1 and H_3, respectively, and G^2 contains hamiltonian cycle C such that no edge of C incident with a is in G by Theorem 2.2 Id).

Subcase 1.2: $s \geq 2$. Then necessarily $1 \leq t \leq 2$. For $i = s + 1, s + 2$, let H_i be the same subgraphs as in Subcase 1.1 and let $\widetilde{G} = G - V(H_{s+1} - a) - V(H_{s+2} - a)$. It is obvious that $\widetilde{G}$ is a star and therefore $(\widetilde{G})^2$ is hamiltonian and $a \in V_{[1]}(\widetilde{G})$. If $t = 1$, then $G = \widetilde{G}[x_1 = x_2]H_{s+1}$, where x_1 and x_2 is the copy of a in $\widetilde{G}$ and H_{s+1}, respectively, and G^2 contains hamiltonian cycle C such that exactly one edge of C incident with a is in G by Theorem 2.2 Ib). Let $t = 2$. If $G_1 = \widetilde{G}[x_1 = x_2]H_{s+1}$, where x_1 and x_2 is the copy of a in $\widetilde{G}$ and H_{s+1}, respectively, then $(G_1)^2$ is hamiltonian and $a \in V_{[2]}(G_1)$ by Theorem 2.2 Ib). Moreover $G = G_1[y_1 = y_2]H_{s+2}$, where y_1 and y_2 is the copy of a in G_1 and H_{s+2}, respectively, and G^2 contains hamiltonian cycle C such that no edge of C incident with a is in G by Theorem 2.2 Id).

Case 2: $r \geq 1$.
For $i = 1, 2, ..., r$, let b_i be the branch of $\mathrm{Bl}(G)$ containing the vertex corresponding to B_i. We denote H_i the subgraph corresponding to b_i. The block graph $\mathrm{Bl}(H_i)$ is a path and therefore $(H_i)^2$ is hamiltonian and $a \in V_{[1]}(H_i)$ either by Theorem 3.3 or by Theorem 2.1. Let b_{r+s+1}, b_{r+s+2} be the branch of $\mathrm{Bl}(G)$ containing the vertex corresponding to B_{r+s+1}, B_{r+s+2} and denote H_{r+s+1}, H_{r+s+2} the subgraph corresponding to b_{r+s+1}, b_{r+s+2}, respectively. For $j = r+s+1, r+s+2$, $\mathrm{Bl}(H_j)$ is a path and therefore $(H_j)^2$ is hamiltonian and $a \in V_{[2]}(H_j)$ by Theorem 3.3.

Let $G_1 = G - V(H_{r+s+1} - a) - V(H_{r+s+2} - a)$ and let ℓ denote the number of branches of $\mathrm{Bl}(G_1)$.

We show that $(G_1)^2$ is hamiltonian and $a \in V_{[1]}(G_1)$.

We proceed by induction on ℓ.

1. For $\ell = 1$ obviously $G_1 = H_1$ and the assertion is true.

2. Suppose the assertion is true for each graph such that its block graph contains at most ℓ branches, let G_1 be a graph without acyclic non-end blocks such that its block graph contains $\ell + 1$ branches and with exactly one vertex in $V_{\geq 3}(Bl(G_1))$ corresponding to a cut vertex a of G_1.

If $\widetilde{G_1} = G_1 - V(H_1 - a)$, then $G_1 = H_1[x_1 = x_2]\widetilde{G_1}$, where x_1 and x_2 is the copy of a in H_1 and $\widetilde{G_1}$, respectively. If $\widetilde{G_1} = B_{r+1}$, then $(G_1)^2$ is hamiltonian and $a \in V_{[1]}(G_1)$ by Theorem 2.2 IIa). Otherwise $\widetilde{G_1}$ is hamiltonian and $a = x_1 \in V_{[1]}(\widetilde{G_1})$ by the induction hypothesis. Then $(G_1)^2$ is hamiltonian and $a \in V_{[1]}(G_1)$ by Theorem 2.2 Ia).

If $G = G_1$, then it is obvious that G^2 contains a hamiltonian cycle C such that both edges of C incident with a are in G. If $t = 1$, then $G = G_1[y_1 = y_2]H_{r+s+1}$, where y_1 and y_2 is the copy of a in G_1 and H_{r+s+1}, respectively, and G^2 contains hamiltonian cycle C such that exactly one edge of C incident with a is in G by Theorem 2.2 Ib). Let $t = 2$. If $G_2 = G_1[y_1 = y_2]H_{r+s+1}$, where y_1 and y_2 is the copy of a in G_1 and H_{r+s+1}, respectively, then $(G_2)^2$ is hamiltonian and $a \in V_{[2]}(G_2)$ by Theorem 2.2 Ib). Moreover $G = G_2[z_1 = z_2]H_{r+s+2}$, where z_1 and z_2 is the copy of a in G_2 and H_{r+s+2}, respectively, and G^2 contains hamiltonian cycle C such that no edge of C incident with a is in G by Theorem 2.2 Id). ■

We will now prove our main result.

Theorem 4.2. *Let G be a connected graph with at least three vertices such that*

i) every vertex $x \in V_{\geq 3}(Bl(G))$ corresponds to a cut vertex of G, and

ii) for any two vertices $x, y \in V_{\geq 3}(Bl(G))$, $dist_{Bl(G)}(x, y) \geq 4$.

Then G^2 is hamiltonian if and only if every cut vertex of G is contained in at most two acyclic non-end blocks of G.

Proof. I) First suppose that we have a vertex $a \in V(G)$ contained in at least three acyclic non-end blocks of G. We show that the graph G^2 is not hamiltonian. Let, to the contrary, C be a hamiltonian cycle in G^2. For $i = 1, 2, 3$, let $B_i = a_i a$ denote three acyclic non-end blocks of G and B_{i+3} a block of G adjacent to the block B_i such that $a \notin V(B_{i+3})$. Then necessarily there is a vertex $c_i \in N_{B_{i+3}}(a_i)$ such that $c_i a \in E(C)$, for $i = 1, 2, 3$. From this it follows that $\deg_C a \geq 3$, contradicting the fact that C is a cycle.

II) Now suppose that every cut vertex of G is contained in at most two acyclic non-end blocks of G. We show that G^2 is hamiltonian.

If G is a cyclic block, then G^2 is hamiltonian by Theorem 2.1, and if $\mathrm{Bl}(G)$ is a path, then G^2 is hamiltonian by Theorem 3.2.

Now suppose that $\mathrm{Bl}(G)$ contains at least one vertex of degree at least three corresponding to a cut vertex of G. For $i = 1, 2, ..., k$, let b_i be a vertex of $\mathrm{Bl}(G)$ in $V_{\geq 3}(\mathrm{Bl}(G))$, let a_i be the vertex of G corresponding to b_i and choose the notation such that $\mathrm{dist}_{Bl(G)}(b_1, b_k)$ is maximum and the (unique) path in $\mathrm{Bl}(G)$ joining b_1 and b_2 has no interior vertices in $V_{\geq 3}(\mathrm{Bl}(G))$. Let $t_G(a_i) \geq 0$ denote the number of acyclic non-end blocks of G containing a_i.

We prove the following statement.

Under the assumptions of Theorem 4.2 the graph G^2 contains a hamiltonian cycle C such that if $t_G(a_i) = 0$, then both edges of C incident with a_i are in G, if $t_G(a_i) = 1$, then exactly one edge of C incident with a_i is in G, if $t_G(a_i) = 2$, then no edge of C incident with a_i is in G, $i = 1, 2, ..., k$.

We proceed by induction on k.

For $k = 1$ the assertion is given by Lemma 4.1.

Suppose the assertion is true for each graph G' such that its block graph $\mathrm{Bl}(G')$ has at most $k-1$ vertices in $V_{\geq 3}(\mathrm{Bl}(G'))$ corresponding to cut vertices of G' and these are at distance at least four in $\mathrm{Bl}(G')$, and let G be a graph such that its block graph $\mathrm{Bl}(G)$ is a tree with k vertices in $V_{\geq 3}(\mathrm{Bl}(G))$ corresponding to cut vertices of G and any two vertices of $\mathrm{Bl}(G)$ in $V_{\geq 3}(\mathrm{Bl}(G))$ are at distance at least four in $\mathrm{Bl}(G)$, $k \geq 2$.

By the notation of a_1, let H be the unique subgraph of G corresponding to the (b_1, b_2)-path in $\mathrm{Bl}(G)$. Let $\widetilde{G} = G - V(H - \{a_1, a_2\})$. We denote the components of $\widetilde{G}$ by G_1, G_2 such that $a_1 \in V(G_1)$ and $a_2 \in V(G_2)$.

If $d_{\mathrm{Bl}(G_1)}(a_1) \geq 3$ and $d_{\mathrm{Bl}(G_2)}(a_2) \geq 3$, then, by the induction hypothesis, $(G_1)^2$, $(G_2)^2$ contains a hamiltonian cycle C_1, C_2 such that if $t_{G_1}(a_1) = 0$, $t_{G_2}(a_i) = 0$, then both edges of C_1, C_2 incident with a_1, a_i are in G_1, G_2, if $t_{G_1}(a_1) = 1$, $t_{G_2}(a_i) = 1$, then exactly one edge of C_1, C_2 incident with a_1, a_i is in G_1, G_2, if $t_{G_1}(a_1) = 2$, $t_{G_2}(a_i) = 2$, then no edge of C_1, C_2 incident with a_1, a_i is in G_1, G_2, $i = 2, 3, ..., k$, respectively.

In the case $d_{\mathrm{Bl}(G_1)}(a_1) = 2$, set $K_2 = vu$ (where $v, u \notin V(G)$), and $\widehat{G_1} = G_1[a_1 = v]K_2$. Then $(\widehat{G_1})^2$ contains a hamiltonian cycle with the required properties by the induction hypothesis and it is obvious that also $(G_1)^2$. Similarly we proceed in the case $d_{\mathrm{Bl}(G_2)}(a_2) = 2$.

Then by the assumption that any two vertices of $\mathrm{Bl}(G)$ in $V_{\geq 3}(\mathrm{Bl}(G))$ are at distance at least four in $\mathrm{Bl}(G)$ and by Theorem 3.3, the graph H^2 contains a hamiltonian cycle C_H such that, for $j = 1, 2$, if a_j is contained in a cyclic block, then both edges of C_H incident with a_j are in H, and if a_j is contained in an acyclic block, then exactly one edge of C_H incident with a_j is in H.

Case 1: $t_{G_2}(a_2) \in \{0, 1\}$.
Let $\widetilde{G_2} = G_2[x_1 = x_2]H$, where x_1 and x_2 is the copy of a_2 in G_2 and H, respectively. Then $(\widetilde{G_2})^2$ contains a hamiltonian cycle $\widetilde{C_2}$ with the required properties by Theorem 2.2 either Ia) or Ib) or Id) (using C_H and C_2). Moreover it is obvious that if a_1 is contained in a cyclic block of $\widetilde{G_2}$, then both edges of $\widetilde{C_2}$ incident with a_1 are in $\widetilde{G_2}$, and if a_1 is contained in an acyclic block of $\widetilde{G_2}$, then exactly one edge of $\widetilde{C_2}$ incident with a_1 is in $\widetilde{G_2}$.

a) If $t_{G_1}(a_1) \in \{0, 1\}$, then $G = G_1[y_1 = y_2]\widetilde{G_2}$, where y_1 and y_2 is the copy of a_1 in G_1 and $\widetilde{G_2}$, respectively, and G^2 contains a hamiltonian cycle C such that if $t_G(a_i) = 0$, then both edges of C incident with a_i are in G, if $t_G(a_i) = 1$, then exactly one edge of C incident with a_i is in G, if $t_G(a_i) = 2$, then no edge of C incident with a_i is in G, $i = 1, 2, ..., k$, by Theorem 2.2 either Ia) or Ib) or Id) (using $\widetilde{C_2}$ and C_1).

b) Let $t_{G_1}(a_1) = 2$. Let B_1 be an acyclic non-end block of G_1 containing the vertex a_1, let F be the subgraph of G_1 corresponding to the

maximal connected subgraph of $\mathrm{Bl}(G_1)$ containing the vertex corresponding to B_1 and not containing the vertex b_1.
Let $F_1 = G_1 - V(F - a_1)$. By the induction hypothesis, G_1 contains a hamiltonian cycle C_1 such that no edge of C_1 incident with a_1 is in G_1 and we can divide C_1 into hamiltonian cycles C_{1a} in F_1 and C_{1b} in F such that exactly one edge of C_{1a} incident with a_1 is in F_1 and exactly one edge of C_{1b} incident with a_1 is in F.
Necessarily both edges of $\widetilde{C_2}$ incident with a_1 are in $\widetilde{G_2}$ (otherwise $t_G(a_1) = 3$, a contradiction). Set $\widetilde{G_1} = F_1[y_1 = y_2]\widetilde{G_2}$, where y_1 and y_2 is the copy of a_1 in F_1 and $\widetilde{G_2}$, respectively. Then $(\widetilde{G_1})^2$ contains a hamiltonian cycle $\widetilde{C_1}$ with the required properties by Theorem 2.2 Ib) (using C_{1a} and $\widetilde{C_2}$).
Then $G = \widetilde{G_1}[z_1 = z_2]F$, where z_1 and z_2 is the copy of a_1 in $\widetilde{G_1}$ and F, respectively, and G^2 contains a hamiltonian cycle C such that if $t_G(a_i) = 0$, then both edges of C incident with a_i are in G, if $t_G(a_i) = 1$, then exactly one edge of C incident with a_i is in G, if $t_G(a_i) = 2$, then no edge of C incident with a_i is in G, $i = 1, 2, ..., k$, by Theorem 2.2 Id) (using $\widetilde{C_1}$ and C_{1b}).

Case 2: $t_{G_2}(a_2) = 2$.
Let B_2 be an acyclic non-end block of G_2 containing the vertex a_2, let S be the subgraph of G_2 corresponding to the maximal connected subgraph of $\mathrm{Bl}(G_2)$ containing the vertex corresponding to B_2 and not containing the vertex b_2.

Let $S_1 = G_2 - V(S - a_2)$. By the induction hypothesis, G_2 contains a hamiltonian cycle C_2 such that no edge of C_2 incident with a_2 is in G and we can divide C_2 into hamiltonian cycles C_{2a} in S_1 and C_{2b} in S such that exactly one edge of C_{2a} incident with a_2 is in S_1 and exactly one edge of C_{2b} incident with a_2 is in S.

Now necessarily both edges of C_H incident with a_2 are in H (otherwise $t_G(a_2) = 3$, a contradiction). Set $\widetilde{S_1} = S_1[x_1 = x_2]H$, where x_1 and x_2 is the copy of a_2 in S_1 and H, respectively. Then $(\widetilde{S_1})^2$ contains a hamiltonian cycle C' with the required properties by Theorem 2.2 Ib) (using C_{2a} and C_H).

Then $\widetilde{G_2} = \widetilde{S_1}[u_1 = u_2]S$, where u_1 and u_2 is the copy of a_2 in $\widetilde{S_1}$ and S, respectively, and $(\widetilde{G_2})^2$ contains a hamiltonian cycle $\widetilde{C_2}$ with the required properties by Theorem 2.2 Id) (using C' and C_{2b}). Moreover it is obvious that if a_1 is contained in a cyclic block of $\widetilde{G_2}$, then both edges of $\widetilde{C_2}$ incident with a_1 are in $\widetilde{G_2}$, and if a_1 is contained in an acyclic block of $\widetilde{G_2}$, then exactly one edge of $\widetilde{C_2}$ incident with a_1 is in $\widetilde{G_2}$. Then we continue similarly as in Subcase 1a) or 1b). ■

It is obvious that the conditions in Theorem 4.2 can be verified in polynomial time. From this it follows that the decision problem, if the square of a graph is hamiltonian, which is NP-complete in general ([3]), can be decided in polynomial time in the class of the graphs G such that every vertex $x \in V_{\geq 3}(\mathrm{Bl}(G))$ corresponds to a cut vertex of G, and for any two vertices $x, y \in V_{\geq 3}(\mathrm{Bl}(G))$, $\mathrm{dist}_{\mathrm{Bl}(G)}(x, y) \geq 4$.

The following theorems are immediate corollaries of Theorem 4.2.

Corollary 4.3. *Let G be a connected graph such that its block graph $Bl(G)$ is homeomorphic to a star in which the center corresponds to a cut vertex a of G. Then the graph G^2 is hamiltonian if and only if the vertex a is contained in at most two acyclic non-end blocks of G.*

Corollary 4.4. *If the block graph of G with at least three vertices is a star, then G^2 is hamiltonian.*

Note that the graph in Figure 1 satisfies the assumptions of Corollary 4.3. Therefore Corollary 4.3 (hence also Theorem 4.2) does not follow from Theorem 3.1.

5 A star in which the center corresponding to a block

Let G be a connected graph such that its block graph $\mathrm{Bl}(G)$ is homeomorphic to a star in which the center corresponds to a block B_c of G. If B_c is acyclic, then $\mathrm{Bl}(G)$ is a path and G^2 is hamiltonian by Theorem 3.2.

Let B_c be cyclic. Let k denote the number of cut vertices of G in $V(B_c)$, let $v_i \in V(B_c)$ be all cut vertices of G in B_c, $i = 1, 2, ..., k$. Let C be a hamiltonian cycle in $(B_c)^2$. We say that C is *acceptable in G* if there are pairwise distinct edges $v_i w_i \in E(C)$ such that $v_i w_i \in E(B_c)$, for any $i = 1, 2, ..., k$.

The following theorem gives only a sufficient condition for hamiltonicity in this class of graphs (in comparison with Theorem 4.2).

Theorem 5.1. *Let G be a connected graph such that its block graph $Bl(G)$ is homeomorphic to a star in which the center corresponds to a block B_c of G. If $(B_c)^2$ contains an acceptable cycle in G, then G^2 is hamiltonian.*

Proof. Let $v_i \in V(B_c)$ be all cut vertices of G in $V(B_c)$, $i = 1, 2, ..., k$. We prove the following slight strengthening of Theorem 5.1.

Let G be a connected graph such that its block graph $Bl(G)$ is homeomorphic to a star in which the center corresponds to a block B_c of G. If $(B_c)^2$ contains an acceptable cycle C in G, then G^2 contains a hamiltonian cycle containing all edges of C except the edges $v_i w_i$, $i = 1, 2, ..., k$.

We prove this assertion by induction on k.

1. Let $k = 0$. Then $G = B_c$ and the assertion is true.

2. Suppose the assertion is true for each graph such that the block B_c contains at most $k - 1$ cut vertices, let G be a graph such that its block graph is homeomorphic to a star in which the center corresponds to a block B_c of G and B_c contains k cut vertices of G, $k \geq 1$.

Let $d \in V(\text{Bl}(G))$ be the vertex corresponding to B_c, let G' be the subgraph of G such that $\text{Bl}(G') = \text{Bl}(G) - d$ and let H_i be the component of G' such that $v_i \in V(H_i)$, for $i = 1, 2, ..., k$. The block graph $\text{Bl}(H_i)$ is a path and therefore either $(H_i)^2$ is hamiltonian and $v_i \in V_{[1]}(H_i) \bigcup V_{[2]}(H_i)$ (either by Theorem 3.3 or by Theorem 2.1) or H_i is isomorphic to one edge.

Let $G_1 = G - V(H_1 - v_1)$ and let x_1 and x_2 denote the copy of v_1 in G_1 and H_1, respectively. Then $G = G_1[x_1 = x_2]H_1$. Let C be an acceptable cycle in G. Then C is also acceptable in G_1 and therefore $(G_1)^2$ contains a hamiltonian cycle C_1 containing all edges of C except $v_i w_i$, $i = 2, 3, ..., k$, by the induction hypothesis. The cycle C is acceptable in G and therefore $x_1 w_1 \in E(G_1)$ and $x_1 w_1 \in E(C_1)$. Then $x_1 \in V_{[1]}(G_1) \bigcup V_{[2]}(G_1)$.

Case 1: $x_2 \in V_{[1]}(H_1) \bigcup V_{[2]}(H_1)$.

Then G^2 contains a hamiltonian cycle containing all edges of C except $v_i w_i$, $i = 1, 2, ..., k$, by Theorem 2.2 I).

Case 2: H_1 is isomorphic to one edge.

Then G^2 contains a hamiltonian cycle containing all edges of C except $v_i w_i$, $i = 1, 2, ..., k$, by Theorem 2.2 either IIa) or IIb). ■

The following theorem is an immediate corollary of Theorem 5.1.

Corollary 5.2. *Let G be a connected graph such that its block graph $Bl(G)$ is homeomorphic to a star in which the center corresponds to a block B_c of G. If B_c is hamiltonian, then G^2 is hamiltonian.*

Let us mention the following theorem by Schaar [4] that motivates the next conjecture.

Theorem 5.3 [4]. *For every block G with $|V(G)| \geq 4$ there exists a hamiltonian cycle in G^2 containing at least four edges of G.*

Conjecture 5.4. *Let G be a connected graph such that its block graph $Bl(G)$ is homeomorphic to a star in which the center c corresponds to a block B_c of G. If $d_{Bl(G)}c \leq k$, $k < 7$, then G^2 is hamiltonian.*

Conjecture 5.4 is true for $k \leq 2$ but for $k = 3, 4, 5, 6$ is an open problem. It is not possible to specify four edges from Theorem 5.3 and therefore Conjecture 5.4 is not an immediate corollary of Theorem 5.3 for $k \leq 4$. If Conjecture 5.4 is true, then the upper bound is sharp as can be seen from Figure 2.

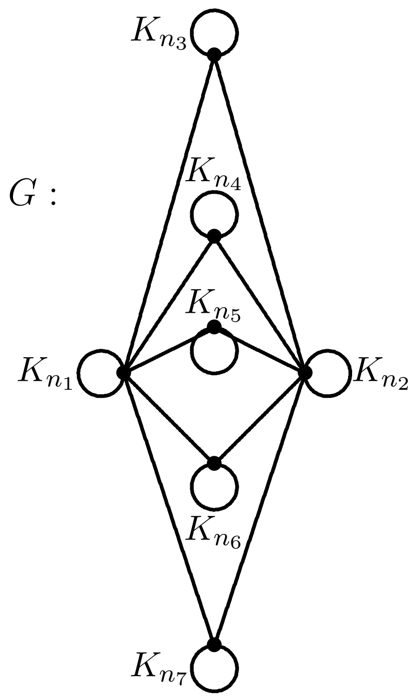

Figure 2:

6 Conclusion and acknowledgment

In conclusion I would like to thank Z. Ryjáček for valuable notes and recommendations.

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