Computability of Width of Submodular Partition Functions

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Abstract

The notion of submodular partition functions generalizes many of well-known tree decompositions of graphs. For fixed k, there are polynomial-time algorithms to determine whether a graph has tree-width, branch-width, etc. at most k. Contrary to these results, we show that there is no sub-exponential algorithm for determining whether the width of a given submodular partition function is at most two. On the other hand, we show that for a subclass of submodular partition functions, which contains tree-width, there exists a polynomial-time algorithm that decides whether the width is at most k.

1 Introduction

Graph decompositions and width-parameters play a very important role in algorithmic graph theory (as well as structural graph theory). The most well-known and studied notions include the tree-width, branch-width and clique-width of graphs. The importance of these notions lie in the fact that many NP-complete problems can be decided for classes of graphs of bounded tree-/branch-width in polynomial time. A classical result of Courcelle [5] (also see [2]) asserts that every problem expressible in the monadic second-order logic can be decided in linear time for the class of graphs with bounded

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tree-/branch-width. An analogous result for matroids with bounded branch-width representable over finite fields have been established by Hliněný [6, 7] and generalized using a more specialized notion of width to all matroids by Král' [10].

Most of the algorithms for classes of graphs of bounded width require a decomposition of an input graph as part of input. Fortunately, optimal tree-decompositions of graphs can be computed in linear time [3] if the width is fixed and there are even simple efficient approximation algorithms [4]. For branch-width, Oum and Seymour [12] recently established that the branch-decompositions of a fixed width of graphs and matroids can be computed in polynomial-time (or decided that they do not exist). Their algorithm actually deals with a more general notion of connectivity functions which are given by an oracle. A fixed-parameter algorithm for computing optimal branch-decompositions for matroids represented over finite fields was designed by Hliněný and Oum [8].

In this paper, we study submodular partition functions introduced by Amini et al. [1]. This general notion includes both graph tree-width and branch-width as special cases. We postpone the formal definition to Section 2. In their paper, Amini et al. [1] presented a duality theorem that implies the known duality theorems for graph tree-width and graph/matroid branch-width of Robertson and Seymour [13].

Since the duality, an essential ingredient for some of the known algorithms for computing decompositions of small width, smoothly translates to this general setting, it is natural to ask whether decompositions of submodular partition functions with fixed width can be computed in polynomial-time. In this paper, we show that such an algorithm cannot be designed in general. In particular, we present an argument that every algorithm deciding whether a partition width of an n-element set is at most two must ask an oracle the number of queries exponential in n. On a positive side, we were able to develop notions of loose tangles and loose tangle kits, a key ingredients of the algorithm of Oum and Seymour [12], and used them to construct a polynomial-time algorithm for class of submodular partition functions with bounded partitions. This class includes tree-width. We hope it will be possible to show that our results can also be adapted for other graph/matroid width parameters.

2 Notation

In this section, we introduce the notation and concepts used in this paper. A function $f : 2^E \to \mathbb{N}$ for a finite set E is said to be *submodular* if the following holds for every pair of subsets $X, Y \subseteq E$:

$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y) \ . \tag{1}$$

A submodular function f is *symmetric* if $f(X) = f(\overline{X})$, for all subsets X of E. Finally, a *connectivity function* is a submodular function that is symmetric and $f(\emptyset) = 0$.

For a connectivity function f on a ground set E, a *branch-decomposition* of f is a pair (T, σ) where T is a *ternary* tree and σ is a bijection between the set of leaves of T and E. Every edge e of T naturally defines a bipartition $(A_e, \overline{A_e})$ of the ground set E, i.e., A_e consists of all elements that corresponds to leaves of T in one of the two components of $T \setminus e$. The *order* of an edge e of T is the value $f(A_e)$ and the *width* of a branch-decomposition (T, σ) is the maximum order of an edge of T. The *branch-width* of f is the minimum width of a branch-decomposition of f. This notion includes the notion of the usual branch-width of graphs and matroids.

There is a dual object to branch-decompositions called a *tangle*, introduced by Robertson and Seymour [13]. A set $\mathcal{T}$ of subsets of E is called an f-tangle of *order* $k+1$ if $\mathcal{T}$ satisfies the following three axioms:

(1) For all $A \subseteq E$, if $f(A) \leq k$, then either $A \in \mathcal{T}$ or $\overline{A} \in \mathcal{T}$.

(2) If $A, B, C \in \mathcal{T}$, then $A \cup B \cup C \neq E$.

(3) For all $e \in E$, we have $E \setminus \{e\} \notin \mathcal{T}$.

Robertson and Seymour [13] proved the following duality theorem between branch-decompositions and tangles.

Theorem 1 (Robertson and Seymour [13])**.** *Let f be a connectivity function on a ground set E. There is no f-tangle of order $k+1$ if and only if the branch-width of f is at most k.*

We now introduce the concept of submodular partition functions that provides a unified view on branch-decompositions of connectivity functions and tree-decompositions of graphs. Throughout the paper, Greek letters will be used for collections of subsets, i.e., α can stand for a collection $A_1, \dots, A_k$ of subsets of a set E. Note, that the sets in a collection are not ordered in any way and a set can occur more than once in a collection. The collection α is a *partition* if the sets A_i are mutually disjoint and their union is the whole set E.

There are shorthands for operations with collections of subsets we want to use: if α is such a collection $A_1, \dots, A_k$ and A is another subset, then $\alpha \cap A$

stands for the collection $A_1 \cap A, \ldots, A_k \cap A$. We use $\alpha \setminus A$ in a similar way. Finally, $[B_1, \ldots, B_p, \alpha]$ stands for the collection obtained from α by inserting sets $B_1, \ldots, B_p$ to the collection. If α is the empty collection, we omit it from the notation. Note that empty sets are allowed in the collections.

A *partition function* is a function from the set of all partitions to non-negative integers that satisfies $\psi([\emptyset, \alpha]) = \psi(\alpha)$ for every partition α, i.e., inserting an empty set to a collection does not change the value of the partition function. A partition function ψ is *submodular* if the following holds for every two partitions $[A, \alpha]$ and $[B, \beta]$:

$$\psi([A, \alpha]) + \psi([B, \beta]) \geq \psi([A \cup \overline{B}, \alpha \cap B]) + \psi([B \cup \overline{A}, \beta \cap A]) \qquad (2)$$

We will further assume that $\psi([E]) = 0$ since shifting all values of a submodular partition function by a constant does not break the property.

Similarly to branch-decompositions, Amini et al. [1] defined a *decomposition tree* of a partition function ψ. A *decomposition tree* on a finite set E is a tree T with a bijection σ between its leaves and E. Every internal node v of T corresponds to the partition of E whose parts are the leaves contained in subtrees of $T \setminus v$. A decomposition tree is *compatible* with a set of partitions $\mathcal{P}$ of E if all partitions corresponding to the internal nodes of T belong to $\mathcal{P}$.

Let $\mathcal{P}_k[\psi]$ denote the set of partitions α of E such that $\psi(\alpha) \leq k$. The *width* of a submodular partition function ψ is the smallest integer k such that there exists a decomposition tree compatible with $\mathcal{P}_k[\psi]$. The concepts of submodular partition functions and decomposition trees include graph tree-width as a special case. Amini et al. [1] generalized submodular partition functions to include branch-width, path-width and other parameters.

There is a dual object to the decomposition tree called a *bramble* introduced by Amini et al. [1]. A *$\mathcal{P}$-bramble* $\mathcal{B}$ on E is a set of pairwise intersecting subsets of E which contains a part of every partition of $\mathcal{P}$. A $\mathcal{P}$-bramble is called *non-principal* if it contains no singleton. The duality theorem for submodular partition functions asserts the following.

Theorem 2 (Amini et al. [1])**.** *Let ψ be a submodular partition function and k a non-negative integer. There is no decomposition tree compatible with $\mathcal{P}_k[\psi]$ if and only if there is a non-principal $\mathcal{P}_k[\psi]$-bramble.*

Note that Theorem 2 is proven in [1] for a larger class of *weakly submodular* partition functions. In this paper, we restrict our attention only to the class of submodular partition functions. In particular, the loose tangles defined in the next section are studied only for submodular partition functions.

3 Loose Tangles

A key ingredient of the algorithm of Oum and Seymour [12] for deciding whether a connectivity function has branch-width at most k (k is fixed) is the notion of a *loose tangle* which we now recall. For a connectivity function f on a ground set E, a *loose f-tangle* of order $k+1$ is a set $\mathcal{T}$ of subsets of E satisfying the following three axioms:

(L1) $\emptyset \in \mathcal{T}$ and $\{e\} \in \mathcal{T}$ for every $e \in E$ such that $f(\{e\}) \leq k$.

(L2) If $A, B \in \mathcal{T}$, $C \subseteq A \cup B$, and $f(C) \leq k$, then $C \in \mathcal{T}$.

(L3) $E \notin \mathcal{T}$.

The following theorem by Oum and Seymour [12] states that the loose f-tangles are also dual objects to branch-decompositions of connectivity functions.

Theorem 3 (Oum and Seymour [12]). *Let f be a connectivity function on a ground set E. Then, no loose f-tangle of order $k+1$ exists if and only if the branch-width of f is at most k.*

Using loose tangles Oum and Seymour [12] managed to construct an algorithm for deciding whether the branch-width of a connectivity function is at most k for a fixed k in polynomial time when f is given by an oracle.

Similarly to the loose tangles of Oum and Seymour we introduce *loose tangles* for submodular partition functions. A *loose $\mathcal{P}$-tangle* is a set $\mathcal{T}$ of subsets of E closed under taking subsets satisfying the following three axioms.

(P1) $\emptyset \in \mathcal{T}$, $\{e\} \in \mathcal{T}$, for all $e \in E$ such that the partition $[\{e\}, \overline{\{e\}}]$ belongs to $\mathcal{P}$.

(P2) If $A_1, A_2, \ldots, A_p \in \mathcal{T}$, $C_i \subseteq A_i$, for $i = 1, \ldots, p$, $[C_1, \ldots, C_p, \overline{\cup_{i=1}^p C_i}] \in \mathcal{P}$, then $\cup_{i=1}^p C_i \in \mathcal{T}$.

(P3) $E \notin \mathcal{T}$.

To prove the main theorem of this section, we need a lemma.

Lemma 4. *Let ψ be a submodular partition function on E and $[A, \alpha]$ a partition. Then $\psi([A, \alpha]) \geq \psi([A, \overline{A}])$.*

Proof. By submodularity of ψ,

$$\begin{aligned}\psi([A, \alpha]) + \psi([\emptyset, E]) &\geq \psi([A \cup E, \alpha \cap \emptyset]) + \psi([\emptyset \cup \overline{A}, E \cap A]) \\ &= \psi([E, \emptyset]) + \psi([\overline{A}, A]).\end{aligned}$$

The result follows. $\square$

In the following theorem, we show that for classes of partitions of bounded width, the loose tangle is a dual object to the decomposition tree.

Theorem 5. *Let ψ be a submodular partition function. There is no decomposition tree compatible with $\mathcal{P}_k[\psi]$ if and only if there is a loose $\mathcal{P}_k[\psi]$-tangle.*

Proof. Suppose there is a decomposition tree (T, σ) compatible with $\mathcal{P}_k[\psi]$ and a loose $\mathcal{P}_k[\psi]$-tangle $\mathcal{T}$. We will show that $\mathcal{T}$ violates (P3). Choose an arbitrary leaf x of T as a root. Every internal node v of T corresponds to a partition α_v. Let C_v be a union of all parts of α_v except the one containing x. Define C_v of a leaf v as the singleton $\sigma(v)$. We will show by backward induction on the distance from x that for every node v of T, the set C_v belongs to $\mathcal{T}$.

Since T is a decomposition tree of E compatible with $\mathcal{P}_k[\psi]$, there is a partition $[\{e\}, \alpha_e]$ in $\mathcal{P}_k[\psi]$, for each $e \in E$. By Lemma 4, $\psi([\{e\}, \overline{\{e\}}]) \leq \psi([\{e\}, \alpha_e])$. Hence, $[\{e\}, \overline{\{e\}}]$ belongs to $\mathcal{P}_k[\psi]$ and $\{e\}$ is in $\mathcal{T}$ by (P1). For an inner node v, all his children $u_1, \ldots, u_p$ are farther from x than v and therefore all C_{u_i} are in $\mathcal{T}$. By (P2), since $[C_{u_1}, \ldots, C_{u_p}, \overline{\cup C_{u_i}}]$ belongs to $\mathcal{P}_k[\psi]$, $C_v \equiv \cup C_{u_i} \in \mathcal{T}$. Finally, let v be the only child of x. Since $C_v \in T$ and $\{\sigma(x)\} \in \mathcal{T}$, by (P2) and Lemma 4, $C_v \cup \{\sigma(x)\} = E$ also belongs to $\mathcal{T}$. (P3) is now violated.

We now prove the opposite implication. A partial decomposition tree for $A \subseteq E$ is a decomposition tree for a partition function ψ' on $A \cup \{a\}$ defined as $\psi'([B, \beta]) = \psi(((B \setminus \{a\}) \cup \overline{A}, \beta))$ for a partition $[B, \beta]$ where B contains a. We say that a set $A \subseteq E$ is k-branched if there is a partial decomposition tree for A compatible with $\mathcal{P}_k[\psi]$.

Define $\mathcal{T}$ to be a subset of 2^E closed under taking subsets, containing all singletons and all k-branched sets. We will show that $\mathcal{T}$ is a loose tangle. (P1) trivially holds since all k-branched singletons are in $\mathcal{T}$. Let $A_1, \ldots, A_p \in \mathcal{T}$ and $C_i \subseteq A_i$, $i = 1, \ldots, p$, such that $[C_1, \ldots, C_p, \overline{\cup C_i}] \in \mathcal{P}_k[\psi]$. We can assume that A_i are k-branched (otherwise take such a superset of it instead). Let $Y_1, \ldots, Y_p$, $Y_i \subseteq A_i$, be such sets that $\cup C_i \subseteq \cup Y_i$ and $\psi([Y_1, \ldots, Y_p, \overline{\cup Y_i}])$ is minimum. We will show that the set $\cup Y_i$ is k-branched.

To this end, we modify the partial decomposition tree T_i for A_i to be a partial decomposition tree for Y_i. At first, we delete from T_i all leaves corresponding to elements not in Y_i. We then repeatedly contract all nodes of degree two or less until we get a ternary tree T_i'. We claim T_i' is compatible with $P_k[\psi]$. Suppose for a contradiction that there is an internal node v' of T_i' corresponding to an internal node v of T_i such that $\alpha_{v'} \notin \mathcal{P}_k[\psi]$. Assume $i = 1$ since we can relabel the parts so. Let $[A, \alpha] = \alpha_v$ such that A is the part of α_v that contains $\overline{A_1}$. We infer from the submodularity of the function

ψ that

$$\begin{aligned}\psi([A,\alpha]) + \psi([Y_1, Y_2, \ldots, Y_p, \overline{\cup Y_i}]) &\geq \psi([A \cup \overline{Y}_1, \alpha \cap Y_1]) \\ + \psi([Y_1 \cup \overline{A}, Y_2 \cap A, \ldots Y_p \cap A, \overline{\cup Y_i} \cap A])\end{aligned}$$

The choice of $Y_1, \ldots, Y_p$ yields that

$$\psi([Y_1 \cup \overline{A}, Y_2 \cap A, \ldots, Y_p \cap A, \overline{\cup Y_i} \cap A]) \geq \psi([Y_1, \ldots, Y_p, \overline{\cup Y_i}]).$$

Hence, $\psi([A \cup \overline{Y}_1, \alpha \cap Y_1]) \leq \psi([A,\alpha]) \leq k$ and T_1' is compatible with $\mathcal{P}_k[\psi]$.

Now, construct a partial decomposition tree T by connecting T_i' to a single node corresponding to a partition $[Y_1, \ldots, Y_p, \overline{\cup Y_i}]$. This partition belongs to $\mathcal{P}_k[\psi]$ since $\psi([Y_1, \ldots, Y_p, \overline{\cup Y_i}]) \leq \psi([C_1, \ldots, C_p, \overline{\cup C_i}]) \leq k$ by the minimality of $\psi([Y_1, \ldots, Y_p, \overline{\cup Y_i}])$. Therefore T is a partial decomposition tree for $\cup Y_i$ compatible with $\mathcal{P}_k[\psi]$ and thus $\cup Y_i \in \mathcal{T}$. Since $\cup C_i \subseteq \cup Y_i$, also $\cup C_i \in \mathcal{T}$ as required.

If $E \in T$, then E is k-branched and the partial decomposition tree for E is actually a decomposition tree for ψ — remove the leaf corresponding to the empty set and suppress the resulting vertex of degree two. This contradicts the fact that ψ does not have a decomposition tree compatible with $\mathcal{P}_k[\psi]$. Therefore, $E \notin \mathcal{T}$ and (P3) holds. We conclude that $\mathcal{T}$ is a loose $\mathcal{P}_k[\psi]$-tangle. $\square$

4 Minimization of submodular functions

Let f be a connectivity function on E. We define a function $f_{\min}$ on pairs of disjoint subsets of E as follows.

$$f_{\min}(A,B) = \min_{A \subseteq Z \subseteq \overline{B}} f(Z)$$

There can be more sets attaining the minimum. Let $\mathcal{M}_f(A,B)$ be the collection of such sets, i.e.,

$$\mathcal{M}_f(A,B) = \{f(Z) = f_{\min}(A,B) \mid A \text{ß} Z \text{ß} \overline{B}\}.$$

The structure of $\mathcal{M}_f(A,B)$ is quite simple as shown in the following lemma. We include a short proof for completeness.

Lemma 6. *Let f be a connectivity function on E, $A, B \text{ß} E$ disjoint, and let $X, Y \in \mathcal{M}_f(A,B)$. Then $X \cup Y \in \mathcal{M}_f(A,B)$ and $X \cap Y \in \mathcal{M}_f(A,B)$.*

Proof. Compute

$$2f_{\min}(A,B) = f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$$

and note that both $f(X \cap Y) \geq f_{\min}(A,B)$, $f(X \cup Y) \geq f_{\min}(A,B)$. Hence both inequalities have to hold with equality and $X \cap Y \in \mathcal{M}_f(A,B)$ and $X \cup Y \in \mathcal{M}_f(A,B)$. □

It follows from Lemma 6 that there is precisely one set containing all other sets in $\mathcal{M}_f(A,B)$. Define $M_f(A,B) = Z$ such that Z is maximal with respect to $f(Z) = f_{\min}(A,B), AßZß\overline{B}$.

Important and very non-trivial results of Iwata [9] and Schrijver [14] state that submodular functions can be minimized in strongly polynomial time. We use an improved result of Orlin [11] saying that a submodular function on set E, $|E| = n$, can be minimized in time $Ø(n^5\gamma + n^6)$ where γ is the time complexity of an oracle query. Orlin's result can be used to compute $f_{\min}(A,B)$ and $\mathcal{M}_f(A,B)$ in polynomial time.

Lemma 7. *Let f be a connectivity function on E, $A, BßE$ disjoint, $|E| = n$. Then $f_{\min}(A,B)$ can be determined in time $Ø(n^5\gamma + n^6)$ where γ is the time complexity of an oracle query. Moreover, in total time $Ø(n^6\gamma + n^7)$, the maximal and the minimal set from $\mathcal{M}_f(A,B)$ can be constructed.*

Proof. First note, that a function $g_{A,B} : 2^{\overline{A \cup B}} \to \mathbb{N}$ defined as $g_{A,B}(X) = f(X \cup A)$ is a submodular function since

$$\begin{aligned} g_{A,B}(X) + g_{A,B}(Y) &= f(X \cup A) + f(Y \cup A) \\ &\geq f(X \cup Y \cup A) + f((X \cup A) \cap (Y \cup A)) \\ &= g_{A,B}(X \cup Y) + g_{A,B}(X \cap Y). \end{aligned}$$

Second, if $Zß\overline{A \cup B}$ is such that $g_{A,B}(Z)$ is the minimum of $g_{A,B}$ then $f(Z \cup A) = f_{\min}(A,B)$. Now, we can use result of Orlin [11] that a submodular function f can be minimized in time $Ø(n^5\gamma + n^6)$ and a set Z such that $f(Z)$ is minimum is provided. Note that Z does not have to be maximal with respect to that. Denote $Z_\emptyset$ the set for $g_{A,B}$.

To get the maximal set Z_m for $g_{A,B}$, obtain Z_e as sets for submodular functions $g_{A\cup\{e\},B}$, $e \in \overline{A \cup B}$. If $g_{A\cup\{e\},B}(Z_e) > g_{A,B}(Z_\emptyset)$, then there is no set Z, such that $A \cup \{e\}ßZß\overline{B}$ and $f(Z)_{A\cup\{e\},B}(Z) = g_{A,B}(Z_\emptyset) = f_{\min}(A,B)$. Hence $e \notin Z_m$. Let $\mathcal{M} = \{Z \mid g_{A\cup\{e\},B}(Z) = f_{\min}(A,B)\}$. By Lemma 6, $\bigcup_{Z\in\mathcal{M}} Z \in \mathcal{M}$. We conclude that $Z_m = \bigcup_{Z_e\in\mathcal{M}} Z_e$. There are at most n sets Z_e so we needed $Ø(n)$ submodular function minimizations giving the claimed time complexity.

The minimal set Z is obtained as the complement of maximal set $M_f(B,A)$, $Z = \overline{M_f(B,A)}$, by symmetry of f. □

The following lemma of Oum and Seymour [12] is critical for our construction of a polynomial-time algorithm for submodular partition functions.

Lemma 8 (Oum and Seymour [12]). *For a connectivity function f on E and a subset Z of E, there exist a subset A of Z and a subset B of $\overline{Z}$ such that*

$$\max\{|A|, |B|\} \leq f_{\min}(A, B) = f(Z).$$

Note that a submodular partition function is a connectivity function when restricted to partitions of size two. We will use $\psi(A)$ as a shorthand for $\psi([A, \overline{A}])$.

$$\begin{aligned}
\psi(A) + \psi(B) &= \psi([A, \overline{A}]) + \psi([B, \overline{B}]) = \psi([A, \overline{A}]) + \psi([\overline{B}, B]) \\
&\geq \psi([A \cup B, \overline{A} \cap \overline{B}]) + \psi([\overline{B} \cup \overline{A}, B \cap A]) \\
&= \psi([A \cup B, \overline{A \cup B}]) + \psi([A \cap B, \overline{A \cap B}]) \\
&= \psi(A \cup B) + \psi(A \cap B)
\end{aligned}$$

We can minimalize ψ as a connectivity function and define

$$\psi_{\min}(A, B) = \min_{A \text{ß} X \text{ß} \overline{B}} \psi(X)$$

We say $(A_1, \ldots, A_p | B_1, \ldots, B_p)$ is a *prepartition* of E if sets A_i and B_i are disjoint for all $i = 1, \ldots, p$, $A_1, \ldots, A_p$ are pairwise disjoint, $A_j \text{ß} B_i$ for all $j \neq i$ and $\cap_{i=1}^{p} B_i = \emptyset$. We write (A_i, B_i, p) as a shorthand for a prepartition $(A_1, \ldots, A_p | B_1, \ldots, B_p)$. The prepartitions can be understood as restrictions for partitions: For every prepartition (A_i, B_i, p), there exists a partition $[C_1, \ldots, C_p]$ satisfying $A_i \text{ß} C_i \text{ß} \overline{B_i}$, $i = 1, \ldots, p$.

For a submodular partition function ψ and a prepartition (A_i, B_i, p) of E define

$$\psi_{\min}(A_i, B_i, p) = \min\{\psi([C_1, \ldots, C_p]) \mid A_i \text{ß} C_i \text{ß} \overline{B_i}\}.$$

Similarly as for connectivity functions, we also define a collection $\mathcal{M}_\psi(A_i, B_i, p)$ of minimal partitions,

$$\mathcal{M}_\psi(A_i, B_i, p) = \{[C_1, \ldots, C_p] \mid A_i \text{ß} C_i \text{ß} \overline{B_i}, \psi([C_1, \ldots, C_p]) = \psi_{\min}(A_i, B_i, p)\}.$$

Note that for two disjoint sets A and B, $\mathcal{M}_\psi(A, B | B, A)$ extends the definition of $\mathcal{M}_f(A, B)$ for a connectivity function f.

The structure of sets $\mathcal{M}_\psi(A_i, B_i, p)$ is richer than that of sets $\mathcal{M}_f(A, B)$. Let f be a connectivity function, A, B disjoint subsets of E and r an integer parameter satisfying $f_{\min}(A, B) \leq r$. In order to describe the structure of $\mathcal{M}_\psi(A_i, B_i, p)$, we consider all maximal sets Z such that $A \text{ß} Z \text{ß} \overline{B}$ and $f(Z) =$

r. Note that Lemma 6 implies that there is only one such a Z for $r = f_{\min}(A, B)$. However, for $r > f_{\min}(A, B)$ there can be more of them. Define an *r-corolla* $F_r^f(A, B)$ of f as a set

$$F_r^f(A, B) = \bigcup_{t=0}^{r} \{Z \mid A \text{ß} Z \text{ß} \overline{B}, Z \text{ inclusion-wise maximal with } f(Z) = t\}.$$

We call sets in $F_r^f(A, B)$ *petals.*

For a partition submodular function ψ, we show that it is enough for minimization to consider partitions created by k-corollas for ψ (viewed as a connectivity function).

Lemma 9. *Let ψ be a submodular partition function on E, (A_i, B_i, p) a prepartition of E such that $\psi_{\min}(A_i, B_i, p) \leq k$, $[C_1, \ldots, C_p] \in \mathcal{M}_\psi(A_i, B_i, p)$ a minimal partition, $A_i \text{ß} C_i \text{ß} \overline{B_i}$, $i = 1, \ldots, p$, and $j \in \{1, \ldots, p\}$ fixed. Then there is a petal $D_j \in F_k^\psi(A_j, B_j)$ such that $C_j \text{ß} D_j$ and*

$$[C_1 \setminus D_j, \ldots, C_{j-1} \setminus D_j, D_j, C_{j+1} \setminus D_j, \ldots, C_p \setminus D_j] \in \mathcal{M}_\psi(A_i, B_i, p).$$

Proof. By symmetry, we can assume that $j = 1$. Let $[C_1, C_2, \ldots, C_p] \in \mathcal{M}_\psi(A_i, B_i, p)$ be a minimal partition, $A_i \text{ß} C_i \text{ß} \overline{B_i}$, $i = 1, \ldots, p$. By Lemma 4, $\psi(C_1) \leq \psi([C_1, C_2, \ldots, C_p]) \leq k$. Let $r = \psi(C_1)$. By definition, there is $Z \in F_k^\psi(A_1, \overline{B_1})$ such that $C_1 \text{ß} Z$, $\psi(Z) = \psi(C_1) = r$. By submodularity of ψ,

$$\begin{aligned}
\psi([C_1, C_2 \ldots, C_p]) + \psi([\overline{Z}, Z]) &\geq \psi([C_1 \cup Z, C_2 \setminus Z, \ldots, C_p \setminus Z]) \\
&\quad + \psi([\overline{Z} \cup \overline{C_1}, Z \setminus \overline{C_1}]) \\
&= \psi([Z, C_2 \setminus Z, \ldots, C_p \setminus Z]) + \psi([\overline{C_1}, C_1]).
\end{aligned}$$

Since $\psi(C_1) = \psi(Z)$ and $[C_1, \ldots, C_p]$ is minimal, we conclude that

$$\psi([C_1, C_2, \ldots, C_p]) = \psi([Z, C_2 \setminus Z, \ldots, C_p \setminus Z]).$$

Hence, $[Z, C_2 \setminus Z, \ldots, C_p \setminus Z]$ is the sought partition. $\square$

Lemma 10. *Let ψ be a submodular partition function on E, (A_i, B_i, p) a prepartition of E. If $\psi_{\min}(A_i, B_i, p) \leq k$, then there is a minimal partition $\alpha \in \mathcal{M}_\psi(A_i, B_i, p)$ of the form*

$$\alpha = [Z_p, Z_{p-1} \setminus Z_p, Z_{p-2} \setminus (Z_{p-1} \cup Z_p), \ldots, Z_1 \setminus \{\cup_{i=2}^p Z_p\}],$$

where $Z_i \in F_k^\psi(A_i, B_i)$.

Proof. Let $\beta \in \mathcal{M}_\psi(A_i, B_i, p)$ be a minimal partition. Using Lemma 9, we will construct a sequence $\beta_0, \beta_1, \dots, \beta_p$ of minimal partitions in $\mathcal{M}_\psi(A_i, B_i, p)$ where β_i is created from β_{i-1} by "exchanging" i-th part of the partition β_{i-1} for a petal. We write $\beta_i = [C^i_1, \dots, C^i_p]$ where $A_j \text{ß} C^i_j \text{ß} \overline{B_j}$, $j = 1, \dots, p$.

Let $\beta_0 = \beta$. Define β_i as the minimal partition obtained from Lemma 9 applied on i-th part of β_{i-1}, i.e.,

$$\beta_i = [C^{i-1}_1 \setminus Z_i, \dots, C^{i-1}_{i-1} \setminus Z_i, Z_i, C^{i-1}_{i+1} \setminus Z_i, \dots, C^{i-1}_p \setminus Z_i],$$

where $C^{i-1}_i \text{ß} Z_i \in F^\psi_k(A_i, B_i)$.

Note that $\beta_i \in \mathcal{M}_\psi(A_i, B_i, p)$, $i = 1, \dots, p$, by Lemma 9. The final partition $\alpha = \beta_p$ have form

$$\alpha = [Z_p, Z_{p-1} \setminus Z_p, Z_{p-2} \setminus (Z_{p-1} \cup Z_p), \dots, Z_1 \setminus \{\cup^p_{i=2} Z_p\}],$$

where $Z_i \in F^\psi_k(A_i, B_i)$. □

The following bound on number of disjoint subsets of E will be used in our subsequent proofs. We leave a straightforward proof to the reader.

Lemma 11. *The number of subsets of size at most k of an n-element set is at most $1 + n^k$. The number of disjoint pairs of subsets of size at most k of an n-element set is at most $1 + n^{2k}$.*

We will see that k-corollas will play important role in devising the minimization algorithm for submodular partition functions. The following lemma shows that k-corollas has always a polynomial size.

Lemma 12. *Let f be a connectivity function on E, A and B disjoint subsets of E. Then*

$$\left| F^f_k(A, B) \right| \leq 1 + n^k,$$

Moreover, $F^f_k(A, B)$ can be constructed in time $\O(n^{k+6}\gamma + n^{k+7})$ where γ is the time complexity of an oracle query.

Proof. Let A and B be fixed disjoint subsets of E. Define $Z_X = \underline{M_f}(A \cup X, B)$ for $X \text{ß} \overline{B}$. Observe that $f(Z_{X \cup \{e\}}) \geq f(Z_X) + 1$ for all $e \in \overline{Z_X \cup B}$. Let $S_0 = \{Z_\emptyset\}$. Define $S_i = \{Z_{X \cup \{e\}} \mid Z_X \in S_{i-1}, e \in \overline{Z_X \cup B}\}$, for $i \geq 1$, and $S = \cup^k_{i=0} S_i$.

Observe that $f(R) \geq i$ for every $R \in S_i$.

We claim that all petals $P \in F^f_k(A, B)$ are contained in S. Let $X \text{ß} P$ be a maximal subset such that $Z_X \in S$. If $Z_X = P$, then we are done.

Otherwise, let $e \in P \setminus Z_X$. The maximality of Z_X with $Z_X = M_f(A \cup X, B)$ and $A \cup X \subseteq P \subseteq \overline{B}$ implies that $f(P) > f(Z_X)$. Consequently, $Z_X \in S_i$ for some $i < k$. Hence, the set $Z_{X \cup \{e\}}$ is included in the set S which contradicts our choice of X.

We have shown that $F_k^f(A, B)$ is contained in the set S. Observe that every set in S can be constructed as Z_X for some X of size at most k. Since the number of choices of X is at most $1 + n^k$ by Lemma 11, we conclude that $|F_k^f(A, B)| \leq 1 + n^k$. The algorithm is now easy to design: for all (at most $1 + n^k$) choices of X, compute $M_f(A \cup X, B)$. Lemma 7 implies that the running time of the algorithm can be bounded by $\emptyset(n^{k+6}\gamma + n^{k+7})$.

□

Now, we will use Lemma 10 and 12 to show that it is possible to minimize a partition function in polynomial time if the number of parts p and the width k of the assumed partitions are fixed integers.

Lemma 13. *Let ψ be a submodular partition function on E, (A_i, B_i, p) a prepartition of E. There is an algorithm running in time $\emptyset(n^{pk}\gamma + pn^{k+6}\gamma + pn^{k+7})$, where γ is the time complexity of an oracle query, that determines whether there exists a partition $\beta \in \mathcal{M}_\psi(A_i, B_i, p)$ with $\psi(\beta) \leq k$ and if so it constructs such a partition with minimal $\psi(\beta)$.*

Proof. By Lemma 10, if there exists a minimal partition $\beta \in \mathcal{M}_\psi(A_i, B_i, p)$ of width at most k, then there exists also one of the form

$$\beta = [Z_p, Z_{p-1} \setminus Z_p, Z_{p-2} \setminus (Z_{p-1} \cup Z_p), \ldots, Z_1 \setminus \{\cup_{i=2}^{p} Z_p\}].$$

Observe that β is constructed only from petals. Hence if we try all possible p-tuples $(Z_1, \ldots, Z_p)$ where $Z_i \in F_k(A_i, B_i)$, then we will find β.

By Lemma 12, there are at most $\emptyset(n^k)$ sets in a k-corolla. There are at most $\emptyset(n^{pk})$ such p-tuples and for each of them we have to call the function oracle. Hence we need time $\emptyset(n^{pk}\gamma)$. We also have to construct the k-corollas. By Lemma 12, $F_k(A_i, B_i)$ can be constructed in time $\emptyset(n^{k+6}\gamma + n^{k+7})$. Thus we can construct all of them in time $\emptyset(pn^{k+6}\gamma + pn^{k+7})$. □

5 Loose Tangle Kits

A loose tangle is a collection of sets that contain (usually) exponentially many sets making it difficult to work with in a polynomial-time algorithm. Hence Oum and Seymour [12] introduced a more compact structure, *loose tangle kits*. A pair (P, μ) is called a *loose f-tangle kit* of order $k + 1$ if

$$P = \{(A, B) \mid A, B \subseteq E, A \cap B = \emptyset, \max\{|A|, |B|\} \leq f_{\min}(A, B) \leq k\}$$

and $\mu : P \to 2^E$ is a function satisfying the following three axioms.

(K1) For every $e \in E$, $f(\{e\}) \leq k$, there exists $(A, B) \in P$ such that $A \subseteq \{e\} \subseteq \overline{B}$, $f(\{e\}) = f_{\min}(A, B)$, and $e \in \mu(A, B)$.

(K2) If $(A, B), (C, D), (F, G) \in P$, $F \subseteq X \subseteq (\mu(A, B) \cup \mu(C, D)) \setminus G$, and $f(X) = f_{\min}(F, G)$, then $X \subseteq \mu(F, G)$.

(K3) $\mu(\emptyset, \emptyset) \neq E$.

The notion of loose tangle kits is a notion dual to branch-decompositions as stated in the next theorem.

Theorem 14 (Oum and Seymour [12]). *Let f be a connectivity function on E. Then, a loose f-tangle of order $k+1$ exists if and only if a loose f-tangle kit of order $k+1$ exists.*

We define a similar structure for submodular partition functions which we also call *loose tangle kits.* A pair (K, μ) is a *loose ψ-tangle kit* of order $k+1$ if

$$K = \{(A, B) \mid A, B \subseteq E, A \cap B = \emptyset, \max\{|A|, |B|\} \leq \psi_{\min}(A, B) \leq k\}$$

and $\mu : K \to 2^E$ is a function satisfying the following three axioms.

(T1) For every $e \in E$, $\psi(\{e\}) \leq k$, there exists $(A, B) \in K$ such that $A \subseteq \{e\} \subseteq \overline{B}$, $\psi(\{e\}) = \psi_{\min}(A, B)$, and $e \in \mu(A, B)$.

(T2) If $(A_1, B_1), \ldots, (A_p, B_p), (A, B) \in K$, $C = \cup_{i=1}^{p} C_i$ such that $C_i \text{ß} \mu(A_i, B_i)$, $i = 1, \ldots, p$, $A \text{ß} C \text{ß} \overline{B}$, $\psi([C_1, \ldots, C_p, \overline{C}]) \leq k$ and $\psi(C) = \psi_{\min}(A, B)$, then $C \text{ß} \mu(A, B)$.

(T3) $\mu(\emptyset, \emptyset) \neq E$.

The following theorem shows that loose tangle kits, similarly to loose tangles, are also dual objects to decomposition trees of submodular partition functions.

Theorem 15. *Let ψ be a submodular partition function on E. Then, a loose $P_k[\psi]$-tangle of order $k+1$ exists if and only if a loose ψ-tangle kit of order $k+1$ exists.*

Proof. Suppose that $\mathcal{T}$ is a loose $P_k[\psi]$-tangle of order $k+1$. We construct a loose ψ-tangle kit of order $k+1$ as follows. Let

$$K = \{(A,B) \mid A,B \subseteq E, A \cap B = \emptyset, \max\{|A|,|B|\} \leq \psi_{\min}(A,B) \leq k\}.$$

For each $(A,B) \in K$, let

$$\begin{aligned} \mathcal{T}_{A,B} &= \{X \mid A \subseteq X \subseteq \overline{B}, \psi_{\min}(A,B) = \psi(X), \text{ and } X \in \mathcal{T}\}, \\ \mu(A,B) &= \bigcup_{X \in \mathcal{T}_{A,B}} X. \end{aligned}$$

If $\mathcal{T}_{A,B} = \emptyset$, then $\mu(A,B) = \emptyset$.

We will show that $\mu(A,B) \in \mathcal{T}$ for every $(A,B) \in K$. If $\mathcal{T}_{A,B}$ is empty then $\mu(A,B) = \emptyset \in \mathcal{T}$. Let $X, Y \in \mathcal{T}_{A,B}$. By Lemma 6, we have $\psi(X \cup Y) = \psi_{\min}(A,B) \leq k$. By (P2), $X \cup Y \in \mathcal{T}$. Hence $X \cup Y \in \mathcal{T}_{A,B}$ and since $\mu(A,B)$ is a union of sets in $\mathcal{T}_{A,B}$, $\mu(A,B) \in \mathcal{T}_{A,B}\text{ß}\mathcal{T}$.

Let $e \in E$ such that $\psi(\{e\}) \leq k$. By Lemma 8, there exists A and B such that $A\text{ß}\{e\}\text{ß}\overline{B}$, $\max\{|A|,|B|\} \leq \psi(\{e\}) = \psi_{\min}(A,B)$. By (P1), $\{e\} \in \mathcal{T}$. Hence $\{e\} \in \mathcal{T}_{A,B}$ and $e \in \mu(A,B)$ as required in property (T1).

Let $(A_1,B_1),\ldots,(A_p,B_p),(A,B) \in K$, $C = \cup_{i=1}^p C_i$ such that $C_i\text{ß}\mu(A_i,B_i)$, $i = 1,\ldots,p$, $A\text{ß}C\text{ß}\overline{B}$, $\psi([C_1,\ldots,C_p,\overline{C}]) \leq k$ and $\psi(C) = \psi_{\min}(A,B)$. As we have shown that all $\mu(A_i,B_i) \in \mathcal{T}$, using (P2), we get that $C \in \mathcal{T}$. Since $\psi(C) = \psi_{\min}(A,B)$ and $(A,B) \in K$, by construction of the loose tangle kit, $C\text{ß}\mu(A,B)$. So (T2) holds.

Since $E \notin \mathcal{T}$ by (P3) and $\mu(\emptyset,\emptyset) \neq E$, we conclude (K,μ) is a loose ψ-tangle kit of order $k+1$.

Conversely, suppose that (K,μ) is a loose ψ-tangle kit of order $k+1$. We define

$$\begin{aligned} \mathcal{T} = \{X \mid {} & \text{there exists } (A,B) \in K \text{ such that } A \subseteq X \subseteq \overline{B}, \\ & \psi_{\min}(A,B) = f(X), \text{ and } X \subseteq \mu(A,B)\}. \end{aligned}$$

We claim that $\mathcal{T}$ is a loose $P_k[\psi]$-tangle of order $k+1$.

If $\psi(e) \leq k$ then by (T1) there exists $(A,B) \in K$ such that $e \in \mu(A,B)$ and $\psi_{\min}(A,B) = \psi(\{e\})$. So $\{e\} \in \mathcal{T}$, ensuring property (P1).

To show (L2), suppose that $A_1,\ldots,A_p \in \mathcal{T}$, $C = \cup_{i=1}^p C_i$, such that $C_i\text{ß}A_i$, $i = 1,\ldots,p$, and $\psi([C_1,\ldots,C_p,\overline{C}]) \leq k$. By construction of $\mathcal{T}$, there are $(U_i,V_i) \in K$ such that $A_i\text{ß}\mu(U_i,V_i)$, $i = 1,\ldots,p$. By Lemma 8, there exist U and V such that $U\text{ß}C\text{ß}\overline{V}$, $\max\{|U|,|V|\} \leq \psi(C) = \psi_{\min}(U,V)$. Using (T2) for $(U_1,V_1),\ldots,(U_p,V_p),(U,V)$ and C, we get that $C\text{ß}\mu(U,V)$. Hence $C \in \mathcal{T}$ by construction of $\mathcal{T}$.

Since $E \neq \mu(\emptyset,\emptyset)$, by definition of $\mathcal{T}$, $E \notin \mathcal{T}$ as required by (T3). We have shown that $\mathcal{T}$ is a loose $P_k[\psi]$-tangle of order $k+1$. □

6 Polynomial-time Algorithm for Submodular Partition Functions with Bounded Partitions

The number of parts play an important role when minimizing submodular partition functions. We say that a class $\mathcal{C}$ of submodular partition functions on E has *k-bounded partitions* if there exists $b(k)$ such that for every $\psi \in \mathcal{C}$ and every partition α with more than $b(k)$ non-empty parts, $\psi(\alpha) > k$.

Theorem 16. *Let $\mathcal{C}$ be a class of submodular partition functions on E, $|E| = n$, with k-bounded partitions. When $\psi \in \mathcal{C}$ is given by an oracle, we can determine in time $\text{Ø}(n^{3bk+2k+2}\gamma + bn^{2bk+3k+8}\gamma + bn^{2bk+2k+9})$, where γ is the time complexity of an oracle query, whether the width of ψ is at most k.*

As in [12], we design an algorithm that either constructs a loose tangle kit of order $k+1$ or shows that no loose tangle kit of order $k+1$ exists.

Algorithm 17. *Decide whether the width of a submodular partition function $\psi \in \mathcal{C}$ is at most k, where $\mathcal{C}$ is a class with k-bounded partitions.*

(A1) *Construct $K = \{(A, B) \mid \max\{|A|, |B|\} \leq \psi_{\min}(A, B) \leq k\}$.*

(A2) *Set $\mu(\emptyset, \emptyset) = \{e \in E \mid \psi(e) = 0\}$.*

(A3) *For each $e \in E$ such that $0 < \psi(e) \leq k$ and all sets B such that $|B| \leq \psi_{\min}(\{e\}, B) = \psi(e)$, set $\mu(\{e\}, B) = \{e\}$.*

(A4) *Test whether (T3) holds. If not, then output that there is no loose ψ-tangle kit of order $k+1$ and stop.*

(A5) *Test whether (T2) holds for all $(A_1, B_1), \ldots, (A_{b-1}, B_{b-1}), (A, B) \in K$. If not, then we have $C = \cup C_i$ such that $A \text{ß} C \text{ß} \overline{B}$, $C_i \text{ß} \mu(A_i, B_i)$, $i = 1, \ldots, b-1$, $\psi([C_1, \ldots, C_{b-1}, \overline{C}]) \leq k$, $C = \psi_{\min}(A, B)$ and $C \not\text{ß} \mu(A, B)$. Add C to $\mu(A, B)$ thus increasing $\mu(A, B)$ by at least one and go back to (A4).*

(A6) *(K, μ) is a loose ψ-tangle kit of order $k+1$. Stop.*

We begin the proof of Theorem 16 by proving the time complexity of Algorithm 17.

Lemma 18. *Algorithm 17 can be implemented to run in time $\text{Ø}(n^{3bk+2k+2}\gamma + bn^{2bk+3k+8}\gamma + bn^{2bk+3k+9})$ where γ is the time complexity of an oracle query.*

Proof. The set K is implemented as a list of valid pairs. With each pair $(A, B) \in K$ we associate $\mu(A, B)$ as an incidence vector.

We will count the time the algorithm spends in each step. By Lemma 11, there are at most Ø(n^{2k}) pairs of disjoint subsets with at most k elements. For each of them we have to determine the minimum separation.By Lemma 7, needs time Ø$(n^{2k+5}\gamma + n^{2k+6})$ for step (A1).

The step (A2) consists of testing all elements which can be done Ø$(n\gamma)$. The step (A3) can be implemented by testing all possible sets B, $|B| \leq k$, in time Ø$(n^k\gamma)$.

In (A4), testing (T3) requires time Ø(n).

Most of the time, the algorithm requires for step (A5). For the algorithm consider all b-tuples of pairs from K, $(A_1, B_1), \dots, (A_{b-1}, B_{b-1}), (A, B)$, AßU where $U = \cup_{i=1}^{b-1} \mu(A_i, B_i) \setminus B$ we compute minimal $Z_e \in \mathcal{M}(A \cup \{e\}, \overline{U})$ for every $e \in U \setminus \mu(A, B)$. This can be done in Ø$(n^6\gamma + n^7)$ time by Lemma 7. The correctness of testing (A5) is shown in the proof of Theorem 16.

If $\psi(Z_e) = \psi_{\min}(A, B)$, then try to find a minimal partition $[P_1, \dots, P_b]$ such that $\overline{Z_e}$ form one part and other parts are subsets of sets $\mu(A_i, B_i)$. That can be done applying Lemma 13 to the prepartition (C_i, D_i, b) where $C_b = \overline{Z_e}$, $D_b = Z_e$, $C_i = \emptyset$, $D_i = \overline{\mu(A_i, B_i)} \cup \overline{Z_e}$ for every $i = 1, \dots, b-1$. Note that necessarily $A \cup \{e\}$ß$Z_e = \overline{P_b}$.

If $\psi([P_1, \dots, P_b]) \leq k$ then all conditions of (T2) are satisfied so we set $\mu(A, B)$ to $\mu(A, B) \cup Z_e$. The existence of a partition $[P_1, \dots, P_b]$ can be determined in Ø$(n^{bk}\gamma + bn^{k+6}\gamma + bn^{k+7})$ by Lemma 13. So step (A5) takes Ø$(n^{3bk+1}\gamma + bn^{2bk+k+7}\gamma + bn^{2bk+k+8})$.

We will show that the algorithm will require time at most Ø$(n^{3bk+2k+2}\gamma + bn^{2bk+3k+8}\gamma + bn^{2bk+3k+9})$ for (A5) in total. Since $e \in Z_e$, we have increased $\mu(A, B)$ by at least one. There are Ø(n^{2k}) pairs in K and for each of them μ can increase at most n times. Hence we can get to step (A5) at most Ø(n^{2k+1}) times. The claim follows.

This finishes the proof. □

Lemma 19. *Let (K, μ) be a loose ψ-tangle kit of order $k + 1$. Let $e \in E$ such that $0 < \psi(\{e\}) \leq k$. For all $(\{e\}, B) \in K$, if $\psi_{\min}(\{e\}, B) = \psi(e)$, then $e \in \mu(\{e\}, B)$.*

Proof. By (T1), there exists $(A, D) \in K$ such that Aß$\{e\}$ß$\overline{D}$, $\psi_{\min}(A, D) = \psi(\{e\})$ and $e \in \mu(A, D)$. Let $(\{e\}, B) \in K$ be a pair such that $\psi_{\min}(\{e\}, B) = \psi(\{e\})$. Using (T2) for pairs (A, D), $(\{e\}, B)$ and set $C = \{e\}$, we get $e \in \mu(\{e\}, B)$ since $\psi(C, \overline{C}) = \psi(\{e\}) \leq k$. □

Of Theorem 16. Lemma 18 asserts that Algorithm 17 runs in time Ø$(n^{3bk+2k+2}\gamma + bn^{2bk+3k+8}\gamma + bn^{2bk+3k+9})$. Hence, it is sufficient to prove that the algorithm is correct, i.e., it constructs a loose ψ-tangle kit of order $k+1$ if it exists or shows that none exists.

Suppose there exists a loose ψ-tangle kit (K, μ'). Let μ_i be the value of μ at the i-th iteration at the beginning of step (A4). We claim $\mu_i(A,B)ß\mu'(A,B)$ for all $(A,B) \in K$ and every i.

First, we check that (T1) holds for (K, μ_i). If $\psi(\{e\}) = 0$ then $e \in \mu(\emptyset, \emptyset)$. If $0 < \psi(\{e\}) \leq k$, then, by Lemma 8, there is $Aß\{e\}ß\overline{B}$ such that $\max\{|A|, |B|\} \leq \psi_{\min}(A,B) = \psi(\{e\})$. Since $\psi(\emptyset) = 0$, we know that $A = \{e\}$. In step (A3), we have tested all possible sets B. So (T1) holds and, since the sets in μ never get smaller, (T1) holds through whole algorithm.

When $i = 1$, by Lemma 19, if $e \in \mu_1(e,B)$ then $e \in \mu'(e,B)$. Then the value of μ changes in step (A5). Let $(A_1,B_1),\ldots,(A_{b-1},B_{b-1}),(A,B) \in K$ be the pairs for which $[P_1,\ldots,P_b]$ is the corresponding partition obtained in step (A5). By assumption we know that $\psi([P_1,\ldots,P_p]) \leq k$, $P_iß\mu(A_i,B_i)ß\mu'(A,B)$, $\cup_{i=1}^{b-1} P_i = Z$ such that $AßZß\overline{B}$ and $\psi(Z) = \psi_{\min}(A,B)$. By (T2), it follows that $Zß\mu'(A,B)$. So we conclude that $\mu_{i+1}(A,B)ß\mu'(A,B)$.

Suppose the algorithm failed at step (A4). Then $E = \mu(\emptyset,\emptyset)ß\mu'(\emptyset,\emptyset)ßE$. Hence no loose ψ-tangle kit (K,μ') of order $k+1$ exists and so ψ has width at most k.

Suppose the algorithm finished at step (A6) with a pair (K,μ). We claim that (K,μ) is a loose ψ-tangle kit. We have shown that (K,μ) satisfies (T1). It also satisfies (T3) since it passed the test at (A4). Now, suppose (T2) is not true and there exists $(A_1,B_1),\ldots,(A_{b-1},B_{b-1}),(A,B) \in K$, $C = \cup_{i=1}^{b-1} C_i$ such that $C_iß\mu(A_i,B_i)$, $i = 1,\ldots,b$, $AßCß\overline{B}$, $\psi(C) = \psi_{\min}(A,B)$, $\psi([C_1,\ldots,C_{b-1},\overline{C}]) \leq k$, and $C \not ß \mu(A,B)$. Take $e \in C \setminus \mu(A,B)$. Let $Z_e \in \mathcal{M}(A \cup e, B)$ be a minimal one. Since $A \cup eßCß\overline{B}$, $\psi(Z_e) \leq \psi(C) = \psi_{\min}(A,B)$. Hence $\psi(Z_e) = \psi_{\min}(A,B)$. Since Z_e minimal, $Z_eßC$. By submodularity of ψ,

$$\psi([C_1,\ldots,C_{b-1},\overline{C}]) + \psi(Z_e) \geq \psi([C_1 \cap Z_e,\ldots,C_{b-1} \cap Z_e, \overline{C} \cup \overline{Z_e}]) + \psi(Z_e \cup C).$$

Since $Z_e \cup C = C$ and $\psi(C) = \psi(Z_e)$, we get $\psi([C_1 \cap Z_e,\ldots,C_{b-1} \cap Z_e, \overline{C} \cup \overline{Z_e}]) \leq k$ and it satisfies all condition required in step (A5) of our algorithm. That contradicts $C \not ß \mu(A,B)$. Hence (K,μ) satisfies (T2) and it is a loose ψ-tangle kit of order $k+1$. $\square$

7 Computing Tree-width

In this section, we show that Theorem 16 implies that it can be determined in polynomial time whether tree-width of a graph is at most k.

We restrict our attention to the following subclass of submodular partition functions. We say that a submodular partition function ψ is *monotone* if for every partiton $[A, B, \alpha]$ of E, $\psi([A, B, \alpha]) \geq \psi([A \cup B, \alpha])$. We assert that it is sufficient to study only partitions with at most three parts for a monotone submodular partition functions.

It is not difficult to show that a monotone submodular partition function has an optimal decomposition tree where all nodes have degree at most 3. On the other hand, it is not clear whether a monotone submodular partition function of width k can be modified to a submodular partition function of the same width with k-bounded partitions for $b = 3$. Despite this, the following lemma shows that Algorithm 17 works for monotone submodular partition functions with parameter $b = 3$.

Lemma 20. *Let ψ be a monotone submodular partition function on E. Then Algorithm 17 with parameters k and $b = 3$ decides correctly whether the width of ψ is at most k.*

Proof. Let (K, μ) be the pair constructed by Algorithm 17. If Algorithm 17 stops in step (A4), then there is no loose ψ-tangle kit of order $k+1$ since the constructed μ is still contained in any loose ψ-tangle kit of order $k + 1$ (see the proof of Theorem 16).

If Algorithm 17 stops in step (A6), then no loose ψ-tangle was found. Suppose there is a loose ψ-tangle and hence there are $(A_1, B_1), \ldots, (A_p, B_p)$, $(A, B) \in K$, $C = \cup_{i=1}^{p} C_i$ such that $C_i \text{ß} \mu(A_i, B_i)$, $A \text{ß} C \text{ß} \overline{B}$, $\psi(C) = \psi_{\min}(A, B)$ and $\psi([C_1, \ldots, C_p, \overline{C}]) \leq k$ but $C \not\text{ß} \mu(A, B)$. Choose such pairs that their number p is as small as possible.

Since $b = 3$, $p > 3$. Let $C' = C_1 \cup C_2$. Since ψ is monotone, $\psi(C') \leq \psi([C_1 \cup C_2, C_3, \ldots, C_p, \overline{C}]) \leq \psi([[C_1, \ldots, C_p, \overline{C}]) \leq k$. By Lemma 8, there is a pair (A', B') in K satisfying $A' \text{ß} C' \text{ß} \overline{B'}$, $\psi(C') = \psi_{\min}(A', B')$. Now, $(A_1, B_1), (A_2, B_2), (A', B'), C' = C_1 \cup C_2$ satisfy all conditions of step (A4). Hence $C' \text{ß} \mu(A', B')$. To derive a contradiction, consider the following pairs from K, $(A', B'), (A_3, B_3), \ldots, (A_p, B_p), (A, B)$, and take $C = C' \cup \bigcup_{i=3}^{p} C_i$. $\psi([C_1 \cup C_2, C_3, \ldots, C_p, \overline{C}]) \leq \psi([[C_1, \ldots, C_p, \overline{C}]) \leq k$, which contradicts the minimality of p. $\square$

Amini et. al. [1] showed that tree-width of a graph $G = (V, E)$ with minimum degree at least 2 is characterized by a submodular partition function δ_G on E, where $\delta_G(\alpha)$ is the size of the *border* of α, $\Delta(\alpha)$, defined as

$$\Delta([A_1, \ldots, A_p]) = \{x \in V(G) \mid \exists xy, xz \in E, xy \in A_i, xz \in A_j, i \neq j\}$$
$$\delta_G(\alpha) = |\Delta(\alpha)|$$

The tree-width of G is the width of δ_G minus one. It is not hard to see that δ_G is a monotone submodular partition function. Therefore, Algorithm 17 can be used to determine whether the tree-width of a graph is at most k in polynomial time.

8 Hardness of Submodular Partition Functions

We first have to define several auxiliary functions before we can establish our hardness result. Let g_n be the function $g_n : 2^E \to \mathbb{N}$ for $E = \{1, \ldots, 2n\}$ defined as $g_n(X) = \min\{|X|, |\overline{X}|\}$. We start our exposition with showing that g_n is submodular.

Lemma 21. *The function g_n is submodular for every n.*

Proof. Consider two subsets X and Y. If both $|X| \leq n$ and $|Y| \leq n$, then

$$\begin{aligned} g_n(X) + g_n(Y) &= |X| + |Y| = |X \cap Y| + |X \cup Y| \\ &\geq g_n(X \cap Y) + g_n(X \cup Y). \end{aligned}$$

If both $|X| > n$ and $|Y| > n$, we get the same result by the symmetry of g.

$$\begin{aligned} g_n(X) + g_n(Y) &= g_n(\overline{X}) + g_n(\overline{Y}) \geq g_n(\overline{X} \cap \overline{Y}) + g_n(\overline{X} \cup \overline{Y}) \\ &= g_n(X \cup Y) + g_n(X \cap Y) \end{aligned}$$

So suppose that $|X| > n$ and $|Y| \leq n$. We get

$$\begin{aligned} g_n(X) + g_n(Y) &= |\overline{X}| + |Y| = |\overline{X} \setminus Y| + |Y \setminus \overline{X}| + 2|\overline{X} \cap Y| \\ &\geq g_n(\overline{X} \setminus Y) + g_n(Y \setminus \overline{X}) = g_n(\overline{X} \cap \overline{Y}) + g_n(X \cap Y) \\ &= g_n(X \cup Y) + g_n(X \cap Y). \end{aligned}$$

This finishes the proof. □

The function g_n can be extended to a partition function ϕ_n on the ground set $E = \{1, \ldots, 2n\}$ by setting

$$\phi_n(\alpha) = \max_{i \in I} g_n(A_i).$$

A part A_i of α is *dominating* if $g_n(A_i) = \phi_n(\alpha)$. Note that, if α has a part with at least n elements, then that part is dominating.

We proceed by showing that the function ϕ_n is submodular.

Lemma 22. *The function ϕ_n is submodular for every n.*

Proof. We check the following inequality for all partitions $[A, \alpha]$ and $[B, \beta]$:

$$\phi_n([A, \alpha]) + \phi_n([B, \beta]) \geq \phi_n([A \cup \overline{B}, \alpha \cap B]) + \phi_n([B \cup \overline{A}, \beta \cap A]).$$

Since one of $A, \overline{A}$ and one of $B, \overline{B}$ has at least n elements, at least one of the parts $A \cup \overline{B}$ or $B \cup \overline{A}$ in this inequality has at least n elements and hence it is dominating. If both $A \cup \overline{B}$ and $B \cup \overline{A}$ are dominating, then the submodularity of ϕ_n follows from the submodularity of g:

$$\begin{aligned}\phi_n([A, \alpha]) + \phi_n([B, \beta]) &\geq g_n(A) + g_n(B) = g_n(A) + g_n(\overline{B}) \\ &\geq g_n(A \cap \overline{B}) + g_n(A \cup \overline{B}) = g_n(\overline{A} \cup B) + g_n(A \cup \overline{B}) \\ &= \phi_n([A \cup \overline{B}, \alpha \cap B]) + \phi_n([B \cup \overline{A}, \beta \cap A])\end{aligned}$$

Suppose that $A \cup \overline{B}$ is not dominating, so take an $A_i \in \alpha$ such that $A_i \cap B$ is dominating. Since $|B| \geq n$ and $A_i \subseteq \overline{A}$, it holds that $g_n(A_i \cup B) \geq g_n(B \cup \overline{A})$. We use this inequality to prove the submodularity as follows:

$$\begin{aligned}\phi_n([A, \alpha]) + \phi_n([B, \beta]) &\geq g_n(A_i) + g_n(B) \geq g_n(A_i \cap B) + g_n(A_i \cup B) \\ &\geq g_n(A_i \cap B) + g_n(B \cup \overline{A}) \\ &= \phi_n([A \cup \overline{B}, \alpha \cap B]) + \phi_n([B \cup \overline{A}, \beta \cap A])\end{aligned}$$

The case when $B \cup \overline{A}$ is not dominating follows by symmetry. $\square$

Values of the function ϕ_n range between 0 and n. We now truncate the function and define the following partition function $\phi_{n,k}$ on $E = \{1, \ldots, 2n\}$ as follows:

$$\phi_{n,k}(\alpha) = \min\{\phi_n(\alpha), k\}.$$

Next, we show that the function ϕ_n stays submodular after the truncation.

Lemma 23. *The function $\phi_{n,k}$ is submodular for every n and k.*

Proof. Let us consider two partitions $[A, \alpha]$ and $[B, \beta]$ that violates the inequality (2):

$$\phi_{n,k}([A, \alpha]) + \phi_{n,k}([B, \beta]) \geq \phi_{n,k}([A \cup \overline{B}, \alpha \cap B]) + \phi_{n,k}([B \cup \overline{A}, \beta \cap A]).$$

Since $\phi_{n,k}(\gamma) \leq \phi_n(\gamma)$ for all partitions γ, at least one of $\phi_n([A, \alpha])$ or $\phi_n([B, \beta])$ is larger than k. If both of them are, then the inequality trivially holds. Suppose that $\phi_n([A, \alpha]) < k$. We will show that at least one of $\phi_n([A \cup \overline{B}, \alpha \cap B])$ or $\phi_n([B \cup \overline{A}, \beta \cap A])$ is smaller or equal to $\phi_n([A, \alpha])$.

If $|A| \geq n$, then $\phi_n([A \cup \overline{B}, \alpha \cap B]) \leq \phi_n([A, \alpha])$ since $A \cup \overline{B}$ is the dominating part and $g_n(A \cup \overline{B}) \leq g_n(A) \leq \phi_n([A, \alpha])$. If $|A| < n$, then $\phi_n([B \cup \overline{A}, \beta \cap A]) \leq \phi_n([A, \alpha])$ since $B \cup \overline{A}$ is the dominating part and $g_n(B \cup \overline{A}) \leq g_n(\overline{A}) \leq \phi_n([A, \alpha])$. This finishes the proof. $\square$

Now, we use the function $\phi_{n,3}$ to construct partition functions ϕ_n^* and $\phi_{n,\beta}^*$ which appear in our hardness result. The function ϕ_n^* is defined as

$$\phi_n^*(\alpha) = \begin{cases} \phi_{n,3}(\alpha) & \text{if } \alpha \text{ has at most three non-empty parts, and} \\ 3 & \text{otherwise.} \end{cases}$$

For a partition β of $\{1, \ldots, 2n\}$ into n two-element subsets, the function $\phi_{n,\beta}^*$ is then defined as

$$\phi_{n,\beta}^*(\alpha) = \begin{cases} \phi_{n,3}(\alpha) & \text{if } \alpha \text{ has at most three non-empty parts,} \\ 2 & \text{if } \alpha = \beta, \text{ and} \\ 3 & \text{otherwise.} \end{cases}$$

First, we show that these functions are submodular.

Lemma 24. *The function ϕ_n^* is submodular for every n.*

Proof. Observe the following:

- If $\phi_{n,3}(\alpha) = 0$, then also $\phi_n^*(\alpha) = 0$.
- If $\phi_{n,3}(\alpha) = 1$, then $\phi_n^*(\alpha) = 1$ unless α is a set of singletons where $\phi_n^*(\alpha) = 3$.
- If $\phi_{n,3}(\alpha) = 2$, then $\phi_n^*(\alpha) = 2$ unless α has more than three non-empty parts and every part of α is a pair or a singleton.

Therefore the functions $\phi_{n,3}$ and ϕ_n^* differ only on partitions consisting of singletons and pairs.

Let us assume for a contradiction that ϕ_n^* is not submodular. Since $\phi_n^*(\alpha) \geq \phi_{n,3}(\alpha)$ for all partitions α, the violation of the submodularity is caused by an increase on the right-hand side of (2). Consider partitions $[A, \alpha]$ and $[B, \beta]$ violating (2). Hence, say, $\gamma = [A \cup \overline{B}, \alpha \cap B]$ is that partition containing only singletons and pairs. Since γ has all parts of size at most two, $|\overline{B}| \leq 2$. If $\overline{A} \cap \overline{B} = \emptyset$, then $\overline{B} \subseteq A$ and $\overline{A} \subseteq B$. Therefore $\gamma = [A, \alpha]$, $[B \cup \overline{A}, \beta \cap A] = [B, \beta]$ and the inequality trivially holds. So we can assume that $|B \cup \overline{A}| > |B|$ and since $2n - 2 \leq |B| < 2n$, by the definition of ϕ_n^*

$$\phi_n^*([B, \beta]) > \phi_n^*([B \cup \overline{A}, \beta \cap A]) \; . \tag{3}$$

Since the number of non-empty parts of γ is at least 4, the number of non-empty parts of $[A, \alpha]$ is at least 3 and therefore $\phi_n^*([A, \alpha]) \geq 2$ by the definition of ϕ_n^*. The submodularity follows from (3) and the fact that $\phi_n^*(\gamma) \leq 3 \leq \phi_n^*([A, \alpha]) + 1$. □

Lemma 25. *The function $\phi^*_{n,\beta}$ is submodular for every $n \geq 4$ and for every partition β consisting only of two-element sets.*

Proof. Since ϕ^*_n and $\phi^*_{n,\beta}$ differ only on the partition β where $\phi^*_n(\beta) > \phi^*_{n,\beta}(\beta)$, β has to be on the left-hand side of the inequality (2) to violate it. Let $[A, \alpha]$ and $\beta = [C, \gamma]$ be the partitions violating (2):

$$\phi^*_{n,\beta}([A, \alpha]) + \phi^*_{n,\beta}([C, \gamma]) \geq \phi^*_{n,\beta}([A \cup \overline{C}, \alpha \cap C]) + \phi^*_{n,\beta}([C \cup \overline{A}, \gamma \cap A])$$

Since β consists only of two-elements sets, $|C| = 2$. Therefore, $\phi^*_{n,\beta}([A \cup \overline{C}, \alpha \cap C]) \leq 2$. To violate (2) it is neccessary to have $\phi^*_{n,\beta}([A, \alpha]) \leq 2$. If $|A| \leq 2$, then $|C \cup \overline{A}| \geq 2n - |A|$ and $\phi^*_{n,\beta}([C \cup \overline{A}, \gamma \cap A]) \leq \phi^*_{n,\beta}([A, \alpha])$, contradicting the assumption. Therefore A has to have at least $2n - 2$ elements and it follows that $\phi^*_{n,\beta}([A \cup \overline{C}, \alpha \cap C]) \leq \phi^*_{n,\beta}([A, \alpha])$.

If $\overline{C} \subseteq A$, then $\overline{A} \subseteq C$ and $\phi^*_{n,\beta}([C \cup \overline{A}, \gamma \cap A]) = \phi^*_{n,\beta}([C, \gamma])$, contradicting the assumption. Therefore $|A \cup \overline{C}| > |A|$ giving $\phi^*_{n,\beta}([A, \alpha]) > \phi^*_{n,\beta}([A \cup \overline{C}, \alpha \cap C])$. Since $\phi^*_{n,\beta}(\beta) + 1 = 3 \geq \phi^*_{n,\beta}([C \cup \overline{A}, \gamma \cap A])$, the inequality (2) holds — a contradiction. $\square$

In the proof of the main theorem we will use the fact that the width of the function ϕ^*_n is three while the width of the modified function $\phi^*_{n,\beta}$ is two. To see that width of $\phi^*_{n,\beta}$ is at most two, just consider the following decomposition tree T of $\phi^*_{n,\beta}$. T has a root x with n children $v_1, \ldots, v_n$ each v_i connected to two leaves corresponding to the two elements in β_i. Since $\phi^*_{n,\beta}(\alpha_x) = \phi^*_{n,\beta}(\beta) = 2$ and $\phi^*_{n,\beta}(\alpha_{v_i}) = 2$, for $i = 1, \ldots, n$, the decomposition tree T has width two. In the next lemma, we show that the width of ϕ^*_n is three.

Lemma 26. *For $n \geq 4$, the width of ϕ^*_n is three.*

Proof. Let T be a decomposition tree of ϕ^*_n of width smaller than three. We assume there are no nodes of degree two in T since we can contract them obtaining a smaller decomposition tree of the same width. Since every internal node v of T of degree larger than three corresponds to a partition α_v of E with more than three parts (thus $\phi^*_n(\alpha_v) = 3$), there are no such vertices in T and T is a ternary tree. Consider an arbitrary internal node v of T with less than two leaves as neighbors. There has to be such a vertex v since there are at most n vertices with two leaves as neighbors but there are $2(n-1)$ internal nodes. For such a vertex v, α_v contains a part with at least three elements and at most $2n - 3$ elements implying $\phi^*_n(\alpha_v) = 3$. This finishes the proof. $\square$

We are now ready to establish our hardness result. We assume the existence of an algorithm and show that it cannot discover a small discrepancy between a submodular partition function having width three and two.

Theorem 27. *There is no sub-exponential algorithm for determining whether the width of an oracle-given submodular partition function on a set with $2n$ elements is at most two.*

Proof. Assume that there exists such a sub-exponential algorithm $\mathcal{A}$ and run $\mathcal{A}$ for the submodular partition function ϕ_n^*. The algorithm $\mathcal{A}$ must clearly output that the width ϕ_n^* is at least three. Since the running time of the algorithm is sub-exponential, for n sufficiently large, there exists a partition β of $\{1, \ldots, 2n\}$ into n two-element subsets such that $\mathcal{A}$ never queries β since the number of such partitions is

$$\frac{(2n)!}{n!2^n} = (2n-1)(2n-3)\cdots 3 \cdot 1 \geq n!$$

and $\mathcal{A}$ cannot query all of them because of its running time. However, the algorithm $\mathcal{A}$ for $\phi_{n,\beta}^*$ performs the same steps and thus it outputs that the width of $\phi_{n,\beta}^*$ is at least three which is not correct. $\square$

Using Yao's principle, Theorem 27 also implies the following lower bound for randomized algorithms:

Corollary 28. *For every randomized algorithm determining whether the width of an oracle-given submodular partition function on a set with $2n$ elements is at most two, there exists a submodular partition function ψ such that the expected running time of the algorithm for ψ is exponential in n.*

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