QUOTIENTS AND COLIMITS OF κ-QUANTALES

RICHARD N. BALL AND ALEŠ PULTR

Dedicated to Eraldo Giuli on the occasion of his 70th birthday

ABSTRACT. Let κ**Qnt** be the category of of κ-quantales, quantales closed under κ-joins in which the monoid identity is the largest element. (κ is an infinite regular cardinal.) Although the lack of lattice completeness in this setting would seem to mitigate against the techniques which lend themselves so readily to the calculation of frame quotients, we show how to easily compute κ**Qnt** quotients by applying generalizations of the frame techniques to suitable extensions of this category.

The second major tool in the analysis is the free κ-quantale over a λ-quantale, $\kappa \geq \lambda$. Surprisingly, these can be characterized intrinsically, and the generating sub-κ-quantale can even be identified. The result that the λ-free κ-quantales coincide with the λ-coherent κ-quantales directly generalizes Madden's corresponding result for κ-frames.

These tools permit a direct and intuitive construction of κ**Qnt** colimits. We provide two applications: an intrinsic characterization of κ**Qnt** colimits, and of free (over sets) κ-quantales. The latter is a direct generalization of Whitman's condition for distributive lattices.

1. Introduction

Factoring frames is a fairly transparent procedure. A frame is a complete lattice satisfying the distributive law

$$(\bigvee_J a_i) \wedge b = \bigvee_J (a_i \wedge b)$$

and frame homomorphisms (and, hence, congruences) respect all joins and finite meets. Thus

Key words and phrases. distributive lattices, κ-quantales and κ-frames, quotients, colimits.

Both authors gratefully acknowldege the support of project 1M0545 of the Ministry of Education of the Czech Republic. In addition, the authors would like to express their thanks to the Faculty Research Fund, and to the Department of Mathematics, of the University of Denver.

- the completeness yields a canonical representation of the congruence classes by their largest elements,
- the Heyting operation following from the distributivity law (preserving suprema by the maps $x \mapsto a \wedge x$ makes them left Galois adjoints) provides a simple technique for extending a generating relation; it often explicitly yields the resulting quotient without really bothering with the congruence itself.

(See, e.g., [5] and [11].) Almost the same holds, more generally, for commutative quantales with top unit with the adjoint to the multiplication in place of the Heyting operation mentioned.

For distributive lattices, σ-frames, q-lattices, etc., neither of these advantageous circumstances obtain. Still, by use of suitable extensions we can exploit the technique of frames/quantales to obtain transparent representations of their respective quotient algebras. One of the purposes of this article is to show how easily this can be done.

Another motivation for our investigation goes back to [3], published in 1993 and related to the much older [2] of 1976. There it was shown how to obtain colimits of distributive algebras in linear categories using the associated colimits of the underlying structures. The important point here is the parallel between phenomena like obtaining coproducts of commutative rings as tensor products of the underlying abelian groups, and the quite analogous construction of coproducts of frames based on coproducts of the underlying meet-semilattices. Using our technique we can in some cases (distributive lattices, σ-frames, q-lattices) replace the abstract categorical construction by a quite explicit one. As an application we present a simple proof of a basic intrinsic fact of the resulting algebras.

2. Preliminaries

In this section we set out the basic definitions and notation, and then develop the machinery of quantale quotients. The latter is a generalization of the corresponding frame technique (see [5] or [11]), and is fundamental to everything that follows.

2.1. **κ-quantales.** If M is a subset of a poset $(X, \leq)$ we will denote the *down-set generated by M* by

$$\downarrow M = \{X : \exists m \in M,\ x \leq m\},$$

and call M a *down-set* if $M = \downarrow M$. We abbreviate $\downarrow \{a\}$ to $\downarrow a$.

Throughout this article κ and λ designate either infinite regular cardinals, the symbol 0, or the symbol ∞. We assume that $0 \leq \kappa \leq \infty$ for *infinite regular cardinals λ, and we assume that $\lambda \le \kappa$.* A *κ-set* is a set of cardinality strictly less than κ; there are no 0-sets, an ω-set is a finite set, an ω_1-set is a countable set, and any set is an ∞-set. A *κ-subset* of a given set A is a subset $B \subseteq A$ which is a κ-set; we write $B \subseteq_\kappa A$. If, in a poset A with subset B, $a = \bigvee B_0$ for some $B_0 \subseteq_\kappa B$ then we will say that *a is a κ-join of elements of B.* When the join operation is the union of subsets, we will speak of a as being a *κ-union.*

We will be concerned with *κ-quantales* $(L, \cdot, 1, \le)$, structures in which $(L, \cdot, 1)$ is a commutative monoid and all κ-subsets possess joins, such that

- 1 is the top of $(L, \le)$, and
- the monoid operation distributes over κ-joins.

If there is no danger of confusion, the operation is denoted simply by juxtaposition. A 0-quantale is simply a commutative monoid, devoid of order. $\aleph_0$-quantales, the counterparts of distributive lattices, are referred to as *q-lattices.* ∞-quantales are referred to as simply quantales. The κ-morphisms preserve all the assumed suprema and the monoid structure, as do the congruences. The resulting category will be designated by κ**Qnt**; at whim we will substitute the synonymous notations **CMon** for **0Qnt** and **Qnt** for ∞**Qnt**. Note that in all cases except $\kappa = 0$ we have the bottom element $0 = \sup \emptyset$, and that it is preserved by homomorphisms.

Observation 2.1.1. *Let L be a κ-quantale, $\kappa > 0$.*

1. *By distributivity, $x \cdot y$ is monotone in both variables.*
2. *$xy \le x, y$, since 1 is the top, and $x \cdot 0 = 0$ since $x \cdot 0 \le 1 \cdot 0 = 0$.*
3. *$xy = x \wedge y$ iff the monoid is idempotent, since in that case $z \le x, y$ implies $z = z \cdot z \le x \cdot y$.*

We use the term κ-quantale as an abbreviation for commutative κ-quantale with top unit. In the general theory of quantales these entities are not necessarily commutative, and the top element does not have to be the unit of the multiplication. (For more about quantales see, e.g., [4], [9], and [10].) In the particularly important case of an idempotent multiplication (that is, of meet), the κ-quantales are precisely the κ-frames; ∞-quantales are usually called frames, $\aleph_1$-quantales are usually called σ-frames, and the $\aleph_0$-quantales are,of course, precisely the bounded distributive lattices. The resulting categories will be denoted by **Frm**, κ**Frm**, (especially σ**Frm**), and **DLat**.

2.2. **Quantale Quotients.** Due to the completeness and to the Heyting structure, quotients of frames are easy to obtain. In this subsection we will generalize the frame factorization procedure to quantales, and in later sections we will use this machinery to factor some structures that do not have the above advantages. *Throughout this section L represents a quantale.*

The distributivity $a \cdot \bigvee b_i = \bigvee (a \cdot b_i)$ in L can be interpreted as saying that the mappings $(x \mapsto a \cdot x) : L \to L$ preserve all suprema, and hence they are left Galois adjoints. This gives rise to an operation $\to$ on L such that

$$ab \le c \quad \text{iff} \quad a \le b \to c.$$

Let R be a binary relation on L. An element $s \in S$ is said to be R-*saturated*, or simply *saturated*, if

$$\forall a, b, c \quad aRb \implies (ac \le s \text{ iff } bc \le s).$$

The set of all saturated elements will be denoted by

$$L/R.$$

Observation 2.2.1. *An arbitrary meet of saturated elements is saturated. And if s is saturated then so is every element of the form $x \to s$, $x \in L$.*

Proof. We have $ac \le x \to s$ iff $acx \le s$ iff $bcx \le s$ iff $bc \le x \to s$. □

Define a mapping

$$\mu_R \equiv (x \mapsto \bigwedge_{x \le s \in L/R} s) : L \longrightarrow L/R.$$

We have

Lemma 2.2.2. (1) *$x \le \mu(x)$, μ is monotone, and $\mu\mu(x) = \mu(x)$,*
(2) *$\mu(xy) = \mu(\mu(x)\mu(y))$.*

Proof. (1) is trivial. (2) For saturated s we have $\mu(xy) \le s$ iff $xy \le s$ iff $x \le y \to s$ iff $\mu(x) \le y \to s$ iff $y \le \mu(x) \to s$ iff $\mu(y) \le \mu(x) \to s$ iff $\mu(x)\mu(y) \le s$ iff $\mu(\mu(x)\mu(y)) \le s$.. □

In the case of frames one has more, namely $\mu(xy) = \mu(x)\mu(y)$. This, together with the property (1), makes μ a *nucleus*, one of the basic means of describing sublocales (generalized subspaces). See, e.g., [5] or [6].

Theorem 2.2.3. *L/R is a complete lattice, and if it is endowed with the multiplication $x * y = \mu(xy)$ it becomes a quantale and μ_R becomes an quantale morphism $L \to L/R$.*

If aRb then $\mu_R(a) = \mu_R(b)$, and for every quantale morphism $h : L \to M$ such that $aRb \Rightarrow h(a) = h(b)$ there is a unique quantale morphism $\overline{h} : L/R \to M$ such that $\overline{h}\mu_R = h$. Moreover, $\overline{h}(a) = h(a)$ for all $a \in L/R$.

Proof. L/R is a complete lattice with the supremum $\bigsqcup a_i = \mu(\bigvee a_i)$: indeed, if $b \geq a_i$ for all i, and if $b \in L/R$ then $b \geq \bigvee a_i$, and $b = \mu(b) \geq \mu(\bigvee a_i)$. μ preserves the multiplication by Lemma 2.2.2(2), and for $a_i \in L$ we have $\mu(\bigvee a_i) \leq \mu(\bigvee \mu(a_i)) = \bigsqcup \mu(a_i) \leq \mu(\bigvee a_i)$. Thus, μ also preserves all joins. Since it is onto, this makes L/R a quantale and μ a quantale morphism. Further, if aRb then $b \leq \mu(a)$ since $a \leq \mu(a)$ and $\mu(a)$ is saturated. Hence $\mu(b) \leq \mu(a)$ and by symmetry $\mu(b) = \mu(a)$.

Let $h : L \to M$ be such that $aRb \;\Rightarrow\; h(a) = h(b)$. We first claim that $h\mu\,(x) = h\,(x)$, $x \in L$. To verify this claim, set

$$\sigma(x) = \bigvee_{h(y) \leq h(x)} y.$$

Obviously

$$(*) \qquad x \leq \sigma(x) \quad \text{and} \quad h\sigma(x) = h(x).$$

Let aRb and $ac \leq \sigma(x)$. Then $h(bc) = h(ac) \leq h\sigma(x) = h(x)$ and hence $bc \leq \sigma(x)$. Thus, $\sigma\,(x)$ is saturated. Combining this fact with $(*)$ we obtain that $x \leq \mu(x) \leq \sigma(x)$ and hence

$$h(x) \leq h\mu(x) \leq h\sigma(x) = h(x),$$

which proves the claim.

To complete the proof of the theorem, define $\overline{h} : L/R \to M$ to be the restriction of h to L/R. Then

$$\overline{h}\left(\bigsqcup_I x_i\right) = h\left(\mu\left(\bigvee_I x_i\right)\right) = h\left(\bigvee_I x_i\right) = \bigvee_I h\,(x_i) = \bigvee_I \overline{h}\,(x_i)\,,$$
$$\overline{h}\,(x * y) = h\,(\mu\,(xy)) = h\,(xy) = h\,(x)\,h\,(y) = \overline{h}\,(x)\,\overline{h}\,(y)\,,$$

so that $\overline{h}$ is the morphism we seek. □

Often it is easy to find transparent formulas characterizing the saturated elements which make the quotient fairly transparent (see Section 4 below). This is sometimes helped by special properties of the initial relation R. We easily deduce the following

Proposition 2.2.4. *Let C be a join basis of L and let $R \subseteq L \times L$ be such that*

$$\forall a, b \in L \;\forall c \in C \quad aRb \;\Rightarrow\; (ac)R(bc).$$

Then $s \in L$ is R-saturated iff

$$aRb \quad \Rightarrow \quad (a \leq s \quad \textit{iff} \quad b \leq s).$$

If, moreover, $aRb \;\Rightarrow\; a \le b$ *this reduces to*

$$aRb \quad\Rightarrow\quad (a \le s \quad\Rightarrow\quad b \le s),$$

or, trivially rewritten, to

$$aRb \,\&\, (a \le s) \quad\Rightarrow\quad b \le s.$$

3. Free κ-quantales

Keeping in mind our convention that $0 \le \lambda \le \kappa \le \infty$, we have the forgetful functor $\mathfrak{U}^\kappa_\lambda : \kappa\mathbf{Qnt} \to \lambda\mathbf{Qnt}$, which we often use but seldom mention, and whose adjoint $\mathfrak{F}^\lambda_\kappa : \lambda\mathbf{Qnt} \to \kappa\mathbf{Qnt}$ we analyze in this section. For a given λ-quantale L, we refer to $\mathfrak{F}^\lambda_\kappa L$ as the *free κ-quantale over L*. We begin by describing $\mathfrak{F}^0_\kappa L$.

3.1. **The free κ-quantale over a commutative monoid .** Fix $\kappa > 0$. A *pre-ideal* in a commutative monoid S is a subset $U \subseteq S$ such that

$$u \in U \,\&\, s \in S \quad\Longrightarrow\quad us \in U.$$

Though a pre-ideal need not be a down-set, a down-set is a pre-ideal in any quantale by Observation 2.1.1(2), and the pre-ideals of a meet-semilattice are exactly the down-sets. The smallest pre-ideal containing an element $a \in S$ is obviously the principal pre-ideal

$$[a] = \{as : s \in S\}\,.$$

In particular, in the semilattice case $[a] = \,\downarrow a$. The pre-ideal generated by an arbitrary subset $A \subseteq S$ is

$$[A] \equiv \{as : a \in A,\ s \in S\} = \bigcup_A [a]\,.$$

Lemma 3.1.1. *Let S be a commutative monoid.*

1. *If U_i, $i \in I$, are pre-ideals then so is $\bigcup_I U_i$.*
2. *If U and V are pre-ideals then $U \cdot V = \{uv : u \in U,\ v \in V\}$ is a pre-ideal. This operation is associative and commutative. If the monoid is idempotent, i.e., a meet semilattice, then $U \cdot U = U$.*
3. *$U \cdot S = U$.*
4. *$U \cdot (\bigcup_I V_i) = \bigcup_I (U \cdot V_i)$.*
5. *$[a] \cdot [b] = [ab]$, and $[1] = S$.*

Proof. (1) is trivial. (2) If $u \in U$, $v \in V$ and $x \in S$ then $(uv)x = u(vx)$ with $vx \in V$. Associativity, commutativity, and the idempotent case are obvious. (3) By definition $US \subseteq U$, but because of the unit we have $US \supseteq U$. (4) $x \in U \cdot \bigcup V_i$ iff $x = uv$ with $u \in U$ and $v \in \bigcup U_i$ iff there is an i such that $x = uv$ with $u \in U$ and $v \in V_i$ iff $x \in \bigcup(U \cdot V_i)$. (5) Obviously $[ab] \subseteq [a]\,[b]$, and if $u \in [a][b]$ then $u = axby = (ab)(xy) \in [ab]$. $[1] = \{1x : s \in S\} = S$. $\square$

For a commutative monoid S set

$$\mathfrak{F}^0_\kappa S \equiv \{[A] : A \subseteq_\kappa S\}\,,$$

endowed with the operations of $U \cdot V$ and κ-unions. We write $\mathfrak{F}^0_\mathfrak{f}$ for $\mathfrak{F}^0_\omega$, and we abbreviate $\mathfrak{F}^0_\infty S$ to $\mathfrak{F}^0 S$. Further, we define the mapping

$$\rho^0_{\kappa S} : S \to \mathfrak{F}^0_\kappa S$$

by setting $\rho^0_{\kappa S}(a) = [a]$. We abbreviate $\rho^0_{\infty S}$ to ρ^0_S. By Lemma 3.1.1, $\mathfrak{F}^0_\kappa S$ is a κ-quantale and $\rho^0_{\kappa S}$ is a κ-morphism, one which is readily seen to be injective. If S is a meet-semilattice (the idempotent case), $\mathfrak{F}^0 S$ is the down-set frame; in particular, $\mathfrak{F}^0_\mathfrak{f} S$ is a distributive lattice.

Proposition 3.1.2. *$\rho^0_{\kappa S} : S \to \mathfrak{F}^0_\kappa S$ is the free κ-quantale over the commutative monoid S. That is, for every κ-quantale L and monoid homomorphism $h : S \to L$ there is precisely one κ-morphism $f : \mathfrak{F}^0_\kappa S \to L$ such that the diagram commutes.*

$$\begin{tikzcd} \mathfrak{F}^0_\kappa S \arrow[r, "f"] & L \\ S \arrow[u, "\rho^0_\kappa"] \arrow[ur, "h"'] & \end{tikzcd}$$

Proof. Since each $U \in \mathfrak{F}^0_\kappa S$ has the form $[A] = \bigcup_A [a]$, $A \subseteq_\kappa L$, the desired f has to satisfy the formula

$$(*) \qquad f(U) = \bigvee_A h(a).$$

This proves the uniqueness of the morphism. Now take $(*)$ for a definition of a mapping $f : \mathfrak{F}^0_\kappa L \to L$. This f obviously preserves the assumed suprema. It preserves the multiplication as well:

$$f(U)\,f(V) = \bigvee_{a\in U, b\in V} h(a)\,h(b) = \bigvee_{a\in U, b\in V} h(ab) = \bigvee_{c\in UV} h(c) = f(UV)\,.$$

Finally, if $b \in [a]$ then $b = ax$ and $h(b) = h(a)h(x) \leq h(a)$, and we conclude that $f([a]) = \bigvee_{[a]} h(b) = h(a)$. $\square$

3.2. **The free quantale over a λ-quantale, $\lambda > 0$.** In order to construct $\mathfrak{F}^{\lambda}L$, the free quantale over a λ-quantale L, $\lambda > 0$, a good place to start might be with the free quantale $\mathfrak{F}^0\mathfrak{U}_0^{\lambda}L$ over the commutative monoid $\mathfrak{U}_0^{\lambda}L$ underlying L. This structure certainly has the freeness we seek, but in the passage from L to $\mathfrak{U}_0^{\lambda}L$ we have lost the order on L, so that the natural embedding $a \longmapsto [a]$ need not preserve the λ-joins in L. We may restore the order given on L by identifying $[A]$ with $[b]$ for all $A \subseteq_{\lambda} L$ with $b = \bigvee A$, that is, by factoring $\mathfrak{F}^0L \equiv \mathfrak{F}^0\mathfrak{U}_0^{\lambda}L$ by the relation

$$R = \left\{([A], [b]) : A \subseteq_{\lambda} L \text{ with } b = \bigvee A\right\}.$$

We denote the resulting quotient $(\mathfrak{F}^0L)/R$, by $\mathfrak{F}^{\lambda}_{\infty}L$ and abbreviate this to $\mathfrak{F}^{\lambda}L$, and we denote the quotient map by $\mu : \mathfrak{F}^0L \to \mathfrak{F}^{\lambda}_{\infty}L$. Because R identifies the join of the images of the elements of a κ-subset of L with the image of its join, the map $\mu\rho^0_L$ is a λ-morphism $L \to \mathfrak{F}^{\lambda}L$; we denote this morphism by $\rho^{\lambda}_{\infty L}$, abbreviated to ρ^{λ}_L.[1]

Proposition 3.2.1. *$\rho^{\lambda}_L : L \to \mathfrak{F}^{\lambda}L$ is the free quantale over a λ-quantale L. That is, for every quantale M and λ-morphism $h : L \to M$ there is precisely one quantale morphism $f : \mathfrak{F}^{\lambda}L \to M$ such that $f\rho^{\lambda}_L = h$.*

Proof. When viewed as the underlying monoid homomorphism, h gives rise (via Proposition 3.1.2 with $\kappa = \infty$) to a unique quantale morphism $h' : \mathfrak{F}^0L \to M$ such that $h'\rho^0_L = h$. Since, for $A \subseteq_{\lambda} L$ with $b = \bigvee A$,

$$\begin{aligned} h'([A]) &= h'\left(\bigvee_A [a]\right) = \bigvee_A h'([a]) = \bigvee_A h'\rho^{\lambda}_L(a) = \bigvee_A h(a) = h\left(\bigvee_A a\right) \\ &= h(b) = h'\rho^0_L(b) = h'([b]), \end{aligned}$$

it follows that h' factors through μ, say $h' = f\mu$. Then $f\rho^{\lambda}_L = f\mu\rho^0_L = h'\rho^0_L = h$. And f is unique with this property, for $f\rho^{\lambda}_L = f'\rho^{\lambda}_L$ implies $f = f'$ since $\rho^0_L[L] = \{[a] : a \in L\}$ generates $\mathfrak{F}^0L$ as a quantale. □

Let us examine the elements of $\mathfrak{F}^{\lambda}L$ in more detail. The explicit description of these elements provided by Proposition 3.2.2 will constitute the working definition of $\mathfrak{F}^{\lambda}L$, and also of the embedding $\rho^{\lambda}_L : L \to \mathfrak{F}^{\lambda}L$. A λ-*ideal* in a λ-quantale L is a down-set $U \subseteq L$ such that $\bigvee A \in U$ for all $A \subseteq_{\lambda} U$. We remind the reader that λ-ideals are pre-ideals because down-sets are pre-ideals.

[1] There is a minor abuse of notation going on here. ρ^0_L is the map $a \longmapsto [a]$ from $\mathfrak{U}_0^{\lambda}L$ to $\mathfrak{U}_0\mathfrak{F}^0\mathfrak{U}_0^{\lambda}L$, and ρ^{λ}_L is the unique λ-morphism for which $\mathfrak{U}_0^{\lambda}\rho^{\lambda}_L = \mathfrak{U}_0^{\lambda}\mu \circ \rho^0_L$.

Proposition 3.2.2. *Let L be a λ-quantale. Then a pre-ideal $U \subseteq L$ is R-saturated iff it is a λ-ideal, i.e.,*

$$\mathfrak{F}^{\lambda} L = \{U : U \text{ is a } \lambda\text{-ideal in } L\}.$$

For $U, V, V_i \in \mathfrak{F}^{\lambda}_{\infty} L$, $i \in I$,

$$U \cdot V = \downarrow \{uv : u \in U,\ v \in V\},$$

$$\bigvee_I V_i = \downarrow \left\{ \bigvee A : A \subseteq_{\lambda} \bigcup_I V_i \right\}$$

And $\rho^{\lambda}_{L}(a) = \downarrow a$ for all $a \in L$.

Proof. The saturation condition for a pre-ideal $U \subseteq L$ is this: for all $A \subseteq_{\lambda} L$ with $b = \bigvee A$ and all pre-ideals V,

$$[A]\, V \subseteq U \text{ iff } bV \subseteq U.$$

Taking $V = [1] = L$ and using the implication from left to right, this condition implies that U is closed under λ-joins. Taking $V = [1]$ and $A = \{a, b\}$ with $a \leq b$ and using the implication from right to left, this condition implies that U is a down-set. Thus a saturated pre-ideal is a λ-ideal. On the other hand, it is straightforward to verify that a λ-ideal is a saturated pre-ideal.

We leave it to the reader to perform the routine verification that the operations in $\mathfrak{F}^{\lambda} L$ are as displayed. And $\rho^{\kappa}_{L}(a) = \downarrow a$ just because $\downarrow a$ is the smallest κ-ideal containing a. □

Note that in a λ-frame, and in a bounded distributive lattice in particular, the pre-ideals are automatically down-sets. However, even in that case the definition of $U \cdot V$ given in Proposition 3.2.2 differs from that given in Lemma 3.1.1. In fact, even in the very simplest instance when $\kappa = \aleph_0$, an element $u_1 v_1 \vee u_2 v_2$ is just majorized by $(u_1 \vee u_2)(v_1 \vee v_2)$, while there is no reason that it should lie in $\{uv : u \in U, v \in V\}$ itself.

3.3. **The free κ-quantale over a λ-quantale, $\lambda > 0$.** With $\mathfrak{F}^{\lambda} L$ in hand, we may now construct $\mathfrak{F}^{\lambda}_{\kappa} L$, the free κ-quantale over a given λ-quantale L. For that purpose, consider a given λ-quantale L. The smallest λ-ideal containing a subset $A \subseteq L$ is

$$\langle A \rangle_{\lambda} \equiv \downarrow \left\{ \bigvee B : B \subseteq_{\lambda} A \right\}.$$

(We drop the subscript λ when it is clear from the context.) A λ-ideal U in L is said to be *κ-generated* if U is of the form $\langle A \rangle$ for some $A \subseteq_{\kappa} L$. Set

$$\mathfrak{F}^{\lambda}_{\kappa} L \equiv \{V : V \text{ is a } \kappa\text{-generated } \lambda\text{-ideal in } L\},$$

a sub-κ-quantale of $\mathfrak{F}^\lambda L$. Let $\rho^\lambda_{\kappa L} : L \to \mathfrak{F}^\lambda_\kappa L$ be the codomain restriction of ρ^λ_L.

Proposition 3.3.1 (cf. [8, Proposition 1.2])**.** *The free κ-quantale over a λ-quantale L is $\mathfrak{F}^\lambda_\kappa L$. That is, for each κ-quantale M and λ-morphism $h : L \to M$ there is precisely one κ-morphism $f : \mathfrak{F}^\lambda_\kappa L \to M$ such that the diagram commutes.*

$$\begin{tikzcd}
\mathfrak{F}^\lambda_\kappa L \arrow[r, "f"] & M \\
L \arrow[u, "\rho^\lambda_\kappa"] \arrow[ur, "h"'] &
\end{tikzcd}$$

In the case of an idempotent multiplication, i.e., κ-frames, this reproves the corresponding result of Madden.

Proof. If f is such a homomorphism then for $U = \langle A \rangle$ we must have

$$f(U) = f(\bigvee_A \downarrow a) = \bigvee_A f\rho^\kappa_\mu(a) = \bigvee_A h(a),$$

hence the only candidate for the morphism in question is the map defined by the rule $f(\langle A \rangle) = \bigvee_A h(a)$, $A \subseteq_\kappa L$. This definition is well-defined, for if $\langle A \rangle = \langle A' \rangle$ then each element of A lies below a λ-join of elements from A' and vice-versa, and from this it follows that $\bigvee_A h(a) = \bigvee_{A'} h(a')$. Clearly f preserves κ-joins, and since, as can be easily checked, $\langle A \rangle \cdot \langle B \rangle = \langle AB \rangle$ for $A, B \subseteq_\mu L$, it follows that, for $U = \langle A \rangle$ and $V = \langle B \rangle$ in $\mathfrak{F}^\lambda_\kappa L$,

$$f(UV) = f(\langle A \rangle \langle B \rangle) = f(\langle AB \rangle) = \bigvee_{AB} h(ab) = \bigvee_A h(a) \cdot \bigvee_B h(b) = f(U)f(V).$$

We have $f(1_{\mathfrak{F}^\lambda_\kappa L}) = f(L) = 1$ because $h(1) = 1$, and $f(\downarrow a) = \bigvee_{b \le a} h(b) = h(a)$. $\square$

Propositions 3.1.2 and 3.3.1 give rise to the functor

$$\mathfrak{F}^\lambda_\kappa : \lambda\mathbf{Qnt} \to \kappa\mathbf{Qnt}.$$

For a λ-morphism $h : L \to M$,

$$\left(\mathfrak{F}^\lambda_\kappa h\right)(\langle A \rangle) = \langle h[A] \rangle, \ A \subseteq_\kappa L.$$

And $\mathfrak{F}^\lambda_\kappa h \circ \rho^\lambda_{\kappa L} = \rho^\lambda_{\kappa L} \circ h$.

It is material to our development that the free functors are compatible in the sense that, for $0 \le \lambda \le \kappa \le \mu \le \infty$,

$$\mathfrak{F}^\kappa_\mu \mathfrak{F}^\lambda_\kappa L \cong \mathfrak{F}^\lambda_\mu L, \ L \in \lambda\mathbf{Qnt}.$$

Proposition 3.3.2. *Let $0 \le \lambda \le \kappa \le \mu \le \infty$. Then for any λ-quantale L, the maps*

$$V \longrightarrow \bigcup V$$
$$\{\langle A\rangle_\lambda : A \subseteq_\kappa U\} \longleftarrow U$$

are inverse isomorphisms between $\mathfrak{F}^\kappa_\mu \mathfrak{F}^\lambda_\kappa L$ and $\mathfrak{F}^\lambda_\mu L$.

Proof. We give the proof for $\lambda > 0$; the proof for $\lambda = 0$ goes along similar lines. The distinction is necessary because $\mathfrak{F}^0_\kappa L$ consists of κ-generated pre-ideals, not κ-generated λ-ideals. That is, we cannot speak of λ-ideals when L has no order.

An element $V \in \mathfrak{F}^\kappa_\mu \mathfrak{F}^\lambda_\kappa L$ is a μ-generated κ-ideal on $\mathfrak{F}^\lambda_\kappa L$, say $V = \langle V_0\rangle_\kappa$ for $V_0 \subseteq_\mu \mathfrak{F}^\lambda_\kappa L$. Let $U \equiv \bigcup V \subseteq L$. We first claim that U is a μ-generated λ-ideal on L. Certainly U is a down-set, for if $a \le u \in U$ then, since $u \in v$ for some $v \in V$ and since v is a λ-ideal and hence a down-set, $a \in v \subseteq U$. To verify that U is closed under λ-joins, consider $a_0 = \bigvee A$ for $A \subseteq_\lambda U$. Then for each $a \in A$ there is some $v_a \in V$ such that $a \in v_a$. Since V is a κ-ideal, $v \equiv \bigvee_A v_a \in V$, and since v is a λ-ideal and $A \subseteq_\lambda v$, $a_0 \in v \subseteq U$. So far we have established that U is a λ-ideal. To show that U is μ-generated, let $A_v \subseteq_\kappa L$ be such that $v = \langle A_v\rangle_\lambda$ for all $v \in V_0$. Then $U = \langle A\rangle_\lambda$ for $A = \bigcup_{V_0} A_v \subseteq_\mu L$. This is true because $A \subseteq U$ implies $\langle A\rangle_\lambda \subseteq \langle U\rangle_\lambda = U$. Moreover, $u \in U$ implies $u \in v$ for some $v \in V = \langle V_0\rangle_\kappa$, which implies $v \le \bigvee V_1$ for some $V_1 \subseteq_\kappa V_0$. But in $\mathfrak{F}^\lambda_\kappa L$, $\bigvee V_1 = \downarrow\!\big\{\bigvee A' : A' \subseteq_\lambda \bigcup_{V_1} A_v\big\}$, so that $u \le \bigvee A'$ for some $A' \subseteq_\lambda \bigcup_{V_1} A_v \subseteq A$, meaning $u \in \langle A\rangle_\lambda$. This proves the first claim.

We next claim that if U is a μ-generated λ-ideal on L, say $U = \langle A\rangle_\lambda$ for $A \subseteq_\mu L$, then $V_U \equiv \{\langle A'\rangle_\lambda : A' \subseteq_\kappa U\}$ is a μ-generated κ-ideal on $\mathfrak{F}^\lambda_\kappa L$. First, V_U is a down-set, for if $\langle A''\rangle_\lambda \le \langle A'\rangle_\lambda \in V_U$ then, since U is a λ-ideal, $A'' \subseteq \langle A'\rangle_\lambda \subseteq \langle U\rangle_\lambda = U$, hence $\langle A''\rangle_\lambda \in V_U$. Secondly, V_U is closed under κ-joins, for if $V_0 \subseteq_\kappa V_U$, say $v = \langle A_v\rangle \subseteq_\kappa U$ for all $v \in V_0$, then $A \equiv \bigcup_{V_0} A_v \subseteq_\kappa U$ and $\bigvee V_0 = \langle A\rangle_\lambda \in V_U$. Finally, V_U is μ-generated, for if $U = \langle A\rangle_\lambda$ for some $A \subseteq_\mu L$ then $\{\downarrow a : a \in A\}$ is a μ-set which generates V_U as a κ-ideal in $\mathfrak{F}^\lambda_\kappa L$.

It remains to show the maps to be inverses of one another. Given $U = \langle A\rangle_\lambda$ for $A \subseteq_\mu L$, let $V_U \equiv \{\langle A'\rangle_\lambda : A' \subseteq_\kappa U\}$. Clearly $U \subseteq \bigcup V_U$, and $\bigcup V_U \subseteq U$ since $A' \subseteq_\kappa U$ implies $\langle A'\rangle_\lambda \subseteq \langle U\rangle_\lambda = U$. Given a μ-generated κ-ideal V on $\mathfrak{F}^\lambda_\kappa L$, put $U \equiv \bigcup V$ and $V_U \equiv \{\langle A'\rangle_\lambda : A' \subseteq_\kappa U\}$. Clearly $V \subseteq V_U$. On the other hand, each $v \in V_U$ is of the form $\langle A'\rangle_\lambda$ for some $A' \subseteq_\kappa U$, so that for each $a \in A'$ there is some $v_a \in V$ such that $a \in v_a$. But since V is closed under κ-joins we have $A' \subseteq v \equiv \bigvee_{A'} v_a \in V$, with the result that $\langle A' \rangle_\lambda \subseteq \langle v \rangle_\lambda = v$, and since V is a down-set, $\langle A' \rangle_\lambda \in V$. $\square$

3.4. **λ-coherent κ-quantales.** We refer to a κ-quantale of the form $\mathfrak{F}^\lambda_\kappa L$ as λ-*free*. It is a remarkable fact that λ-free κ-quantales, and even their generating elements, can be characterized internally. This result is due to Madden in the case of κ-frames ([8]); we generalize it here to κ-quantales.

Definition 3.4.1 (cf. [8, Definition 1.3])**.** *Let L be a κ-quantale. An element $a \in L$ is called a* λ-element *if for all $A \subseteq_\kappa L$ such that $\bigvee A \geq a$ there is some $A_0 \subseteq_\lambda A$ such that $\bigvee A_0 \geq a$. The set of λ-elements of L is designated $\mathfrak{E}^\kappa_\lambda L$. This set is evidently closed under λ-joins, and we call L* λ-coherent *if it forms a generating sub-λ-frame of L. More explicitly, L is λ-coherent if*

- *every element of L is a supremum of a κ-set of λ-elements,*
- *the product of finitely many λ-elements is a λ-element,*
- *and* 1 *is a λ-element.*

Proposition 3.4.2 (cf. [8, Proposition 1.4])**.** *A κ-quantale is λ-free iff it is λ-coherent. More precisely, we have the following.*

(1) *For any λ-quantale L, $\mathfrak{F}^\lambda_\kappa L$ is λ-coherent and*
$$\mathfrak{E}^\kappa_\lambda \mathfrak{F}^\lambda_\kappa L = \{\downarrow a : a \in L\}.$$
(2) *For any λ-coherent κ-frame L, the inclusion $\mathfrak{E}^\kappa_\lambda L \to L$ lifts to an isomorphism $\mathfrak{F}^\lambda_\kappa \mathfrak{E}^\kappa_\lambda L \to L$..*

Proof. (1) If the displayed equation holds then it is clear that $\mathfrak{F}^\lambda_\kappa L$ is λ-coherent. Now any element of $\mathfrak{F}^\lambda_\kappa L$ has the form $\langle A \rangle$ for some $A \subseteq_\kappa L$. If this is a λ-element then it may be expressed as $\langle A_0 \rangle$ for some $A_0 \subseteq_\lambda A$, and hence is of the form $\downarrow b$ for $b = \bigvee A_0$. On the other hand, if $\downarrow a \leq \bigvee_I U_i$ for some κ-family $\{U_i : i \in I\}$ of elements of $\mathfrak{F}^\lambda_\kappa L$, then, according to the description of the join operation provided by Proposition 3.2.2, $a \leq \bigvee A$ for some $A \subseteq_\lambda \bigcup_I U_i$. This fact implies the existence of some $I_0 \subseteq_\lambda I$ such that $A \subseteq_\lambda \bigcup_{I_0} U_i$, i.e., $\downarrow a \leq \bigvee_{I_0} U_i$.

(2) The lifted map is $\langle A \rangle \longmapsto \bigvee A$ for $A \subseteq_\kappa \mathfrak{E}^\kappa_\lambda L$, and its inverse is $b \longmapsto \{a \in \mathfrak{E}^\kappa_\lambda L : a \leq b\}$, $b \in L$. For $U \equiv \{a \in \mathfrak{E}^\kappa_\lambda L : a \leq b\}$ is generated by any $A \subseteq_\kappa U$ for which $\bigvee A = b$, and such a set A exists because L is λ-coherent. We have
$$b \longmapsto \{a \in \mathfrak{E}^\kappa_\lambda L : a \leq b\} = \langle A \rangle \longmapsto \bigvee A = b,\ b \in L.$$

On the other hand,

$$U = \langle A\rangle \longmapsto \bigvee A \equiv b \longmapsto \{a \in \mathfrak{E}^\kappa_\lambda L : a \le b\}\,,$$

and we claim that $\{a \in \mathfrak{E}^\kappa_\lambda L : a \le b\} = U$. For if $c \in U = \langle A\rangle$ it is only because c is a λ-element such that $c \le \bigvee A_0$ for some $A_0 \subseteq_\lambda A$, hence $c \le \bigvee A = b$. And if c is a λ-element such that $c \le b = \bigvee A$ then $c \le \bigvee A_0$ for some $A_0 \subseteq_\lambda S$, hence $c \in \langle A\rangle$. □

4. κ-QUANTALE QUOTIENTS, $\kappa > 0$

The factorization procedure of Subsection 2.2 can now be adjusted for κ-quantales, and in particular for κ-frames and bounded distributive lattices, by a simple application of the functor $\mathfrak{F}^\kappa$.

4.1. **Construction.** Let L be a κ-quantale, $\kappa > 0$, and let R be a binary relation on L. Embed L in $\mathfrak{F}^\kappa L$ via ρ^κ_L as in Propositions 3.2.1 and 3.2.2, and then factor $\mathfrak{F}^\kappa L$ by the relation

$$\widetilde{R} = \{(\downarrow a, \downarrow b) : (a,b) \in R\} \subseteq \mathfrak{F}^\kappa L \times \mathfrak{F}^\kappa L,$$

as per Theorem 2.2.3, resulting in the quotient map μ. Factor $\mu\rho^\kappa_L$ into $j\mu'$ for an injection j and surjection μ', and denote $\mu\rho^\kappa_L[L]$ by L/R.

$$\begin{tikzcd}
L \arrow[rrr, "\rho^\kappa_L"'] \arrow[dr, "\mu'"'] \arrow[dd, "h"'] & & & \mathfrak{F}^\kappa L \arrow[dl, "\mu"] \arrow[dd, "\mathfrak{F}^\kappa h"] \\
& L/R \arrow[r, "j"] \arrow[dl, "\overline{h}"] & (\mathfrak{F}^\kappa L)/\widetilde{R} \arrow[dr, "\widetilde{h}"'] & \\
M \arrow[rrr, "\rho^\kappa_M"] & & & \mathfrak{F}^\kappa M
\end{tikzcd}$$

Proposition 4.1.1. *Let L be a κ-quantale, $\kappa > 0$, and let R be a binary relation on L. Then $\mu' : L \to L/R$ is the quotient of L factored by the smallest κ-congruence containing R.*

Proof. To verify the claim we must show that an arbitrary κ-morphism $h : L \to M$ such that

$$(a,b) \in R \implies h(a) = h(b),\ a,b \in L,$$

factors through μ'. Since $\mathfrak{F}^\kappa h(\downarrow a) = \downarrow h(a)$ for all $a \in L$, it follows that for $(\downarrow a, \downarrow b) \in \widetilde{R}$ we have $\mathfrak{F}^\kappa h(\downarrow a) = \downarrow h(a) = \downarrow h(b) = \mathfrak{F}^\kappa h(\downarrow b)$, and hence there is an $\widetilde{h}$ such that $\widetilde{h}\mu = \mathfrak{F}^\kappa h$. Now if $b \in \mu\rho^\kappa_L[L]$, that is, if $b = \mu({\downarrow} a)$ for some $a \in L$ then

$$\widetilde{h}j(b) = \widetilde{h}(\mu({\downarrow} a)) = \mathfrak{F}^\kappa h({\downarrow} a) = {\downarrow} h(a)$$

is in $\rho^\kappa_M[M]$ and hence, since ρ^κ_M is one-one, there is a κ-morphism $\overline{h} : \mu\rho^\kappa_L[L] \to M$ such that $\rho^\kappa_M\overline{h} = \widetilde{h}j$ and we have

$$\rho^\kappa_M\overline{h}\mu' = \widetilde{h}j\mu' = \widetilde{h}\mu\rho^\kappa_L = \mathfrak{F}^\kappa h \circ \rho^\kappa_L = \rho^\kappa_M h$$

and since ρ^κ_M is one-one, $\overline{h}\mu' = h$. □

A κ-ideal U on L is R-saturated in the sense of Subsection 2.2 iff

$$\forall\, a, b, c \in L \ \ (aRb \Longrightarrow (ac \in U \Longleftrightarrow bc \in U))\,.$$

We denote by $\langle A\rangle_R$ the smallest R-saturated κ-ideal containing a subset $A \subseteq L$.

Corollary 4.1.2. *Let L be a κ-quantale, $\kappa > 0$, and let R be a binary relation on L. Then the map*

$$(a \longmapsto \langle a\rangle_R) : L \to \{\langle a\rangle_R : a \in L\}$$

is the quotient of L by the smallest κ-congruence containing R.

Remark 4.1.3. *There is nothing like saturation in a κ-quantale. Note, however, that the quotient above is made up of some of the saturated elements in $\mathfrak{F}^\kappa L$. Thus, if these elements are well understood we again have a transparent description of L/R.*

5. Colimits

In this section we describe colimits in the category of κ-quantales, $\kappa > 0$. Since the $\mathfrak{F}^\lambda_\kappa$-construction from Section 3 preserves idempotence of multiplication, if we start in κ**Frm** (in particular, in **DLat**) we obtain colimits in κ**Frm** as well. An abstract construction of colimits in categories of a similar and more general, nature was presented in [3]. The description we obtain here can, in many cases, be fairly explicit and transparent. An observation similar to Remark 4.1.3 can be made here as well. We will see two easy but important applications in Section 6.

5.1. **Construction.** Let $D = (L_i, \phi_{ij})_I$ be a diagram in $\kappa\mathbf{Qnt}$. Consider the colimit $(\delta_i : L_i \to S)_I$ in $\mathbf{CMon}$, embed S in $\mathfrak{F}^0_\kappa S$ via $\rho^0_{\kappa S}$ as per Proposition 3.1.2, and then factor $\mathfrak{F}^0_\kappa S$ by the relation

$$R = \left\{ \left([\delta_i (b)], \bigcup_A [\delta_i (a)] \right) : A \subseteq_\kappa L_i \text{ with } b = \bigvee A,\ i \in I \right\}$$

as per Section 4. Label the quotient map μ, and denote the sub-κ-quantale of $\mathfrak{F}^0_\kappa S/R$ generated by $\bigcup_I \mu\rho^0_\kappa\delta_i [L_i]$ by L. Observe that factoring by this particular relation R forces the maps $\mu\rho^0_\kappa\delta_i : L_i \to \mathfrak{F}^0_\kappa S/R$ to preserve κ-joins; let $\gamma_i : L_i \to L$ be the unique κ-morphism whose underlying monoid homomorphism agrees with $\mu\rho^0_\kappa\delta_i$.

$$\begin{tikzcd}
 & S \arrow[r, "\rho^0_{\kappa S}"] \arrow[d, "h'"] & \mathfrak{F}^0_\kappa S \arrow[dl, "f"] \arrow[d, "\mu'"] \\
L_i \arrow[ur, "\delta_i"] \arrow[r, "h_i"'] & M & \widetilde{L} \arrow[l, "j"]
\end{tikzcd}$$

Proposition 5.1.1. *$(\gamma_i : L_i \to L)_I$ is a colimit of the diagram $D = (L_i, \phi_{ij})_I$ in $\kappa\mathbf{Qnt}$.*

Proof. Consider an upper bound $(h_i : L_i \to M)_I$ of D in $\kappa\mathbf{Qnt}$. First, forget the join structure and take the colimit $(\delta_i : L_i \to S)$ in $\mathbf{CMon}$, thereby obtaining a unique monoid homomorphism h' such that $h'\delta_i = h_i$ for all i. Then, since $\mathfrak{F}^0_\kappa S$ is the free κ-quantale over S, find the unique κ-morphism f such that $f\rho^0_{\kappa S} = h'$. Now for all $i \in I$ and all $A \subseteq_\kappa L_i$ with $b = \bigvee A$,

$$\begin{aligned} f\left(\bigcup_A [\delta_i (a)]\right) &= \bigvee_A f\rho^0_{\kappa S}\delta_i (a) = \bigvee_A h'\delta_i (a) = \bigvee_A h_i (a) = h_i \left(\bigvee A\right) = h_i (b) \\ &= h'\delta_i (b) = f\rho^0_{\kappa S}\delta_i (b) = f ([\delta_i (b)]), \end{aligned}$$

with the result that f factors through μ, say $f = j\mu$. Then, for all $i \in I$,

$$j\gamma_i = j\mu\rho^0_{\kappa S}\delta_i = f\rho^0_{\kappa S}\delta_i = h'\delta_i = h_i$$

as desired. The map j is unique with respect to the condition just displayed, for if $k\gamma_i = h_i$ for all i then $j\mu\rho^0_{\kappa S} = h' = k\mu\rho^0_{\kappa S}$ by virtue of the uniqueness of h', which implies that $j\mu = k\mu$ because $\rho^0_{\kappa S} [S]$ generates $\mathfrak{F}^0_\kappa S$ as a κ-quantale, and this, in turn, implies $j = k$ because μ is surjective. □

Proposition 5.1.1 gives the colimit L as a sub-κ-quantale of $\mathfrak{F}^0_\kappa S/R$, and this quotient is literally $\mathfrak{F}^\kappa\mathfrak{F}^0_\kappa S/\widetilde{R}$ according to Proposition 4.1.1. But it is simpler to work with pre-ideals on S, and we might as well since $\mathfrak{F}^\kappa\mathfrak{F}^0_\kappa S$ is isomorphic to $\mathfrak{F}^0 S$ by Proposition 3.3.2. The question then naturally arises as to which pre-ideals on S correspond to, i.e., are unions of, $\widetilde{R}$-saturated element of $\mathfrak{F}^\kappa \mathfrak{F}^0_\kappa S$. We refer to such pre-ideals as being *R-saturated.*

Lemma 5.1.2. *A pre-ideal $U \subseteq S$ is R-saturated iff it satisfies the following conditions.*

1. *For all $i \in I$ and all $a \leq b$ in L_i, and for all $s \in S$, if $\delta_i(b)\, s \in U$ then $\delta_i(a)\, s \in U$..*
2. *For all $i \in I$ and $A \subseteq_\kappa L_i$ with $b = \bigvee A$, and for all $s \in S$, if $\delta_i(a)\, s \in U$ for all $a \in A$ then $\delta_i(b)\, s \in U$.*

Proof. Let T be an $\widetilde{R}$-saturated element of $\mathfrak{F}^\kappa \mathfrak{F}^0_\kappa S$. Then T is a κ-ideal of $\mathfrak{F}^0_\kappa S$, the κ-quantale of κ-generated pre-ideals of S, such that

$$(*) \qquad (\downarrow [\delta_i(b)]) \cdot V \subseteq T \text{ iff } \left(\downarrow \bigcup_A [\delta_i(a)] \right) \cdot V \subseteq T$$

for all $i \in I$ and $A \subseteq_\kappa L_i$ with $b = \bigvee A$, and for all $V \in \mathfrak{F}^\kappa \mathfrak{F}^0_\kappa S$. (The down-sets here are taken in $\mathfrak{F}^0_\kappa S$.) Let $U \equiv \bigcup T$, so that, by Proposition 3.3.2, $T = \{W \in \mathfrak{F}^0_\kappa S : W \subseteq U\}$. Fix $i \in I$ and $s \in S$. Taking $V = \downarrow [s]$ and using the implication from right to left in $(*)$, we get that, for $A \subseteq L_i$ with $b = \bigvee A$,

$$\{\delta_i(a)\, s : a \in A\} \subseteq U \Longrightarrow \left(\downarrow \bigcup_A [\delta_i(a)] \right) \cdot V \subseteq T \Longrightarrow (\downarrow [\delta_i(b)]) \cdot V \subseteq T \Longrightarrow$$
$$\Longrightarrow \delta_i(b)\, s \in U,$$

which is condition (2) above. Taking $V = \ \downarrow [s]$ and $A = \{a, b\}$ with $a \leq b$ in L_i and using the implication from left to right in $(*)$, we get

$$\delta_i(b)\, s \in U \Longrightarrow (\downarrow [\delta_i(b)]) \cdot V \subseteq T \Longrightarrow (\downarrow \{[\delta_i(a)] \cup [\delta_i(b)]\}) \cdot V \subseteq \text{-}T \Longrightarrow$$
$$\Longrightarrow \delta_i(a)\, s \in U,$$

which is condition (1) above. On the other hand, it is straightforward to verify that if U satisfies (1) and (2) then $T \equiv \{W \in \mathfrak{F}^0_\kappa S : W \subseteq U\}$ satisfies $(*)$. □

Let $[A]_R$ designate the smallest R-saturated pre-ideal containing a subset $A \subseteq S$. An R-saturated pre-ideal $U \subseteq S$ is said to be *κ-generated* if it is of the form $[A]_R$ for some $A \subseteq_\kappa S$. We denote the κ-quantale of κ-generated R-saturated pre-ideals of A by $\widetilde{L}$, and, by abuse of notation, we denote the κ-morphism $a \longmapsto [a]_R$ by $\gamma_i : L_i \to \widetilde{L}$.

Proposition 5.1.3. $\left(\gamma_i : L_i \to \widetilde{L}\right)_I$ *is a colimit of the diagram* $D = (L_i, \phi_{ij})_I$ *in* κ**Qnt**.

6. Application: coproducts

In this section we apply the results of Section 5 to coproducts of κ-quantales in order to characterized them in Theorem 6.2.2. This requires that we begin by outlining coproducts in **CMon**.

6.1. **Coproducts in CMon.** Let L_i, $i \in J$, be a family of monoids. Set

$$\prod_J{}' L_i = \{(x_i) \in \prod_J L_i : x_i = 1 \text{ for all but finitely many } i\},$$

a submonoid of the product monoid $\prod_J L_i$. Let $j \in J$ be fixed. For $y \in L_j$ and $x \in \prod' L_i$ set

$$y *_j x = v, \quad v_i = \begin{cases} y \text{ for } i = j \\ x_i \text{ for } i \neq j \end{cases}.$$

That is, $y *_j x$ is the result of replacing the j^{th} coordinate of x by y and leaving the other coordinates unchanged. Denote the identity element by $\overline{1} \in \prod' L_i$, i.e., $\overline{1}_i = 1$ for all i. To avoid confusion with the (categorical) product $\prod_J S_i$ of monoids, and with other structures, we will use the symbol

$$\sqcap_{i=1}^n x_i \quad \text{or just} \quad \sqcap_i x_i \quad \text{for} \quad x_1 \cdot x_2 \cdot \dots \cdot x_n.$$

Consider the mappings

$$\delta_j = (x \mapsto x *_j \overline{1}) : L_j \to \prod_J{}' L_i.$$

Obviously the δ_j's are homomorphisms. We have

Proposition 6.1.1. $(\delta_j : L_j \to \prod'_J L_i)_J$ *is a coproduct in* **CMon**.

Proof. We have to prove that for any family $h_j : L_j \to M$ of homomorphisms there is precisely one homomorphism $h : \prod'_J L_i \to M$ such that $h\delta_i = h_i$ for all i. First, we see that there is at most one such h. For $(x_i) \in \prod'_J L_i$ let $x_{j_1}, \dots, x_{j_n}$ be all the coordinates that are not 1. Then necessarily

$$h((x_i)) = h(\sqcap_{k=1}^n (x_{j_k} *_{j_k} \overline{1})) = \sqcap h_{j_k}(x_{j_k}).$$

Now define

$$h((x_i)) = \sqcap_J h_i(x_i).$$

This is essentially a finite product since all but finitely many of the $h_i(x_i)$'s are 1. Then $h(\delta_j(x)) = h_j(x)$ for all $j \in J$ and $x \in L_j$, $h(\overline{1}) = 1$, and if $x = (x_i)$ and $y = (y_i)$ then, by commutativity,

$$h(x \cdot y) = h((x_i y_i)) = \sqcap_i h_i(x_i y_i) = \sqcap_i h_i(x_i) \cdot \sqcap_i h_i(y_i) = h(x) \cdot h(y).$$

□

6.2. **Coproducts in κQnt, $\kappa > 0$.** Now let L_i, $i \in J$, be κ-quantales (in particular bounded distributive lattices). If we view the L_i's for the moment as their underlying monoids, we may form their **CMon** coproduct $\prod'_J L_i$. The binary relation R of Subsection 5.1 can be written as

$$\left\{ \left(\left[b *_j \overline{1} \right], \bigcup_A \left[a *_j \overline{1} \right] \right),\ A \subseteq_\kappa L_j \text{ with } b = \bigvee A,\ j \in J \right\}.$$

According to Propositions 5.1.3 and 6.1.1, the κ**Qnt** coproduct consists of those pre-ideals $U \subseteq \prod'_J L_i$ which are R-saturated in the sense of Lemma 5.1.2. At this point it becomes both relevant and useful to view $\prod'_J L_i$ as a partially ordered set in the product order, and this permits the conditions of Lemma 5.1.2 to be nicely simplified:

(1) part (1) becomes the condition that U be a down-set, and
(2) part (2) becomes the condition that, for all $j \in J$ and $A \subseteq_\kappa L_j$ with $b = \bigvee A$, and for all $x \in \prod'_J L_i$,

$$A *_j x \subseteq U \Longrightarrow b *_j x \in U.$$

The set A can be empty, and hence we have, in particular, that any R-saturated down-set contains

$$\mathbb{O} \equiv \{(x_i) : \exists\, j \ \ (x_j = 0)\},$$

and $\mathbb{O}$ itself is R-saturated. Denote the κ-quantale of κ-generated R-saturated down-sets of $\prod'_J L_i$ by $\bigoplus_J L_i$, denote the smallest R-saturated down-set containing a given subset $A \subseteq \prod'_J L_i$ by $\downarrow_R A$, and denote by $\gamma_i : L_i \to \bigoplus_J L_i$ the κ-morphism

$$a \longmapsto\ \downarrow_R \left(a *_i \overline{1}\right),\ a \in L_i.$$

An important observation in this connection is that

$$\downarrow_R \left(a *_i \overline{1}\right) = \ \downarrow \left(a *_i \overline{1}\right) \cup \mathbb{O},$$

since $\downarrow \left(a *_i \overline{1}\right) \cup \mathbb{O}$ clearly satisfies properties (1) and (2) above.

Our development is summarized in Theorem 6.2.1, a direct generalization to κ-quantales of Johnstone's description of the frame coproduct ([5, p. 59]).

Theorem 6.2.1. *Let $\kappa > 0$. The family $\left(\gamma_i : L_i \to \bigoplus_J L_i\right)_J$ is a κ**Qnt** coproduct of the family $(L_i)_J$.*

Theorem 6.2.1 permits a characterization of the coproduct in κ**Qnt** in a manner independent of its construction.

Theorem 6.2.2. *Let $\kappa > 0$. A family $(\upsilon_i : L_i \to L)_J$ of κ-morphisms is a κ**Qnt** coproduct of the family $(L_i)_J$ iff it has these properties.*

(1) *$\bigcup_J \upsilon_i [L_i]$ generates L.*
(2) *For any $I_0 \subseteq_\omega J$ and $I_1 \subseteq_\kappa J$, and for any $a_i \in L_i$, $i \in I_0$, and $b_j \in L_j$, $j \in I_1$,*

$$\sqcap_{I_0} \upsilon_i (a_i) \leq \bigvee_{I_1} \upsilon_j (b_j) \Longrightarrow \exists\, i \in I_0 \cap I_1 \ \ (a_i \leq b_i) .$$

Proof. To verify the forward direction we must show that $\left(\gamma_i : L \to \bigoplus_J L_i\right)$ has the second property above, since it clearly has the first. Since $\sqcap_{I_0} \gamma_i (a_i) = \left(\downarrow \sqcap_{I_0} \left(a_i *_i \overline{1}\right)\right) \cup \mathbb{O}$, this follows from the fact that

$$\bigvee_J \gamma_j (b_j) = \bigcup_J \gamma_j (b_j) = \left(\bigcup_J \downarrow \left(b_j *_j \overline{1}\right)\right) \cup \mathbb{O}.$$

i.e., that $\left(\bigcup_J \downarrow \left(b_j *_j \overline{1}\right)\right) \cup \mathbb{O}$ is R-saturated. This is a consequence of the fact that different b_j's are chosen from different L_j's, and is easily verified.

Now suppose that $(\upsilon_i : L_i \to L)$ is a family of κ-morphisms satisfying (1) and (2), and let $\upsilon : \bigoplus_J L_i \to L$ be the unique κ-morphism such that $\upsilon\gamma_i = \upsilon_i$ for all i. This map is surjective as a consequence of the assumption that $\bigcup_J \upsilon_i [L_i]$ generates L; it remains only to show that it is injective as well.

A member of $\bigoplus_J L_i$ has the form $\downarrow_R S$ for $S \subseteq_\kappa \prod'_J L_i$, and if we write each $s \in S$ in the form $s = \sqcap_{I_s} \left(a_i *_i \overline{1}\right)$ for $I_s \subseteq_\omega I$, where $a_i \in L_i$ for $i \in I_s$, then by necessity

$$\upsilon (\downarrow_R S) = \bigvee_S \sqcap_{I_s} \upsilon_i (a_i) .$$

Suppose $\upsilon (\downarrow_R S) = \upsilon (\downarrow_R T)$ for $S, T \subseteq_\kappa \prod'_J L_i$, i.e.,

$$\bigvee_S \sqcap_{I_s} \upsilon_i (a_i) = \bigvee_T \sqcap_{I_t} \upsilon_j (b_j)$$

for $I_s, I_t \subseteq_\omega I$, $t \in T$, $s \in S$. Fix $s_0 \in S$, and denote the set of choice functions by

$$\Theta \equiv \left\{ \theta : T \to \bigcup_T I_{tr} : \theta(t) \in I_t,\ t \in T \right\}.$$

For each $\theta \in \Theta$ we have

$$\sqcap_{I_{s_0}} v_i(a_i) \leq \bigvee_S \underset{I_s}{\sqcap} v_i(a_i) = \bigvee_T \underset{I_t}{\sqcap} v_j(b_j) \leq \bigvee_T v_{\theta(t)}\left(b_{\theta(t)}\right),$$

so that by (2) there is some $i \in I_{s_0}$ and $t \in T$ such that $\theta(t) = i$ and $a_i \leq b_i$. It follows that there must be some $t_0 \in T$ for which $I_{t_0} \subseteq I_{s_0}$ and $a_i \leq b_i$ for all $i \in I_{t_0}$. This implies

$$s_0 = \sqcap_{I_{s_0}} a_i \leq \sqcap_{I_{t_0}} a_i \leq \sqcap_{I_{t_0}} b_j = t_0.$$

Since s_0 was arbitrarily chosen from S, we conclude that $\downarrow_R S \leq\ \downarrow_R T$, and since the argument is symmetrical in S and T, that $\downarrow_R S \leq\ \downarrow_R T$. □

6.3. **Free κ-quantales (over sets).** When specialized to the coproduct of free κ-quantales over a single generator, Theorem 6.2.2 yields 6.3.1, the generalization to κ-quantales of Whitman's condition for the free generation of a lattice ([12]).

Theorem 6.3.1. *Let L be a κ-quantale, $\kappa > 0$, generated by a subset X. Then L is freely generated by X iff for any $X_0 \subseteq_\omega X$ and $Y \subseteq_\kappa X$, and for any choice of integers $n_x, m_y \in \mathbb{Z}^+$, $x \in X_0$, $y \in Y$,*

$$\underset{X_0}{\sqcap} x^{n_x} \leq \bigvee_Y y^{m_y} \Longrightarrow \exists\ x \in X_0 \cap Y \ \ (n_x \geq m_y).$$

Proof. L is freely generated by S iff L is isomorphic to the free κ-quantale on $|X|$ generators, i.e., the coproduct of $|X|$ many copies of the the free κ-quantale on a single generator. Since the latter is clearly $\mathfrak{F}_k^0 S$, where S is the free commutative monoid on one generator, and since S is clearly the multiplicative monoid $\left\{ \left(\frac{1}{2}\right)^n : n \in \mathbb{Z}^+ \right\}$, the result follows from Theorem 6.2.1. □

References

[1] R. Balbes and P. Dwinger, *Distributive Lattices*, University of Missouri Press, Missouri, 1974.

[2] B. Banaschewski and E. Nelson, *Tensor products and bimorphisms,* Can. Math. Bull. **19** (1976), 385-402.

[3] B. Banaschewski and A. Pultr, *Distributive algebras in linear categories,* Algebra Universalis **30** (1993), 101-118.

[4] F.Borceux and G. van den Bossche, *Quantales and their sheaves,* Order, **3,1** (1986), 69-87.

[5] P.T. Johnstone, *Stone Spaces,* Cambridge Sudies in Advanced Math. no 3, Cambridge University Press, 1983.

[6] A. Joyal and M. Tierney, *An extension of the Galois theory of Grothendieck,* Memoirs of the AMS **51** Number 309 (1984).

[7] S. Mac Lane, *Categories for the Working Mathematician,* Springer-Verlag, New York, 1971.

[8] J. Madden, *κ-frames*, J. Pure and Appl. Alg. 70 (1991), 107–127.

[9] C. J. Mulvey, *Quantales,* in: Encyclopedia of Math.(ed. M. Hazewinkel), Springer 2001.

[10] J. Paseka and J. Rosicky, *Quantales,* Current Res. in Oper. Quantum Logic: Algebras, Categories and Languages, Fund. Theories Phys, **111**, Kluwer 2000, 245-262.

[11] A. Pultr, *Frames,* Chapter in: Handbook of Algebra, Vol.**3**, (Ed. by M. Hazewinkel), Elsevier 2003, 791-858.

[12] P. M. Whitman, *Free Lattices II*, Ann. Math. **43** (1941), 104–115.

[Ball]Department of Mathematics, University of Denver, Denver, Colorado 80208, U.S.A., [Pultr]Department of Applied Mathematics and ITI, MFF, Charles University, CZ 11800 Praha 1, Malostranské nám. 25