Covering a graph by forests and a matching

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Abstract

We prove that for any positive integer k and for $\varepsilon = 1/((k+2)(3k^2+1))$, the edges of any graph whose fractional arboricity is at most $k+\varepsilon$ can be decomposed into k forests and a matching.

1 Introduction

The *arboricity* $\Upsilon(G)$ of a graph G is the least number k such that the edge set of G can be covered by k forests. A classical result of Nash-Williams [11] states that a trivial lower bound to arboricity actually gives the right value:

Theorem 1. *For any graph G,*

$$\Upsilon(G) = \max_H \left\lceil \frac{|E(H)|}{|V(H)|-1} \right\rceil,$$

where the maximum is taken over all subgraphs H of G.

(Here and in the rest of the paper, we write $V(H)$ and $E(H)$ for the vertex set and the edge set of a graph H, respectively, and the graphs may contain loops and parallel edges. For any graph-theoretical notions not defined here, we refer the reader to Diestel [6].)

Payan [14] defined the *fractional arboricity* $\Upsilon_f(G)$ of G by

$$\Upsilon_f(G) = \max_{H \subseteq G} \frac{|E(H)|}{|V(H)|-1}.$$

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Thus, $\Upsilon(G) = \lceil \Upsilon_f(G) \rceil$, and one may ask whether $\Upsilon_f(G)$ is a finer measure of the properties of G than $\Upsilon(G)$. In particular, suppose that $\Upsilon_f(G) = k + \varepsilon$ for some integer k and small $\varepsilon > 0$. By Theorem 1, $E(G)$ can be covered by $k + 1$ forests, but it is natural to ask whether one of these forests can be restricted to have, for instance, bounded maximum degree or bounded maximum component size. The problem was studied by Montassier et al. [10] and in two cases, an affirmative answer was obtained. These are summarized in the following theorem which improves earlier results on decompositions of planar graphs [5, 8]:

Theorem 2. *Let G be a graph.*

(i) If $\Upsilon_f(G) \le \frac{4}{3}$, then $E(G)$ can be covered by a forest and a matching.

(ii) If $\Upsilon_f(G) \le \frac{3}{2}$, then $E(G)$ can be covered by two forests, one of which has maximum degree at most 2.

In [10], a general conjecture was proposed which would include Theorem 2 as a special case:

Conjecture 3. *Let k and d be positive integers. If G is a graph with $\Upsilon_f(G) \le k + \frac{d}{k+d+1}$, then $E(G)$ can be decomposed into $k+1$ forests, one of which has maximum degree at most d.*

A. V. Kostochka and X. Zhu (personal communication) proved Conjecture 3 for $k = 1$ and $3 \le d \le 6$. For $k \ge 2$ or $d \ge 7$, the conjecture is open.

Another partial result of [10] toward Conjecture 3 is the following:

Theorem 4. *If G has fractional arboricity $\Upsilon_f(G) \le k + \varepsilon$, where k is an integer and $0 \le \varepsilon < 1$, then $E(G)$ can be covered by k forests and a graph of maximum degree at most d, where*

$$d = \left\lceil \frac{(k+1)(k-1+2\varepsilon)}{1-\varepsilon} \right\rceil.$$

Theorem 4 provides a value of d for any choice of ε, but it does not ensure a suitable value of ε for an arbitrary d. In particular, for $k \ge 2$ it leaves open the question whether there is $\varepsilon = \varepsilon(k)$ such that the edges of any graph with fractional arboricity at most $k + \varepsilon$ can be covered by k forests and a matching. In the present paper, we answer this question in the affirmative:

Theorem 5. *Let $k \ge 1$ be an integer and G a graph with $\Upsilon_f(G) \le k + \varepsilon$, where $\varepsilon = 1/((k+2)(3k^2+1))$. Then $E(G)$ can be decomposed into k forests and a matching.*

Our proof is based on an extension of the matroid intersection theorem [7] due to Aharoni and Berger [1]. The structure of the paper is as follows. In Section 2, we recall the necessary notions of matroid theory. Section 3 gives an overview of the topological preliminaries. The pieces are assembled in Section 4, where we prove Theorem 5.

2 Matroids

The purpose of this section to introduce the relevant terminology and facts from matroid theory. For more details, the reader may consult the book of Oxley [13] or Part IV of Schrijver [15].

A *matroid* is a pair $(E, \mathcal{I})$, where E is a finite set and $\mathcal{I}$ is a nonempty collection of subsets of E satisfying the following axioms:

(M1) if $A \subseteq B \subseteq E$ and $B \in \mathcal{I}$, then $A \in \mathcal{I}$, and

(M2) if $A, B \in \mathcal{I}$ and $|A| < |B|$, then for some $x \in B \setminus A$, $A \cup x \in \mathcal{I}$.

(For brevity, we write $A \cup x$ in place of $A \cup \{x\}$, and $A \setminus x$ instead of $A \setminus \{x\}$.)

Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid. The sets in $\mathcal{I}$ are called *independent sets* of $\mathcal{M}$ (the other subsets of E being *dependent*), and $\mathcal{M}$ is said to be a matroid *on* E.

It is easy to prove from (M2) that all inclusionwise maximal subsets of a set $X \subseteq E$ that are independent in $\mathcal{M}$ have the same cardinality. This cardinality is called the *rank* of X in $\mathcal{M}$ and denoted by $\mathrm{rank}_{\mathcal{M}}(X)$. By definition, the rank of $\mathcal{M}$ is $\mathrm{rank}_{\mathcal{M}}(E)$. Any independent set of size $\mathrm{rank}_{\mathcal{M}}(E)$ is a *base* of $\mathcal{M}$.

The *dual matroid* $\mathcal{M}^*$ of $\mathcal{M}$ is a matroid on E whose independent sets are all the subsets of E that are disjoint from some base of $\mathcal{M}$. Thus, the bases of $\mathcal{M}^*$ are precisely the conplements of the bases of $\mathcal{M}$. The rank function of the dual matroid is given by the following lemma (see [13, Proposition 2.1.9]):

Lemma 6. *If $\mathcal{M}$ is a matroid on E and X is a subset of E, then the rank of X in the dual matroid $\mathcal{M}^*$ is*

$$\mathrm{rank}_{\mathcal{M}^*}(X) = |X| + \mathrm{rank}_{\mathcal{M}}(E \setminus X) - \mathrm{rank}_{\mathcal{M}}(E).$$

Each graph G has an associated matroid, the *cycle matroid* of G. This is a matroid on $E(G)$ and its independent sets are the edge sets of forests in G. For $X \subseteq E(G)$, let $G[X]$ be the subgraph of G induced by the edge set X (that is, its vertices are all the vertices incident with an edge of X, and its edge set is X). The rank function of X is easily interpreted in terms of $G[X]$:

Lemma 7. *Let $\mathcal{M}$ be the cycle matroid of a graph G and $X \subseteq E(G)$. If the subgraph $G[X]$ has $n(X)$ vertices and $c(X)$ components, then*

$$\mathrm{rank}_{\mathcal{M}}(X) = n(X) - c(X).$$

Theorem 1 has a natural proof using matroid theory, based on the following important result of Nash-Williams [12] (see also [13, Proposition 12.3.1]):

Theorem 8 (Matroid union theorem)**.** *Let $\mathcal{M}$ and $\mathcal{N}$ be matroids on E. Let $\mathcal{I}$ be the collection of all sets $I \cup J$, where I is an independent set of $\mathcal{M}$ and J is an independent set of $\mathcal{N}$. Then $(E, \mathcal{I})$ is a matroid and the rank of a set $X \subseteq E$ in this matroid equals*

$$\min_{T \subseteq X} \Big(|X \setminus T| + \mathrm{rank}_{\mathcal{M}}(T) + \mathrm{rank}_{\mathcal{N}}(T) \Big).$$

The matroid from Theorem 8 is called the *union* of $\mathcal{M}$ and $\mathcal{N}$ and is denoted by $\mathcal{M} \vee \mathcal{N}$.

Let $\mathcal{M}^{(k)}$ be the union of k copies of $\mathcal{M}$. The Matroid union theorem implies that the rank function of this matroid is as follows:

Corollary 9. *The rank of a set $X \subseteq E$ in $\mathcal{M}^{(k)}$ is*

$$\mathrm{rank}_{\mathcal{M}^{(k)}}(X) = \min_{T \subseteq X} \Big(|X \setminus T| + k \cdot \mathrm{rank}_{\mathcal{M}}(T) \Big).$$

Closely related to Theorem 8 is the following result of Edmonds [7] (see also [13, Theorem 12.3.15]):

Theorem 10 (Matroid intersection theorem)**.** *Let $\mathcal{M}$ and $\mathcal{N}$ be matroids on E. The maximum size of a subset of E that is independent in both $\mathcal{M}$ and $\mathcal{N}$ equals*

$$\min_{X \subseteq E} \Big(\mathrm{rank}_{\mathcal{M}}(X) + \mathrm{rank}_{\mathcal{N}}(E \setminus X) \Big).$$

A *circuit* of $\mathcal{M}$ is any inclusionwise minimal dependent subset of E. A set $X \subseteq E$ is a *flat* of $\mathcal{M}$ if for every $x \in E \setminus X$, $\mathrm{rank}_{\mathcal{M}}(X \cup x) = \mathrm{rank}_{\mathcal{M}}(X) + 1$. We will need the following lemma which relates circuit and flats (see [13, Proposition 1.4.10] for a proof):

Lemma 11. *If X is a flat of $\mathcal{M}$ and $e \in E$ is contained in a circuit C such that $C \subseteq X \cup e$, then $e \in X$.*

3 Complexes

In this section, we review the topological machinery needed in our proof. A more complete account can be found in Section 2 of [1]. A standard reference on topological methods in combinatorics is Björner [3].

A *simplicial complex* (or just *complex*) on a finite set E is any nonempty collection $\mathcal{C}$ of subsets of E such that if $A \subseteq B \in \mathcal{C}$, then $A \in \mathcal{C}$. The subsets belonging to $\mathcal{C}$ are called the *faces* (or *simplices*) of $\mathcal{C}$.

Any complex $\mathcal{C}$ has an associated geometric realization $\|\mathcal{C}\|$ called the *polyhedron* of $\mathcal{C}$. This is a topological space obtained as follows. To each $e \in E$ contained in some face of $\mathcal{C}$, assign a vector v_e in $\mathbb{R}^{|E|}$ in such a way that all the vectors v_e are linearly independent. Every face A of $\mathcal{C}$ then has an associated geometric simplex σ_A, namely the convex hull of the set $\{v_e : e \in A\}$. The polyhedron of $\mathcal{C}$ is obtained as the union of all the simplices σ_A, where A ranges over $\mathcal{C}$.

Next, we recall the notion of connectivity of complexes and topological spaces in general. For $d \geq 0$, a topological space X is *d-connected* if every continuous mapping f from the d-dimensional sphere S^d to X can be extended to the $(d+1)$-dimensional closed ball B^{d+1} in a continuous way. Every nonempty space is considered to be (-1)-connected. The *connectivity* of a nonempty space X is the largest integer k such that X is d-connected for all d, where $-1 \leq d \leq k$. (If X is d-connected for all integers $d \geq -1$, then its connectivity is infinite.) The property of being 0-connected is equivalent to the usual arcwise connectedness of X. For higher d, d-connected spaces can be intuitively thought of as those which have no 'holes' of dimension less than d.

The connectivity of a complex $\mathcal{C}$ is defined as the connectivity of its polyhedron $\|\mathcal{C}\|$. For technical reasons, it is useful to work with a slight modification of this parameter, denoted by $\eta(\mathcal{C})$ and defined as the connectivity of $\mathcal{C}$ plus 2.

Matroids can be viewed as complexes of a special type: if $\mathcal{M}$ is a matroid on a set E, then the independent sets of $\mathcal{M}$ form a complex on E. While most properties of matroids do not carry over to the more general world of complexes, one important matroid-theoretical result that has a partial extension to complexes is Theorem 10. This extension is due to Aharoni and Berger [1, Theorem 4.5] and we will use it in the following formulation:

Theorem 12. *Let $\mathcal{N}$ be a matroid on E and let $\mathcal{C}$ be a complex whose vertex set is also E. If*

$$\eta(\mathcal{C}[X]) \geq \mathrm{rank}_{\mathcal{N}}(E) - \mathrm{rank}_{\mathcal{N}}(E \setminus X) \tag{1}$$

for every $X \subseteq E$ which is the complement of a flat of $\mathcal{N}$, then $\mathcal{N}$ has a base which is a face of $\mathcal{C}$.

It can be shown [4] that if $\mathcal{C}$ is the complex of independent sets of a matroid $\mathcal{M}$, then $\eta(\mathcal{C})$ equals the rank of $\mathcal{M}$ (unless the dual of $\mathcal{M}$ contains a loop, in which case $\eta(\mathcal{C})$ is infinite). From this, one can easily derive that Theorem 12 implies the nontrivial direction of Theorem 10.

Another class of complexes that is relevant in this paper is that of independence complexes. The *independence complex* $\mathcal{I}(H)$ of a graph H is a complex on $V(H)$ whose faces are the independent sets of H (that is, sets I such that the induced subgraph of H on I has no edges). When H is the line graph of a graph G, the independent sets of H are the matchings of G and this construction produces the *matching complex* of G.

Since the connectivity of a complex is in general difficult to establish, it is very useful that there are several results relating the connectivity of an independence complex $\mathcal{I}(H)$ to the properties of the graph H. Typically, these properties concern some variant of the notion of domination in H (a useful overview is given in [2, Section 2]). In our case, the bound involves the edge-domination number which we recall next.

A set D of edges of the graph H is said to *dominate* H if for every vertex v of H, v or at least one of its neighbors is incident with an edge of D. The least cardinality of a set of edges dominating H is the *edge-domination number* $\gamma^E(H)$ of H. (If H contains an isolated vertex, then there is no dominating set of edges and $\gamma^E(H)$ is defined to be infinite.) The following result is implicitly proved in several papers on independence complexes (for references, see [2]):

Theorem 13. *If H is a graph, then*

$$\eta(\mathcal{I}(H)) \geq \gamma^E(H).$$

If we specialize this result to line graphs (and matching complexes), we obtain a notion previously used in [9]. A *2-path* in a graph is a path of length 2. A set P of 2-paths in G *dominates* a set F of edges if every edge of F is incident with a 2-path in P. The *2-path domination number* $\gamma^{\mathrm{v}}(G)$ is the minimum size of a set of 2-paths dominating $E(G)$ (or ∞ if G contains a component with exactly one edge). Since $\gamma^{\mathrm{v}}(G)$ is equal to $\gamma^E(L(G))$, we have the following observation:

Observation 14. *If G is a graph and $\mathcal{C}$ is its matching complex, then*

$$\eta(\mathcal{C}) \geq \gamma^{\mathrm{v}}(G).$$

We conclude this section with the definition of the induced subcomplex. If $\mathcal{C}$ is a complex on E and $X \subseteq E$, then the *induced subcomplex* $\mathcal{C}[X]$ of $\mathcal{C}$ on X is the complex on X consisting of all the faces of $\mathcal{C}$ contained in X.

4 Proof of Theorem 5

We now prove Theorem 5. Let k be a positive integer and G be a graph with $\Upsilon_f(G) \leq k + \varepsilon$, where

$$\varepsilon = \frac{1}{(k+2)(3k^2+1)}.$$

Throughout this section, we write E for $E(G)$, $\mathcal{M}$ for the cycle matroid of G and $\mathcal{C}$ for the matching complex of G. We also let $\mathcal{N}$ denote the matroid $(\mathcal{M}^{(k)})^*$. Our aim is to use Theorem 12 to decompose E into k forests and a matching in G. The following easy lemma provides the link:

Lemma 15. *The set E can be covered by k forests and a matching if and only if there exists a base of $\mathcal{N}$ which is a matching of G (i.e., a face of $\mathcal{C}$).*

Proof. We prove necessity first. Let a matching M of G be a base of $\mathcal{N}$. Then $E \setminus M$ is a base of $\mathcal{N}^* = \mathcal{M}^{(k)}$. In particular, $E \setminus M$ is the union of k forests in G.

To prove sufficiency, let $E = F \cup M'$, where F is (the edge set of) a union of k forests and M' is a matching. Since F is independent in $\mathcal{M}^{(k)}$, there is a base B of $\mathcal{M}^{(k)}$ containing F. Its complement $E \setminus B$ is a base of $\mathcal{N}$ that is disjoint from F and hence contained in M'. As a subset of a matching, $E \setminus B$ is a matching itself. This proves the lemma. $\square$

We first characterize the independent sets of $\mathcal{N}$. Let us say that a set of edges $B \subseteq E$ is *basic* if the subgraph induced by B can be covered by k forests and has the maximum possible number of edges among the subgraphs of G with this property. This is just another way of saying that B is a base of $\mathcal{M}^{(k)}$. From the definition of the dual matroid, we get the following observation:

Observation 16. *A set $X \subseteq E$ is independent in $\mathcal{N}$ if and only if it is disjoint from some basic set $B \subseteq E$.*

In Theorem 12, the restriction to sets that are complements of flats will be crucial for us. The reason is given by the following lemma and Lemma 18 below:

Lemma 17. *If $X \subseteq E$ is a flat in $\mathcal{N}$, then the subgraph $G[E \setminus X]$ of G has minimum degree at least $k+1$.*

Proof. Suppose that $G[E \setminus X]$ contains a vertex v of degree $d(v) \leq k$; let $E(v)$ denote the set of edges of G incident with v. Let I be an inclusionwise maximal independent set of $\mathcal{N}$ contained in $X \cap E(v)$. (Such a set exists since $\emptyset$ is independent.)

Since v has nonzero degree in $G[E \setminus X]$, we may choose an edge e of $E(v) \setminus X$. We claim that $I \cup e$ is dependent in $\mathcal{N}$. Suppose that this is not the case. By Lemma 16, some basic set $B \subseteq E$ is disjoint from $I \cup e$. Let us choose edge-disjoint forests $F_1, \ldots, F_k$ such that $B = E(F_1 \cup \cdots \cup F_k)$.

Since $|E(v) \setminus X| \leq k-1$, one of the forests (say, F_1) does not contain any edge of $E(v) \setminus X$. Let F_1' be obtained by adding e to F_1. By the maximality of B, F_1' is not a forest. Thus, F_1' contains a unique cycle C and $e \in E(C)$. Let f be the other edge of C incident with v. We know that $f \in X$ and $f \notin I$. Since the set $I \cup f$ is disjoint from the set $B \cup e \setminus f$ which is clearly basic, $I \cup f$ is an independent set of $\mathcal{N}$. This contradiction with the choice of I proves that $I \cup e$ is dependent as claimed.

Let I' be an inclusionwise minimal subset of I such that $I' \cup e$ is dependent. Then $I' \cup e$ is a circuit of $\mathcal{N}$ contained in $X \cup e$, contradicting Lemma 11. This finishes the proof. $\square$

We will now see that Lemma 17 makes it possible to lower bound the connectivity of the matching complexes of the subgraphs that appear in our application of Theorem 12 in terms of their order. Without a minimum degree condition, such a bound would not be possible, as is seen by considering the star $K_{1,n}$, whose matching complex has $\eta = 0$ for every $n \geq 2$.

Lemma 18. *Let H be a graph with $\Upsilon_f(H) \leq k + \varepsilon$ (where ε is as defined at the beginning of this section) and with minimum degree at least $k+1$. Let $\mathcal{K}$ be the matching complex of H. Then*

$$\eta(\mathcal{K}) \geq \varepsilon \cdot |V(H)| \,.$$

Proof. Let m and n be the number of edges and vertices of H, respectively. By Observation 14, it suffices to prove that $E(H)$ cannot be dominated by fewer than εn 2-paths.

Let A be the set of vertices of degree at most $k(k+2)$ in H and B be the set of edges with both endvertices in A. We write $a = |A|$ and $b = |B|$. We will first prove a lower bound on b, and then deduce a lower bound on the size of a dominating set of 2-paths.

Since vertices in A have degree at least $k+1$ and the degree of any other vertex is at least $(k+1)^2$, we can lower-bound the number of edges of H as

$$m \geq \frac{k+1}{2} a + \frac{(k+1)^2}{2} (n-a) \tag{2}$$

as well as

$$m \geq (k+1)a - b. \tag{3}$$

Multiplying (3) by k and summing with the inequality (2) multiplied by 2, we find

$$m(k+2) \geq (k+1)^2 n - kb. \tag{4}$$

On the other hand, an upper bound to m is given by $n(k+\varepsilon)$. In combination with (4), this gives us the following lower bound for b:

$$\begin{aligned} b &\geq n \cdot \frac{(k+1)^2 - (k+2)(k+\varepsilon)}{k} \\ &= n \cdot \frac{1 - \varepsilon(k+2)}{k} \\ &= n \cdot \frac{3k}{3k^2+1} \end{aligned} \tag{5}$$

by the definition of ε.

We finish the proof by observing that B takes many 2-paths to dominate. More precisely, it is easy to see that any 2-path dominates at most $3k(k+2)$ edges from B. By (5), the minimum size of a set of 2-paths dominating B is at least

$$\frac{b}{3k(k+2)} \geq n \cdot \frac{3k}{3k^2+1} \cdot \frac{1}{3k(k+2)} = \varepsilon n.$$

Observation 14 implies that $\eta(\mathcal{K}) \geq \varepsilon n$ as claimed. □

We will need one more auxiliary result, an observation on the definition of fractional arboricity:

Lemma 19. *If G is a graph and $H \subseteq G$, then*

$$|E(H)| \leq \Upsilon_f(G) \cdot (|V(H)| - c(H)),$$

where $c(H)$ denotes the number of components of H.

Proof. The lemma holds for a connected subgraph H. If H has components $H_1, \ldots, H_\ell$, then for each $i = 1, \ldots, \ell$, we know that

$$|E(H_i)| \leq \Upsilon_f(G) \cdot (|V(H_i)| - 1)$$

and the claim follows by summing these inequalities. □

Let us finish the proof of Theorem 5. By Lemma 15, we will be done if we can find a base of $\mathcal{N}$ which is a face of $\mathcal{C}$ (recall that $\mathcal{C}$ is the matching complex of G). In view of Theorem 12, we consider an arbitrary set X which is the complement of a flat of $\mathcal{N}$, and aim to verify condition (1).

Using Lemma 6 and then Corollary 9 (for the equality on the last line), the right hand side of (1) can be rewritten as

$$\begin{aligned}\operatorname{rank}_{\mathcal{N}}(E) - \operatorname{rank}_{\mathcal{N}}(E \setminus X) &= \Big(|E| + \operatorname{rank}_{\mathcal{M}^{(k)}}(\emptyset) - \operatorname{rank}_{\mathcal{M}^{(k)}}(E)\Big) \\ &\quad - \Big(|E \setminus X| + \operatorname{rank}_{\mathcal{M}^{(k)}}(X) - \operatorname{rank}_{\mathcal{M}^{(k)}}(E)\Big) \\ &= |X| - \operatorname{rank}_{\mathcal{M}^{(k)}}(X) \\ &= |X| - \min_{T \subseteq X}\Big(|X \setminus T| + k \cdot \operatorname{rank}_{\mathcal{M}}(T)\Big).\end{aligned}$$

Consequently, to be able to apply Theorem 12, it suffices to verify that

$$\eta(\mathcal{C}\,[X]) \geq |T| - k \cdot \operatorname{rank}_{\mathcal{M}}(T) \tag{6}$$

for every $T \subseteq X$.

Consider a fixed $T \subseteq X$. Let $n(T)$ and $n(X)$ denote the number of vertices of $G[T]$ and $G[X]$, respectively, and let $c(T)$ be the number of components of $G[T]$. By Lemma 7, $\operatorname{rank}_{\mathcal{M}}(T) = n(T) - c(T)$.

As for the left hand side of (6), observe that $\mathcal{C}\,[X]$ is just the matching complex of $G[X]$. Since X is the complement of a flat of $\mathcal{N}$, $G[X]$ has minimum degree at least $k+1$ (Lemma 17). By Lemma 18, $\eta(\mathcal{C}\,[X]) \geq \varepsilon n(X)$.

With the aim of establishing (6) (for the given X and T) in mind, we write

$$\begin{aligned}|T| &\leq (k + \varepsilon) \cdot (n(T) - c(T)) \\ &\leq k \cdot (n(T) - c(T)) + \varepsilon n(X) \\ &\leq k \cdot \operatorname{rank}_{\mathcal{M}}(T) + \eta(\mathcal{C}\,[X]),\end{aligned}$$

where we use the fractional arboricity assumption and Lemma 19 for the first inequality, the inclusion $T \subseteq X$ for the second one, and the above interpretation of $\operatorname{rank}_{\mathcal{M}}(T)$ together with Lemma 18 for the last one. The resulting upper bound for $|T|$ is equivalent to (6). Thus, we have verified (6) for any set X which is the complement of a flat of $\mathcal{N}$ and any $T \subseteq X$, and a final invocation of Theorem 12 completes the proof.

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