EXPONENTIALLY MANY PERFECT MATCHINGS IN CUBIC GRAPHS

L. ESPERET, F. KARDOŠ, A. D. KING, D. KRÁL', AND S. NORINE

Abstract. We show that every cubic bridgeless graph G has at least $2^{|V(G)|/3656}$ perfect matchings. This confirms an old conjecture of Lovász and Plummer.

1. Introduction

Given a graph G, let $\mathcal{M}(G)$ denote the set of perfect matchings in G. A classical theorem of Petersen [14] states that every cubic bridgeless graph has at least one perfect matching, i.e. $\mathcal{M}(G) \neq \emptyset$. Indeed, it can be proven that any edge in a cubic bridgeless graph is contained in some perfect matching [13], which implies that $|\mathcal{M}(G)| \geq 3$.

In the 1970s, Lovász and Plummer conjectured that the number of perfect matchings of a cubic bridgeless graph G should grow exponentially with its order (see [11, Conjecture 8.1.8]). It is a simple exercise to prove that G contains at most $2^{|V(G)|}$ perfect matchings, so we can state the conjecture as follows:

Lovász-Plummer conjecture. There exists a universal constant $\epsilon > 0$ such that for any cubic bridgeless graph G,

$$2^{\epsilon|V(G)|} \leq |\mathcal{M}(G)| \leq 2^{|V(G)|}.$$

The problem of computing $|\mathcal{M}(G)|$ is connected to problems in molecular chemistry and statistical physics (see e.g. [11, Section 8.7]). In general graphs, this problem is $\sharp P$-complete [16]. Thus we are interested in finding good bounds on the number of perfect matchings for various classes of graphs such as the bounds in the conjecture above.

The first author is partially supported by ANR Project Heredia under Contract ANR-10-JCJC-HEREDIA.

The second author is partially supported by Slovak Research and Development Agency under Contract No. APVV-0007-07.

The third author is supported by an NSERC Postdoctoral Fellowship.

The Institute for Theoretical Computer Science is supported by Ministry of Education of Czech Republic as projects 1M0545. The work leading to this invention has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ERC grant agreement no. 259385.

The fifth author is partially supported by NSF under Grant No. DMS-0803214.

For bipartite graphs, $|\mathcal{M}(G)|$ is precisely the permanent of the graph biadjacency matrix. Voorhoeve proved the conjecture for cubic bipartite graphs in 1979 [17]; Schrijver later extended this result to all regular bipartite graphs [15]. We refer the reader to [10] for an exposition of this connection and of an elegant proof of Gurvits generalizing Schrijver's result. For *fullerene graphs*, a class of planar cubic graphs for which the conjecture relates to molecular stability and aromaticity of fullerene molecules, the problem was settled by Kardoš, Král', Miškuf and Sereni [8]. Chudnovsky and Seymour recently proved the conjecture for all cubic bridgeless planar graphs [1].

The general case has until now remained open. Edmonds, Lovász and Pulleyblank [4] proved that any cubic bridgeless G contains at least $\frac{1}{4}|V(G)| + 2$ perfect matchings (see also [12]); this bound was later improved to $\frac{1}{2}|V(G)|$ [9] and then $\frac{3}{4}|V(G)| - 10$ [6]. The order of the lower bound was not improved until Esperet, Kardoš, and Král' proved a superlinear bound in 2009 [5]. The first bound, proved in 1982, is a direct consequence of a lower bound on the dimension of the perfect matching polytope, while the more recent bounds combine polyhedral arguments with analysis of brick and brace decompositions.

In this paper we solve the general case. To avoid technical difficulties when contracting sets of vertices, we henceforth allow graphs to have multiple edges, but not loops. Let $m(G)$ denote $|\mathcal{M}(G)|$, and let $m^\star(G)$ denote the minimum, over all edges $e \in E(G)$, of the number of perfect matchings containing e. Our result is the following:

Theorem 1. *For every cubic bridgeless graph G we have*

$$m(G) \geq 2^{|V(G)|/3656}.$$

We actually prove that at least one of two sufficient conditions applies:

Theorem 2. *For every cubic bridgeless graph G, at least one of the following holds:*

[**S1**] $m^\star(G) \geq 2^{|V(G)|/3656}$, *or*
[**S2**] *there exist $M, M' \in \mathcal{M}(G)$ such that $M \triangle M'$ has at least $|V(G)|/3656$ components.*

To see that Theorem 2 implies Theorem 1, we can clearly assume that [**S2**] holds since $m^\star(G) \leq m(G)$. Choose $M, M' \in \mathcal{M}(G)$ such that the set $\mathcal{C}$ of components of $M \triangle M'$ has cardinality at least $|V(G)|/3656$, and note that each of these components is an even cycle alternating between M and M'. Thus for any subset $\mathcal{C}' \subseteq \mathcal{C}$, we can construct a perfect matching $M_{\mathcal{C}'}$ from M by flipping the edges on the cycles in $\mathcal{C}'$, i.e. $M_{\mathcal{C}'} = M \triangle \bigcup_{C \in \mathcal{C}'} C$. The $2^{|\mathcal{C}|}$ perfect matchings $M_{\mathcal{C}'}$ are distinct, implying Theorem 1.

We cannot discard either of the sufficient conditions [**S1**] or [**S2**] in the statement of Theorem 2. To see that [**S2**] cannot be omitted,

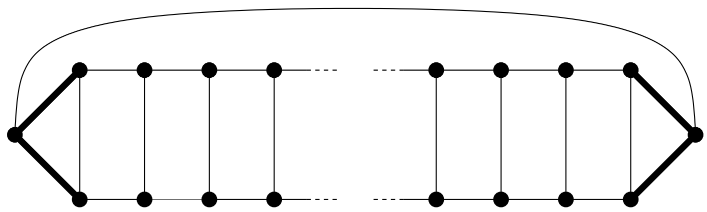

FIGURE 1. A graph cubic bridgeless graph G with $m^\star(G) = 1$.

consider the graph depicted in Figure 1 and observe that each of the four bold edges is contained in a unique perfect matching. To see that [**S1**] cannot be omitted, it is enough to note that there exist cubic graphs with girth logarithmic in their size (see [7] for a construction). Such graphs cannot have linearly many disjoint cycles, so condition [**S2**] does not hold.

1.1. **Definitions and notation.**

For a graph G and a set $X \subseteq V(G)$, $G|X$ denotes the subgraph of G induced by X. For a set $X \subseteq V(G)$, let $\delta(X)$ denote the set of edges with exactly one endpoint in X, and let E_X denote the set of edges with at least one endpoint in X, i.e. $E_X = E(G|X) \cup \delta(X)$. The set $C = \delta(X)$ is called an *edge-cut*, or a *k-edge-cut*, where $k = |C|$, and X and $V(G) \setminus X$ are the *sides* of C. A k-edge-cut is said to be *even* (resp. *odd*) if k is even (resp. odd). Observe that the parity of an edge-cut $\delta(X)$ in a cubic graph is precisely that of $|X|$. An edge-cut $\delta(X)$ is *cyclic* if both $G|X$ and $G|(V(G) \setminus X)$ contain a cycle. Observe that every 2-edge-cut in a cubic graph is cyclic. If G contains no edge-cut (resp. cyclic edge-cut) of size less than k, we say that G is *k-edge-connected* (resp. *cyclically k-edge-connected*).

Observe that the number of perfect matchings of a graph is the product of the number of perfect matchings of its connected components. Hence, in order to prove Theorem 1, we restrict ourselves to connected graphs for the remainder of this paper (this means, for example, that we can consider the terms *2-edge-connected* and *bridgeless* to be interchangeable, and the sides of a cut are well-defined).

For a matching M and vertex set X, we say that M *covers* X or that X is *M-covered* if every vertex in X is an endpoint of an edge in M. Further, we use $M|X$ to denote the set $M \cap E(G|X)$.

1.2. **Constants.** Let $x := \log(\frac{4}{3})/\log(2)$. The following constants appear throughout the paper:

$$\alpha := \tfrac{x}{314}, \qquad \beta_1 := \tfrac{154x}{314}, \qquad \beta_2 := \tfrac{74x}{314}, \qquad \gamma := \tfrac{312x}{314}.$$

We avoid using the numerical values of these constants for the sake of clarity. Throughout the paper we make use of the following inequalities, which can be routinely verified:

$$0 < \alpha \le \beta_2 \le \beta_1, \tag{1}$$

$$1/3656 \le \frac{\alpha}{9\beta_1 + 3}, \tag{2}$$

$$\beta_2 + 6\alpha \le \beta_1, \tag{3}$$

$$74\alpha \le \beta_2, \tag{4}$$

$$146\alpha \le \beta_1, \tag{5}$$

$$\beta_2 + 80\alpha \le \beta_1, \tag{6}$$

$$6\alpha + \gamma \le \log(6)/\log(2), \tag{7}$$

$$\gamma + 2\beta_1 + 7\alpha - \beta_2 \le 1, \tag{8}$$

$$6\alpha + 2\beta_1 \le \log(\tfrac{4}{3})/\log(2), \tag{9}$$

$$2\beta_1 + 4\alpha \le \gamma. \tag{10}$$

The integer 3656 is chosen minimum so that the system of inequalities above has a solution. Inequalities **(4)**, **(6)**, **(9)**, and **(10)** are tight.

2. The proof of Theorem 2

In this section we sketch the proof of Theorem 2, postponing the proofs of two main lemmas until later sections. Our general approach to Theorem 2 is to reduce on cyclic 2-edge-cuts and cyclic 3-edge-cuts and prove inductively that either [**S1**] or [**S2**] holds. Dealing with [**S1**] is relatively straightforward – perfect matchings containing a given edge behave well with reductions on a cut, which is our main motivation for considering $m^\star(G)$. To deal with [**S2**], we do not directly construct perfect matchings M and M' for which $M \triangle M'$ has many components. Instead, we prove the existence of a vector $\mathbf{w}$ in the perfect matching polytope that in turn guarantees the existence of such perfect matchings M and M'. In order to do this, we define a special type of vertex set in which a given random perfect matching admits an alternating cycle with high probability (i.e. at least $\frac{1}{3}$). We call these sets *burls* and we call a set of disjoint burls a *foliage* – a large foliage will guarantee the existence of two perfect matchings with many components in their symmetric difference. In the end, the vector $\mathbf{w}$ we seek in the perfect matching polytope will be uniformly valued $\frac{1}{3}$ except inside the burls.

2.1. Alternating sets and the perfect matching polytope.

To define burls properly, we must first define three notions of a vertex set X being *alternating*. The first is simple. Given a matching M such that X is M-covered, we say that X is *M-alternating* if there is another matching M' such that X is M'-covered and $M \triangle M' \subseteq (G|X)$. The other two notions require consideration of random variables in $\mathcal{M}(G)$.

Let $\mathbf{M}$ be a random perfect matching, i.e. a random variable $\mathbf{M}$ in $\mathcal{M}(G)$, and let $\mathbf{w}$ be a real edge weighting in $\mathbb{R}^{E(G)}$. We say that $\mathbf{M}$ *corresponds to* $\mathbf{w}$ (and vice-versa) if for every edge e, we have $\Pr[e \in \mathbf{M}] = \mathbf{w}(e)$. The *perfect matching polytope* $\mathcal{PMP}(G)$ is the set of edge weightings $\mathbf{w}$ with at least one corresponding random variable $\mathbf{M_w}$ on $\mathcal{M}(G)$. The second notion of an alternating set involves a weighting $\mathbf{w} \in \mathcal{PMP}(G)$. For such $\mathbf{w}$ we say that X is $\mathbf{w}$*-alternating* if for every $\mathbf{M_w}$ corresponding to $\mathbf{w}$, we have

$$\Pr[X \text{ is } \mathbf{M_w}\text{-alternating}] \geq 1/3.$$

If $\{X_1, \dots, X_k\}$ is a collection of disjoint $\mathbf{w}$-alternating sets, then for a random variable $\mathbf{M_w}$ in $\mathcal{M}(G)$ corresponding to $\mathbf{w}$, the probability that $\mathbf{M_w}$ is X_i-alternating for at least $k/3$ values of i is non-zero. Thus [**S2**] is satisfied as long as we have a vector $\mathbf{w} \in \mathcal{PMP}(G)$ and a collection of at least $\frac{3}{3656} \cdot |V(G)|$ disjoint $\mathbf{w}$-alternating sets. Unfortunately the notion of $\mathbf{w}$-alternating sets has a troublesome shortcoming: When deciding whether or not X is a $\mathbf{w}$-alternating set, we want the freedom to ignore the weighting $\mathbf{w}$ on edges not intersecting X.

Thus for the third notion of an alternating set, we look at partial edge weightings. Given a vertex set X, let $\mathbf{w}_X$ be a weighting on the edges of E_X, i.e. those edges with at least one endpoint in X. Let $\mathcal{M}(G, X)$ denote the set of matchings contained in E_X and covering X. As with edge weightings in $\mathbb{R}^{E(G)}$, we say that a random variable $\mathbf{M}^X_\mathbf{w} \in \mathcal{M}(G, X)$ *corresponds to* $\mathbf{w}_X$ (and vice-versa) if for every edge $e \in E_X$, we have $\Pr[e \in \mathbf{M}] = \mathbf{w}(e)$. We say that the set X is *strongly* $\mathbf{w}_X$*-alternating* if for every random variable $\mathbf{M}^X_\mathbf{w}$ on $\mathcal{M}(G, X)$ corresponding to $\mathbf{w}_X$, we have

$$\Pr[X \text{ is } \mathbf{M}^X_\mathbf{w}\text{-alternating}] \geq 1/3.$$

Given an edge weighting $\mathbf{w}$ and an edge set E' such that $\mathbf{w}$ gives each edge in E' a weight, let $\mathbf{w}|E'$ denote the restriction of $\mathbf{w}$ to E'. Clearly, if we have a total edge weighting $\mathbf{w} \in \mathcal{PMP}(G)$ such that a vertex set X is strongly $(\mathbf{w}|E_X)$-alternating, then X is $\mathbf{w}$-alternating.

We now extend this idea. We wish to take a collection of disjoint vertex sets $\{X_1, \dots, X_k\}$ and partial edge weightings $\mathbf{w}_{X_i}$ such that each X_i is strongly $\mathbf{w}_{X_i}$-alternating, and construct from them a total edge weighting $\mathbf{w}$ such that each X_i is $\mathbf{w}$-alternating. To do this as simply as possible we want $\mathbf{w}$, which must be in $\mathcal{PMP}(G)$, to agree with each $\mathbf{w}_{X_i}$. Thus we certainly want the partial weightings to agree – this only concerns edges on the boundaries of the vertex sets – but we need more restrictions. To determine a sufficient set of restrictions for $\mathbf{w}_{X_i}$, we use Edmonds' characterization of the perfect matching polytope:

Theorem 3 (Edmonds [3]). *Let G be a graph and let $\mathbf{w}$ be a vector in $\mathbb{R}^{E(G)}$. Then $\mathbf{w}$ is in $\mathcal{PMP}(G)$ precisely if the following hold:*

(i) $0 \leq \mathbf{w}(e) \leq 1$ *for each* $e \in E(G)$,
(ii) $\mathbf{w}(\delta(\{v\})) = 1$ *for each vertex* $v \in V$, *and*
(iii) $\mathbf{w}(\delta(X)) \geq 1$ *for each* $X \subseteq V$ *of odd cardinality.*

This characterization immediately tells us that for any bridgeless cubic graph, the vector $\mathbf{\frac{1}{3}}$, i.e. the vector valued $\frac{1}{3}$ on each edge, is in $\mathcal{PMP}(G)$. Given a vertex set X, let ∂X denote the set of vertices in X incident to edges in $\delta(X)$. We say that a partial edge weighting $\mathbf{w}_X$ on E_X is *extendable from* X if it satisfies the following sufficient restrictions:

EXT1: $\mathbf{w}_X(e) \in \{0, \frac{1}{3}, \frac{2}{3}\}$ for each $e \in E_X$,
EXT2: $\mathbf{w}_X(\delta(\{v\})) = 1$ for each vertex $v \in X$,
EXT3: $\mathbf{w}_X(e) = \frac{1}{3}$ for each $e \in \delta(X)$,
EXT4: $\mathbf{w}_X(C) \geq \frac{1}{3}$ for every non-empty edge-cut C in $G|X$,
EXT5: if $\mathbf{w}_X(C) < 1$ for some edge-cut C in $G|X$ with $|C|$ odd then either $|C| = 1$ or one of the sides of C contains exactly one vertex in ∂X.

We are finally ready to formally define burls and foliages. A vertex set X is a *burl* if there exists a vector $\mathbf{w}_X \in \mathbb{R}^{E_X}$ such that (1) X is $\mathbf{w}_X$-alternating, and (2) $\mathbf{w}_X$ is extendable from X. In this case we say that $\mathbf{w}_X$ is a *certificate* for the burl X. Again, a collection of disjoint vertex sets $\{X_1, \ldots, X_k\}$ is a *foliage* if each X_i is a burl.

We already noted that $\mathbf{\frac{1}{3}} \in \mathcal{PMP}(G)$. We can also verify that for any vertex set X, the partial weighting $\mathbf{\frac{1}{3}}|E_X$ is extendable from X. Actually, much more is true. The following lemma clarifies our motivation for the definition of a foliage:

Lemma 4. *Let G be a cubic bridgeless graph, let $\mathcal{X} = \{X_1, \ldots, X_k\}$ be a foliage, and for each i let $\mathbf{w}_{X_i}$ be a certificate for X_i. Let $\mathbf{w}$ be an edge weighting for G defined as*

$$\mathbf{w}(e) = \begin{cases} \mathbf{w}_{X_i}(e) & \text{if } \ e \in E(G|X_i) \\ 1/3 & \text{if } \ e \notin \bigcup_i E(G|X_i) \end{cases}$$

Then every set $X_i \in \mathcal{X}$ is $\mathbf{w}$-alternating.

Proof. Since every partial weighting $\mathbf{w}_{X_i}$ is equal to $\frac{1}{3}$ on the boundary of X_i, we know that each X_i is strongly $(\mathbf{w}|E_{X_i})$-alternating. Therefore each X_i is $\mathbf{w}$-alternating. It remains to confirm that $\mathbf{w} \in \mathcal{PMP}(G)$. By Theorem 3 it suffices to check that $\mathbf{w}$ satisfies conditions (i), (ii) and (iii). The first two conditions are satisfied by (EXT1), (EXT2) and (EXT3). To verify (iii), consider an odd $Y \subseteq V(G)$. We show that $\mathbf{w}(\delta(Y)) \geq 1$.

It follows from (EXT1) and (EXT2) that $3\mathbf{w}(\delta(Y))$ is an odd integer. Therefore, it is sufficient to verify that $\mathbf{w}(\delta(Y)) > 1/3$. Let $X_i \in \mathcal{X}$ be such that $C = \delta(Y) \cap E(G|X_i)$ is a non-empty edge-cut in $G|X_i$. (If no such X_i exists then $\mathbf{w}(\delta(Y)) \geq \frac{1}{3}|\delta(Y)| \geq 1$ by (EXT3).) It follows from (EXT1), (EXT2) and (EXT3) that $|C|$ and $3\mathbf{w}(C)$ have the same parity. Therefore, $\mathbf{w}(\delta(Y)) > 1/3$ by (EXT3) and (EXT4), unless $|C|$ is odd and $\delta(Y) = C$. In this last case, we have $|C| > 1$, as G is bridgeless and by (EXT5) one of the sides of C, without loss of generality $X_i \cap Y$, contains exactly one vertex in ∂X_i. Then $\delta(Y \setminus X_i)$ consists only of edges incident to this vertex, contradicting once again the fact that G is bridgeless. □

In light of what we have already discussed, we get the following key fact as a consequence:

Corollary 5. *If a cubic bridgeless graph G contains a foliage $\mathcal{X}$, then there exist perfect matchings $M, M' \in \mathcal{M}(G)$ such that $M \triangle M'$ has at least $|\mathcal{X}|/3$ components.*

2.2. Burls, twigs, and foliage weight.

We now introduce a special class of burls. Let G be a cubic bridgeless graph and let $X \subseteq V(G)$. We say that X is a 2-*twig* if $|\delta(X)| = 2$, and X is a 3-*twig* if $|\delta(X)| = 3$ and $|X| \geq 5$ (that is, X is not a triangle or a single vertex). A *twig* in G is a 2- or 3-twig. Before we prove that every twig is a burl, we need a simple lemma.

Lemma 6. *Let G be a cubic bridgeless graph. Then*

1. *$m(G - e) \geq 2$ for every $e \in E(G)$, and*
2. *$m(G) \geq 4$ if $|V(G)| \geq 6$. In particular, for any $v \in V(G)$ there is an $e \in \delta(\{v\})$ contained in at least two perfect matchings.*

Proof. The first item follows from the classical result mentioned in the introduction: every edge of a cubic bridgeless graph is contained in a perfect matching. The second is implied by the bound $m(G) \geq \frac{1}{4}|V(G)| + 2$ from [4]. □

Lemma 7. *Every twig X in a cubic bridgeless graph G is a burl.*

Proof. We show that $\mathbf{w}_X = \mathbf{\frac{1}{3}}|E_X$ is a certificate for X. As we already noted, $\mathbf{w}_X$ is extendable from X. Let $\mathbf{M}^X_\mathbf{w}$ be a random matching in $\mathcal{M}(G, X)$ corresponding to $\mathbf{w}_X$, as in the definition of a strongly alternating set.

If X is a 2-twig, let H be obtained from $G|X$ by adding an edge e joining the vertices in ∂X. Then H is cubic and bridgeless. By applying Lemma 6(1) to H, we see that the set X is M-alternating for every $M \in \mathcal{M}(G, X)$ such that $M \cap \delta(X) = \emptyset$. As $\Pr[\mathbf{M}^X_\mathbf{w} \cap \delta(X) = \emptyset] \geq 1 - \mathbf{M}^X_\mathbf{w}(\delta(X)) = 1/3$, we conclude that X is strongly $\mathbf{w}_X$-alternating.

Suppose now that X is a 3-twig. Let $\delta(X) = \{e_1, e_2, e_3\}$. Let H be obtained from G by identifying all the vertices in $V(G) - X$ (removing loops but preserving multiple edges). We apply Lemma 6(2) to H, which is again cubic and bridgeless. It follows that for some $1 \leq i \leq 3$, the edge e_i is in at least two perfect matchings of H. Therefore X is M-alternating for every $M \in \mathcal{M}(G, X)$ such that $M \cap \delta(X) = \{e_i\}$.

Finally, $\Pr[\mathbf{M}_{\mathbf{w}}^X \cap \delta(X) = \{e_i\}] = 1/3$ and thus X is strongly $\mathbf{w}_X$-alternating. □

The *weight* of a foliage $\mathcal{X}$ containing k twigs is defined as $fw(\mathcal{X}) := \beta_1 k + \beta_2(|\mathcal{X}| - k)$, that is each twig has weight β_1 and each non-twig burl has weight β_2. Let $fw(G)$ denote the maximum weight of a foliage in a graph G.

2.3. **Reducing on small edge-cuts.**

We now describe how we reduce on 2-edge-cuts and 3-edge-cuts, and consider how these operations affect $m^\star(G)$ and foliages. Let C be a 3-edge-cut in a cubic bridgeless graph G. The two graphs G_1 and G_2 obtained from G by identifying all vertices on one of the sides of the edge-cut (removing loops but preserving multiple edges) are referred to as *C-contractions* of G and the vertices in G_1 and G_2 created by this identification are called *new*.

We need a similar definition for 2-edge-cuts. Let $C = \{e, e'\}$ be a 2-edge-cut in a cubic bridgeless graph G. The two *C-contractions* G_1 and G_2 are now obtained from G by deleting all vertices on one of the sides of C and adding an edge joining the remaining ends of e and e'. The resulting edge is now called *new*.

In both cases we say that G_1 and G_2 *are obtained from G by a cut-contraction*. The next lemma provides some useful properties of cut-contractions.

Lemma 8. *Let G be a graph, let C be a 3- or a 2-edge-cut in G, and let G_1 and G_2 be the two C-contractions. Then*

(1) *G_1 and G_2 are cubic bridgeless graphs,*
(2) *$m^\star(G) \geq m^\star(G_1)\, m^\star(G_2)$, and*
(3) *For $i = 1, 2$ let $\mathcal{X}_i$ be a foliage in G_i such that for every $X \in \mathcal{X}_i$, if $|C| = 3$ then X does not contain the new vertex, and if $|C| = 2$ then $E(G_i|X)$ does not contain the new edge. Then $\mathcal{X}_1 \cup \mathcal{X}_2$ is a foliage in G. In particular, we have $fw(G) \geq fw(G_1) + fw(G_2) - 2\beta_1$.*

Proof.

(1) This can be confirmed routinely.
(2) Consider first the case of the contraction of a 2-edge-cut $C = \delta(X)$ in G. Let e be an edge with both ends in $X = V(G_1)$. Every perfect matching of G_1 containing e combines either with $m^\star(G_2)$ perfect matchings of G_2 containing the new edge of G_2, or with $2m^\star(G_2)$ perfect matchings of G_2 avoiding the new edge of G_2. If e lies in C, note that perfect matchings of G_1 and G_2 containing the new edges can be combined into perfect matchings of G containing C. Hence, e is in at least $m^\star(G_1)\, m^\star(G_2)$ perfect matchings of G.

Now consider a 3-edge-cut $C = \delta(X)$. If e has both ends in $X \subset V(G_1)$, perfect matchings of G_1 containing e combine with perfect matchings of G_2 containing either of the 3 edges of C. If e is in C, perfect matchings containing e in G_1 and G_2 can also be combined into perfect matchings of G. In any case, e is in at least $m^\star(G_1)\, m^\star(G_2)$ perfect matchings of G.

(3) By (EXT3), the coordinates of the new elements (if they are defined) in their respective certificates are precisely 1/3, so assigning 1/3 (if necessary) to the edges of C in G yields valid certificates for the elements of $\mathcal{X}_1 \cup \mathcal{X}_2$. Since $\beta_1 \geq \beta_2$, this implies $fw(G) \geq fw(G_1) + fw(G_2) - 2\beta_1$. □

It is not generally advantageous to reduce on a 3-edge-cut arising from a triangle, unless this reduction leads to a chain of similar reductions. Thus we wish to get rid of certain triangles from the outset. We say that a triangle sharing precisely one edge with a cycle of length three or four in a graph G is *relevant*, and otherwise it is *irrelevant*. A graph G is *pruned* if it contains no irrelevant triangles. The following easy lemma shows that we can prune a bridgeless cubic graph by repeated cut-contraction without losing too many vertices.

Lemma 9. *Let G be a cubic bridgeless graph, and let k be the size of maximum collection of vertex-disjoint irrelevant triangles in G. Then one can obtain a pruned cubic bridgeless graph G' from G with $|V(G')| \geq |V(G)| - 2k$ by repeatedly contracting irrelevant triangles.*

Proof. We proceed by induction on k. Let a graph G'' be obtained from G by contracting an irrelevant triangle T. The graph G'' is cubic and bridgeless by Lemma 8(1). Since T is irrelevant in G, the unique vertex of G'' obtained by contracting T is not in a triangle in G''. Therefore if $\mathcal{T}$ is a collection of vertex disjoint irrelevant triangles in G'' then $\mathcal{T} \cup \{T\}$ is such a collection in G. (After the contraction of an irrelevant triangle, triangles that were previously irrelevant might become relevant, but the converse is not possible.) It follows that $|\mathcal{T}| \leq k - 1$. By applying the induction hypothesis to G'', we see that the lemma holds for G. □

Corollary 10. *Let G be a cubic bridgeless graph. Then we can obtain a cubic bridgeless pruned graph G' from G with $|V(G')| \geq |V(G)|/3$ by repeatedly contracting irrelevant triangles.*

We wish to restrict our attention to pruned graphs, so we must make sure that the function $m^\star(G)$ and the maximum size of a foliage does not increase when we contract a triangle.

Lemma 11. *Let G' be obtained from a graph G by contracting a triangle. Then $m^\star(G') \leq m^\star(G)$ and the maximum size of a foliage in G' is at most the maximum size of a foliage in G.*

Proof. Let xyz be the contracted triangle, and let e_x, e_y, and e_z be the edges incident with x, y, z and not contained in the triangle in G. Let t be the vertex of G' corresponding to the contraction of xyz. Every perfect matching M' of G' has a canonical extension M in G: assume without loss of generality that e_x is the unique edge of M' incident to t. Then M consists of the union of M' and yz. Observe that perfect matchings in G containing yz necessarily contain e_x, so every edge of G is contained in at least $m^\star(G')$ perfect matchings.

Now consider a burl X' in G' containing t, and let $\mathbf{w}'$ a the certificate for X'. Let $\mathbf{w}$ be the vector $\mathbf{w}'$ with three new coordinates $\mathbf{w}(xy) = \mathbf{w}'(e_z)$, $\mathbf{w}(yz) = \mathbf{w}'(e_x)$ and $\mathbf{w}(xz) = \mathbf{w}'(e_y)$, then $\mathbf{w}$ is a certificate showing that $X = X' \cup \{x, y, z\} \setminus t$ is a burl in G. Properties (EXT1), (EXT2), and (EXT3) are trivially satisfied. Now consider an edge-cut C in $G|X$. If $B = C \cap \{xy, yz, xz\}$ is empty, (EXT4) and (EXT5) follow directly from the fact that $\mathbf{w}'$ is a certificate for X'. Otherwise B contains precisely two elements, say xy and yz. Then we have $\mathbf{w}(C) \geq \mathbf{w}(xy) + \mathbf{w}(yz) \geq \frac{1}{3}$ by (EXT1) and (EXT2), and therefore, (EXT4) follows. If $|C| \geq 3$ is odd and $\mathbf{w}(C) < 1$, then without loss of generality $\mathbf{w}(xy) = 0$. Using (EXT4) it can be checked that only one of the following two cases applies:

If $C \cap \{e_x, e_z\} = \emptyset$ then $C' = C \cup \{e_x, e_z\} \setminus \{xy, yz\}$ is an edge-cut of the same weight and cardinality as C in $G|X$, but also in $G'|X'$, and consequently, (EXT5) follows.

If $C \cap E_{\{x,y,z\}} = \{xy, yz, e_z\}$ then C has cardinality at least five and $C'' = C \cup \{e_x\} \setminus \{xy, yz, e_z\}$ is an odd edge-cut in $G|X$, but also in $G'|X'$ of cardinality at least 3 and weight $\mathbf{w}(C'') = \mathbf{w}(C)$. Since $\mathbf{w}'$ satisfies (EXT5), $\mathbf{w}$ also satisfies (EXT5) in this case.

Since a burl avoiding t in G' is also a burl in G, it follows from the analysis above that the maximum size of a foliage cannot increase when we contract a triangle. □

2.4. **Proving Theorem 2.**

We say that G *has a core* if we can obtain a cyclically 4-edge-connected graph G' with $|V(G')| \geq 6$ by applying a (possibly empty) sequence of cut-contractions to G (recall that this notion was defined in the previous subsection).

We will deduce Theorem 2 from the next two lemmas. This essentially splits the proof into two cases based on whether or not G has a core.

Lemma 12. *Let G be a pruned cubic bridgeless graph. Let $Z \subseteq V(G)$ be such that $|Z| \geq 2$ and $|\delta(Z)| = 2$, or $|Z| \geq 4$ and $|\delta(Z)| = 3$. Suppose that the $\delta(Z)$-contraction G' of G with $Z \subseteq V(G')$ has no core. Then there exists a foliage $\mathcal{X}$ in G with $\bigcup_{X \in \mathcal{X}} X \subseteq Z$ and*

$$fw(\mathcal{X}) \geq \alpha|Z| + \beta_2.$$

By applying Lemma 12 to a cubic graph G without a core and $Z = V(G) \setminus \{v\}$ for some $v \in V(G)$, we obtain the following.

Corollary 13. *Let G be a pruned cubic bridgeless graph without a core. Then*

$$fw(G) \geq \alpha(|V(G)| - 1) + \beta_2.$$

On the other hand, if G has a core, we will prove that either $fw(G)$ is linear in the size of G or every edge of G is contained in an exponential number of perfect matchings.

Lemma 14. *Let G be a pruned cubic bridgeless graph. If G has a core then*

$$m^\star(G) \geq 2^{\alpha|V(G)| - fw(G) + \gamma}.$$

We finish this section by deriving Theorem 2 from Lemmas 12 and 14.

Proof of Theorem 2. Let $\epsilon := 1/3656$. By Corollary 10 there exists a pruned cubic bridgeless graph G' with $|V(G')| \geq |V(G)|/3$ obtained from G by repeatedly contracting irrelevant triangles. Suppose first that G' has a core. By Corollary 10 and Lemmas 11 and 14, condition [**S1**] holds as long as $\epsilon|V(G)| \leq \alpha|V(G)|/3 - fw(G')$. Therefore we assume $fw(G') \geq (\frac{\alpha}{3} - \epsilon)|V(G)|$. It follows from the definition of $fw(G')$ that G' has a foliage containing at least $(\frac{\alpha}{3} - \epsilon)|V(G)|/\beta_1$ burls. If G' has no core then by Corollary 13 and the fact that $\alpha \leq \beta_2$, $fw(G') \geq \alpha(|V(G')| - 1) + \beta_2 \geq \alpha|V(G')|$, so G' contains a foliage of size at least $\alpha|V(G')|/\beta_1 \geq \alpha|V(G)|/3\beta_1$. In both cases condition [**S2**] holds by Corollary 5 and Lemma 11, since Equation **(2)** tells us that $3\epsilon \leq (\frac{\alpha}{3} - \epsilon)/\beta_1$. □

3. Cut decompositions

In this section we study cut decompositions of cubic bridgeless graphs. We mostly follow notation from [1], however we consider 2- and 3-edge-cuts simultaneously. Cut decompositions play a crucial role in the proof of Lemma 12 in the next section.

Let G be a graph. A *non-trivial cut-decomposition* of G is a pair (T, ϕ) such that:

- T is a tree with $E(T) \neq \emptyset$,
- $\phi : V(G) \to V(T)$ is a map, and
- $|\phi^{-1}(t)| + \deg_T(t) \geq 3$ for each $t \in V(T)$.

For an edge f of T, let T_1, T_2 be the two components of $T \setminus f$, and for $i = 1, 2$ let $X_i = \phi^{-1}(T_i)$. Thus (X_1, X_2) is a partition of $V(G)$ that induces an edge-cut denoted by $\phi^{-1}(f)$. If $|\phi^{-1}(f)| \in \{2, 3\}$ for each $f \in E(T)$ we call (T, ϕ) a *small-cut-decomposition* of G.

Let (T, ϕ) be a small-cut-decomposition of a 2-edge-connected cubic graph G, and let T_0 be a subtree of T such that $\phi^{-1}(V(T_0)) \neq \emptyset$. Let $T_1, \ldots, T_s$ be the components of $T \setminus V(T_0)$, and for $1 \leq i \leq s$ let f_i be the unique edge of T with an end in $V(T_0)$ and an end in $V(T_i)$. For $0 \le i \le s$, let $X_i = \phi^{-1}(V(T_i))$. Thus $X_0, X_1, \dots, X_s$ form a partition of $V(G)$. Let G' be the graph obtained from G as follows. Set $G_0 = G$. For $i = 1, \dots, s$, take G_{i-1} and let G_i be the $(\phi^{-1}(f_i))$-contraction containing X_0. Now let G' denote G_s. Note that G' is cubic. We call G' *the hub of* G *at* T_0 (with respect to (T, ϕ)). If $t_0 \in V(T)$ and $\phi^{-1}(t_0) \neq \emptyset$, by the *hub of* G *at* t_0 we mean the hub of G at T_0, where T_0 is the subtree of T with vertex set $\{t_0\}$.

Let $\mathcal{Y}$ be a collection of disjoint subsets of $V(G)$. We say that a small-cut-decomposition (T, ϕ) of G *refines* $\mathcal{Y}$ if for every $Y \in \mathcal{Y}$ there exists a leaf $v \in V(T)$ such that $Y = \phi^{-1}(v)$. Collections of subsets of $V(G)$ that can be refined by a small-cut decomposition are charaterized in the following easy lemma.

Lemma 15. *Let G be a cubic bridgeless graph. Let $\mathcal{Y}$ be a collection of disjoint subsets of $V(G)$. Then there exists a small-cut-decomposition refining $\mathcal{Y}$ if $|Y| \ge 2$ and $|\delta(Y)| \in \{2,3\}$ for every $Y \in \mathcal{Y}$, and either*

(1) *$\mathcal{Y} = \emptyset$ and G is not cyclically 4-edge-connected, or*
(2) *$\mathcal{Y} = \{Y\}$, and $|V(G) \setminus Y| > 1$, or*
(3) *$|\mathcal{Y}| \ge 2$.*

Proof. We only consider the case $|\mathcal{Y}| \ge 3$, as the other cases are routine. Take T to be a tree on $|\mathcal{Y}| + 1$ vertices with $|\mathcal{Y}|$ leaves $\{v_Y \mid Y \in \mathcal{Y}\}$ and a non-leaf vertex v_0. The map ϕ is defined by $\phi(u) = v_Y$, if $u \in Y$ for some $Y \in \mathcal{Y}$, and $\phi(u) = v_0$, otherwise. Clearly, (T, ϕ) refines $\mathcal{Y}$ and is a small-cut-decomposition of G. □

We say that (T, ϕ) is *$\mathcal{Y}$-maximum* if it refines $\mathcal{Y}$ and $|V(T)|$ is maximum among all small-cut decompositions of G refining $\mathcal{Y}$. The following lemma describes the structure of $\mathcal{Y}$-maximum decompositions. It is a variation of Lemma 4.1 and Claim 1 of Lemma 5.3 in [1].

Lemma 16. *Let G be a cubic bridgeless graph. Let $\mathcal{Y}$ be a collection of disjoint subsets of $V(G)$ and let (T, ϕ) be a $\mathcal{Y}$-maximum small-cut-decomposition of G. Then for every $t \in V(T)$ either $\phi^{-1}(t) = \emptyset$, or $\phi^{-1}(t) \in \mathcal{Y}$, or the hub of G at t is cyclically 4-edge-connected.*

Proof. Fix $t \in V(T)$ with $\phi^{-1}(t) \neq \emptyset$ and $\phi^{-1}(t) \notin \mathcal{Y}$. Let $f_1, \dots, f_k$ be the edges of T incident with t, and let $T_1, \dots, T_k$ be the components of $T \setminus \{t\}$, where f_i is incident with a vertex t_i of T_i for $1 \le i \le k$. Let $X_0 = \phi^{-1}(t)$, and for $1 \le i \le k$ let $X_i = \phi^{-1}(V(T_i))$. Let G' be the hub of G at t, and let G'' be the graph obtained from G' by subdividing precisely once every new edge e corresponding to the cut-contraction of a cut C with $|C| = 2$. The vertex on the subdivided edge e is called *the new vertex corresponding to the cut-contraction of* C, by analogy with the new vertex corresponding to the cut-contraction of a cyclic 3-edge-cut.

Note that G' is cyclically 4-edge-connected if and only if G'' is cyclically 4-edge-connected. Suppose for the sake of contradiction that $C = \delta(Z)$ is a cyclic edge-cut in G'' with $|C| \leq 3$. Then $|C| \in \{2, 3\}$ by Lemma 8(1), as G'' is a subdivision of G' and G' can be obtained from G by repeated cut-contractions. Let T' be obtained from T by by splitting t into two vertices t' and t'', so that t_i is incident to t' if and only if the new vertex of G'' corresponding to the cut-contraction of $\phi^{-1}(f_i)$ is in Z. Let $\phi'(t') = X_0 \cap Z$, $\phi'(t'') = X_0 \setminus Z$, and $\phi'(s) = \phi(s)$ for every $s \in V(T') \setminus \{t', t''\}$.

We claim that (T', ϕ') is a small-cut-decomposition of G contradicting the choice of T. It is only necessary to verify that $|\phi^{-1}(s)| + \deg_{T'}(s) \geq 3$ for $s \in \{t', t''\}$. We have $|\phi^{-1}(t')| + \deg_{T'}(t') - 1 = |Z \cap V(G'')| \geq 2$ as C is a cyclic edge-cut in G''. It follows that $|\phi^{-1}(t')| + \deg_{T'}(t') \geq 3$ and the same holds for t'' by symmetry. □

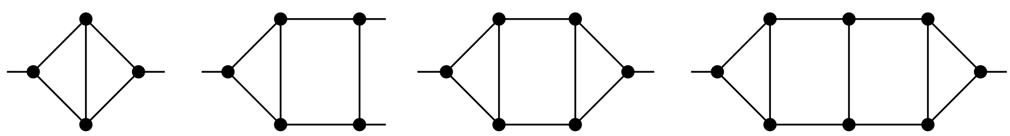

FIGURE 2. Isomorphism classes of subgraphs induced by elementary twigs.

We finish this section by describing a collection $\mathcal{Y}$ to which we will be applying Lemma 16 in the sequel. In a cubic bridgeless graph G a union of the vertex set of a relevant triangle with the vertex set of a cycle of length at most four sharing an edge with it is called *a simple twig.* Note that simple twigs corresponding to distinct relevant triangles can intersect, but one can routinely verify that each simple twig intersects a simple twig corresponding to at most one other relevant triangle. *An elementary twig* is either a simple twig, that intersects no simple twig corresponding to a relevant triangle not contained in it, or the union of two intersecting simple twigs, corresponding to distinct relevant triangles. An elementary twig is, indeed, a twig, unless it constitutes the vertex set of the entire graph. Figure 2 shows all possible elementary twigs. The next corollary follows immediately from the observations above and Lemmas 15 and 16.

Corollary 17. *Let G be a cubic bridgeless graph that is not cyclically 4-edge-connected with $|V(G)| \geq 8$. Then there exists a collection $\mathcal{Y}$ of pairwise disjoint elementary twigs in G such that every relevant triangle in G is contained in an element of $\mathcal{Y}$. Further, there exists a $\mathcal{Y}$-maximum small-cut-decomposition (T, ϕ) of G and for every $t \in V(T)$ either $\phi^{-1}(t) = \emptyset$, or $\phi^{-1}(t)$ is an elementary twig, or the hub of G at t is cyclically 4-edge-connected.*

4. Proof of Lemma 12.

The proof of Lemma 12 is based on our ability to find burls locally in the graph. The following lemma is a typical example.

Lemma 18. *Let G be a cubic bridgeless graph and let $X \subseteq V(G)$ be such that $|\delta(X)| = 4$ and $m(G|X) \geq 2$. Then X contains a burl.*

Proof. Let $\mathbf{w} = \frac{\mathbf{1}}{\mathbf{3}}|E_X$. We already observed that $\mathbf{w}$ is extendable from X. Note that if $M \in \mathcal{M}(G, X)$ contains no edges of $\delta(X)$ then X is M-alternating. As $M \cap \delta(X)$ is even for every $M \in \mathcal{M}(G, X)$ we have

$$\frac{4}{3} = \mathbb{E}\left[|\mathbf{M_w} \cap \delta(X)|\right] \geq 2\Pr[\mathbf{M_w} \cap \delta(X) \neq \emptyset].$$

Therefore $\Pr[\mathbf{M_w} \cap \delta(X) = \emptyset] \geq 1/3$, and so X is strongly $\mathbf{w}$-alternating. □

The proof of Lemma 12 relies on a precise study of the structure of small-cut trees for graphs with no core. The following two lemmas indicate that long paths in such trees necessarily contain some burls.

Lemma 19. *Let (T, ϕ) be a small-cut-decomposition of a cubic bridgeless graph G, and let P be a path in T with $|V(P)| = 10$. If we have*

- $\deg_T(t) = 2$ *for every* $t \in V(P)$,
- *the hub of G at t is isomorphic to K_4 for every $t \in V(P)$, and*
- $|\phi^{-1}(f)| = 3$ *for every edge $f \in E(T)$ incident to a vertex in $V(P)$,*

then $\phi^{-1}(P)$ contains a burl.

Proof. Let $P' = v_{-1}v_0 \dots v_9v_{10}$ be a path in T such that $P = v_0 \dots v_{10}$. Let $f_i = v_{i-1}v_i$ and let $C_i = \phi^{-1}(f_i)$, $0 \leq i \leq 10$. Let $X := \phi^{-1}(V(P))$. We assume without loss of generality that $G|X$ contains no cycles of length 4, as otherwise the lemma holds by Lemma 18. Let A be the set of ends of edges in C_0 outside of X, and let B be the set of ends of edges in C_{10} outside of X. Observe that E_X consists of 3 internally vertex-disjoint paths from A to B, as well as one edge in $G|\phi^{-1}(\{v_i\})$ for $0 \leq i \leq 9$. Let R_1, R_2 and R_3 be these three paths from A to B, and let u_j be the end of R_j in A for $j = 1, 2, 3$. For $0 \leq i \leq 9$, we have $\phi^{-1}(v_i) = \{x_i, y_i\}$ so that $x_i \in V(R_j), y_i \in V(R_{j'})$ for some $\{j, j'\} \subseteq \{1, 2, 3\}$ with $j \neq j'$, and $e_i := x_iy_i \in E(G)$. Let the *index* of i be defined as $(\{j, j'\}, \operatorname{sgn}(i))$, where $\operatorname{sgn}(i) = 0$ if the number of vertices in R_j between u_j and x_i and the number of vertices in $R_{j'}$ between $u_{j'}$ and y_i have the same parity, and $\operatorname{sgn}(i) = 1$ otherwise. There are 6 possible indices, so there exist $1 \leq i < i' \leq 7$ with the same indices. Without loss of generality we assume that those indices are $(\{1, 2\}, 0)$ or $(\{1, 2\}, 1)$.

To show that X is a burl, we construct a certificate $\mathbf{w}$ on E_X. We first set $\mathbf{w}(e) = \frac{1}{3}$ for every $e \in \delta(X)$. We then set $\mathbf{w}(e_{i''}) = 0$ for $i < i'' < i'$ and $\mathbf{w}(e_{i''}) = \frac{1}{3}$ for $0 \leq i'' \leq i$ and $i' < i'' \leq 9$. On the edges of R_1, R_2, R_3, and $e_{i'}$, we let $\mathbf{w}$ be the unique assignment of weights that satisfies conditions (EXT2) and (EXT3), which gives each such edge weight $\frac{1}{3}$ or $\frac{2}{3}$ on the paths and gives $\mathbf{w}(e_{i'})$ weight either 0 or $\frac{1}{3}$, depending on the parity of $i'-i$. Two examples are shown in Figure 3.

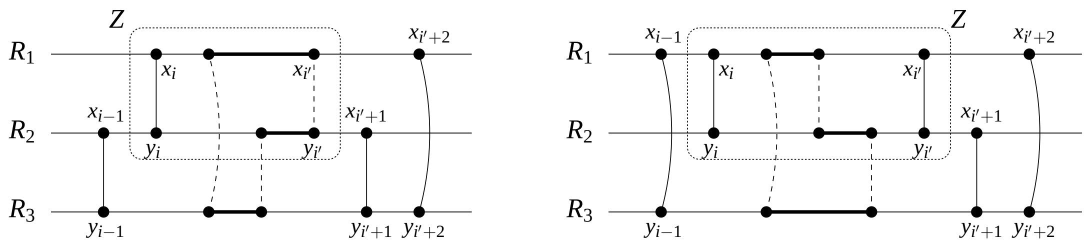

FIGURE 3. Certificates for the burl X when $i'-i$ is odd (left) and when $i'-i$ is even (right). Horizontal paths are R_1, R_2 and R_3, solid edges correspond to the value 1/3 of $\mathbf{w}$, bold edges to value 2/3 and dashed edges to value 0.

We claim that $\mathbf{w}$ is a certificate for X. Let Z consist of $x_i, y_i, x_{i'}, y_{i'}$ and vertex sets of segments of R_1 and R_2 between these vertices. The only edges in support of $\mathbf{w}$ in $\delta(Z)$ belong to either R_1 or R_2. As $|Z|$ is even, repeating the argument in the proof of Lemma 18, we deduce that $\Pr[\mathbf{M}_\mathbf{w} \cap \delta(Z) = \emptyset] \geq 1/3$. As $G|Z$ contains a spanning even cycle, and therefore at least two perfect matchings, we conclude that Z, and consequently X, are strongly $\mathbf{w}$-alternating. It is easy to see that $\mathbf{w}$ satisfies (EXT4). By our assumption that $\phi^{-1}(P)$ contains no cycles of length 4, the edges $e_{i-1}, e_{i'+1}$ have ends on R_3 and both R_1 and R_2 contain an end of one of the edges $e_{i'+1}$ and $e_{i'+2}$ (we insist that P contains 10 rather than 7 vertices to ensure this property). Using this fact, one can routinely verify that $\mathbf{w}$ satisfies (EXT5) and is therefore extendable from X. □

Lemma 20. *Let (T, ϕ) be a small-cut-decomposition of a cubic bridgeless graph G. Let $t_1, t_2 \in V(T)$ be a pair of adjacent vertices of degree 2. Suppose that $|\phi^{-1}(f)| = 2$ for every edge $f \in E(T)$ incident to t_1 or t_2. Then $\phi^{-1}(\{t_1, t_2\})$ contains a burl.*

Proof. Let $t_0t_1t_2t_3$ be a subpath of T and let $C_i = \phi^{-1}(t_{i-1}t_i)$ for $i = 1, 2, 3$ be an edge-cut of size 2. Assume that both $G|\phi^{-1}(t_1)$ and $G|\phi^{-1}(t_2)$ have at most one perfect matching. By Lemma 18 it suffices to show that $G|\phi^{-1}(\{t_1, t_2\})$ has at least two perfect matchings. As the hub G_1 over t_1 is cubic and bridgeless it contains at least 2 perfect matching avoiding any edge. Let $e_1, e_2 \in E(G_1)$ be the edges in $E(G_1) - E(G)$ corresponding to C_1- and C_2-contraction, respectively. By assumption, at most one perfect matching of G_1 avoids both e_1 and e_2. It follows that either two perfect matchings of G_1 avoid e_1 and contain e_2, or one avoids e_1 and e_2 and one avoids e_1 and contains e_2. Let G_2 be the hub over t_2. The symmetric statement holds for G_2.

In any case, the perfect matchings in G_1 and G_2 can be combined to obtain at least two perfect matchings of $G|\phi^{-1}(\{t_1, t_2\})$. □

From the definition of a small-cut-decomposition, we immediately get the following corollary:

Corollary 21. *Let (T, ϕ) be a small-cut-decomposition of a cubic bridgeless graph G, and let P be a path in T in which every vertex has degree 2. Suppose there exist three edges f_1, f_2, f_3 of T incident to vertices of P such that $|\phi^{-1}(f_1)| = |\phi^{-1}(f_2)| = |\phi^{-1}(f_3)| = 2$. Then $\phi^{-1}(P)$ contains a burl.*

Let B_3 denote the cubic graph consisting of two vertices joined by three parallel edges. Lemmas 19 and 20 imply the following.

Corollary 22. *Let (T, ϕ) be a small-cut-decomposition of a cubic bridgeless graph G and let P be a path in T with $|V(P)| = 32$. If for every $t \in V(P)$, $\deg_T(t) = 2$ and the hub of G at t is isomorphic to K_4 or B_3, then $\phi^{-1}(P)$ contains a burl.*

Proof. If at least three edges incident to vertices in $V(P)$ correspond to edge-cuts of size 2 in G then the corollary holds by Corollary 21. Otherwise, since there are 33 edges of T incident to vertices of P, there must be 11 consecutive edges incident to vertices in P corresponding to edge-cuts of size 3. In this case, the result follows from Lemma 19. □

Proof of Lemma 12. We proceed by induction on $|Z|$. If $|Z| \leq 6$ then Z is a twig. In this case the lemma holds since $\beta_1 \geq \beta_2 + 6\alpha$ by **(3)**. We assume for the remainder of the proof that $|Z| \geq 7$. It follows that G' is not cyclically 4-edge-connected, as G' has no core. Therefore Corollary 17 is applicable to G'. Let $\mathcal{Y}$ be a collection of disjoint elementary twigs in G' such that every relevant triangle in G' is contained in an element of $\mathcal{Y}$, and let (T, ϕ) be a $\mathcal{Y}$-maximum small-cut decomposition of G'. By Corollary 17, the hub at every $t \in V(T)$ with $|\phi^{-1}(t)| \neq \emptyset$ is either an elementary twig, in which case t is a leaf of T, or is cyclically 4-edge-connected, in which case it is isomorphic to either K_4 or B_3.

In calculations below we will make use of the following claim: If $\deg_T(t) = 2$ for some $t \in V(T)$, then $|\phi^{-1}(t)| \leq 2$. If this is not the case, the hub at t is isomorphic to K_4, and at least three of its vertices must be vertices of G. It follows that there is an edge $f \in E(T)$ incident to t for which $|\phi^{-1}(f)| = 2$. Let $v \in \phi^{-1}(t)$ be a vertex incident to an edge in $\phi^{-1}(f)$. Then $C = \phi^{-1}(f) \triangle \delta(v)$ is a 3-edge-cut in G. As in the proof of Lemma 16 we can split t into two vertices t', t'' with $\phi^{-1}(t') = \{v\}$ and $\phi^{-1}(t'') = \phi^{-1}(t) \setminus v$. We now have $\phi^{-1}(t't'') = C$ and the new small-cut-decomposition contradicts the maximality of (T, ϕ). This completes the proof of the claim.

Let $t_0 \in V(T)$ be such that $\phi^{-1}(t_0)$ contains the new vertex or one of the ends of the new edge in G'. Since G is pruned, G' contains at most one irrelevant triangle, and if such a triangle exists, at least one of its vertices lies in $\phi^{-1}(t_0)$. As a consequence, for any leaf $t \neq t_0$ of T, $\phi^{-1}(t)$ is a twig. Let $t^* \in V(T) \setminus \{t_0\}$ be such that $\deg_T(t^*) \geq 3$ and, subject to this condition, the component of $T \setminus \{t^*\}$ containing t_0 is maximal. If $\deg_T(t) \leq 2$ for every $t \in V(T) \setminus \{t_0\}$, we take $t^* = t_0$ instead.

Let $T_1, \ldots, T_k$ be all the components of $T \setminus \{t^*\}$ not containing t_0. By the choice of t^*, each T_i is a path. If $|V(T_i)| \geq 33$ for some $1 \leq i \leq k$ then let T' be the subtree of T_i containing a leaf of T and exactly 32 other vertices. Let f be the unique edge in $\delta(T')$. Let H (resp. H') be the $\phi^{-1}(f)$-contraction of G (resp. G') containing $V(G') \setminus \phi^{-1}(T')$, and let Z' consist of $V(H') \cap Z$ together with the new vertex created by $\phi^{-1}(f)$-contraction (if it exists). If H is not pruned then it contains a unique irrelevant triangle and we contract it, obtaining a pruned graph. By the induction hypothesis, either $|Z'| \leq 6$ or we can find a foliage $\mathcal{X}'$ in Z' with $fw(\mathcal{X}') \geq \alpha(|Z'| - 2) + \beta_2$. If $|Z'| \leq 6$ let $\mathcal{X}' := \emptyset$.

Let t' be a vertex of T' which is not a leaf in T. Since $\deg_T(t') = 2$, $|\phi^{-1}(t')| \neq \emptyset$. Therefore $\phi^{-1}(t')$ is isomorphic to B_3 or K_4 and we can apply Corollary 22. This implies that $\phi^{-1}(T')$ contains an elementary twig and a burl that are vertex-disjoint, where the elementary twig is the preimage of the leaf. Further, we have $|\phi^{-1}(T')| \leq 8 + 2 \cdot 32 = 72$, since an elementary twig has size at most 8 and the preimage of every non-leaf vertex of T' has size at most 2 by the claim above. By Lemma 8(3), we can obtain a foliage $\mathcal{X}$ in Z by adding the twig and the burl to $\mathcal{X}'$ and possibly removing a burl (which can be a twig) containing the new element of H' created by $\phi^{-1}(f)$-contraction. It follows that if $|Z'| \geq 7$ then

$$fw(\mathcal{X}) \geq \alpha(|Z'| - 2) + 2\beta_2 \geq (\alpha|Z| + \beta_2) - 74\alpha + \beta_2 \geq \alpha|Z| + \beta_2,$$

by **(4)**, as desired. If $|Z'| \leq 6$ then $|Z| \leq 78$ and

$$fw(\mathcal{X}) \geq \beta_1 + \beta_2 \geq 78\alpha + \beta_2 \geq \alpha|Z| + \beta_2,$$

by **(5)**.

It remains to consider the case when $|V(T_i)| \leq 32$ for every $1 \leq i \leq k$. Suppose first that $t^* \neq t_0$ and that $|\phi^{-1}(T_0)| \geq 7$, where T_0 denotes the component of $T \setminus t^*$ containing t_0. Let f_0 be the edge incident to t^* and a vertex of T_0. We form the graphs H, H' and a set Z' by a $\phi^{-1}(f_0)$-contraction as in the previous case, and possibly contract a single irrelevant triangle. As before, we find a foliage $\mathcal{X}'$ in Z' with $fw(\mathcal{X}') \geq \alpha(|Z'| - 2) + \beta_2$. Note that $\phi^{-1}(T_i)$ contains a twig for every $1 \leq i \leq k$. By Lemma 8(3), we now obtain a foliage $\mathcal{X}$ in Z from $\mathcal{X}'$, adding $k \geq 2$ twigs and possibly removing one burl (which can be a twig) from $\mathcal{X}'$. We have $|\phi^{-1}(T_i)| \leq 8 + 31 \cdot 2 = 70$ for every $1 \leq i \leq k$, and $|\phi^{-1}(t^*)| \leq 4$. Therefore $|Z| \leq |Z'| + 70k + 4$. It follows from **(5)** that

$$fw(\mathcal{X}) \geq \alpha(|Z'| - 2) + \beta_2 + (k-1)\beta_1 \geq$$
$$\geq \alpha|Z| + \beta_2 - 76\alpha + (k-1)(\beta_1 - 70\alpha) \geq \alpha|Z| + \beta_2.$$

Now we can assume $t^* = t_0$ or $|\phi^{-1}(T_0)| \leq 6$. First suppose $t^* \neq t_0$ but $|\phi^{-1}(T_0)| \leq 6$. Then again $|\phi^{-1}(t^*)| \leq 4$, so we have $|Z| \leq 70k + 10$. Let $\mathcal{X}$ be the foliage consisting of twigs in $T_1, \ldots, T_k$. Thus by **(6)**, we have

$$fw(\mathcal{X}) = k\beta_1 \geq (\alpha|Z| + \beta_2) + k(\beta_1 - 70\alpha) - 10\alpha - \beta_2 \geq \alpha|Z| + \beta_2.$$

Finally we can assume $t^* = t_0$. Then $|\phi^{-1}(t^*)| \leq 4$, unless $k = 1$ and $\phi^{-1}(t^*)$ is an elementary twig. In either case, $|Z| \leq 70k + 8$ and the equation above applies. □

5. Proof of Lemma 14

The following lemma is a direct consequence of a theorem of Kotzig, stating that any graph with a unique perfect matching contains a bridge (see [6]).

Lemma 23. *Every edge of a cyclically* 4*-edge-connected cubic graph with at least six vertices is contained in at least two perfect matchings.*

Let G be a cubic graph. For a path $v_1v_2v_3v_4$, the graph obtained from G by *splitting along the path* $v_1v_2v_3v_4$ is the cubic graph G' obtained as follows: remove the vertices v_2 and v_3 and add the edges v_1v_4 and $v_1'v_4'$ where v_1' is the neighbor of v_2 different from v_1 and v_3 and v_4' is the neighbor of v_3 different from v_2 and v_4. The idea of this construction (and its application to the problem of counting perfect matchings) originally appeared in [17]. We say that a perfect matching M of G is a *canonical extension* of a perfect matching M' of G' if $M \triangle M' \subseteq E(G) \triangle E(G')$, i.e. M and M' agree on the edges shared by G and G'.

Lemma 24. *Let G be a cyclically 4-edge-connected cubic graph with $|V(G)| \geq 6$. If G' is the graph obtained from G by splitting along some path $v_1v_2v_3v_4$, then*

(1) *G' is cubic and bridgeless;*
(2) *G' contains at most* 2 *irrelevant triangles;*
(3) *$fw(G) \geq fw(G') - 2\beta_1$;*
(4) *Every perfect matching M' of G' avoiding the edge v_1v_4 has a canonical extension in G.*

Proof.

(1) The statement is a consequence of an easy lemma in [5], stating that the cyclic edge-connectivity can drop by at most two after a splitting.

(2) Since G is cyclically 4-edge-connected and has at least six vertices, it does not contain any triangle. The only way an irrelevant triangle can appear in G' is that v_1 and v_4 (or v_1' and v_4') have precisely one common neighbor (if they have two common neighbors, the two arising triangles share the new edge v_1v_4 or $v_1'v_4'$ and hence, are relevant).
(3) At most two burls from a foliage of G' intersect the edge v_1v_4 or the edge $v_1'v_4'$. Therefore, a foliage of G can be obtained from any foliage of G' by removing at most two burls.
(4) The canonical extension is obtained (uniquely) from $M' \cap E(G)$ by adding either v_2v_3 if $v_1'v_4' \notin M'$ or $\{v_1'v_2, v_3v_4'\}$ if $v_1'v_4' \in M'$. □

Proof of Lemma 14. We proceed by induction on $|V(G)|$. The base case $|V(G)| = 6$ holds by Lemma 23 and **(7)**.

For the induction step, consider first the case that G is cyclically 4-edge-connected. Fix an edge $e = uv \in E(G)$. Our goal is to show that e is contained in at least $2^{\alpha|V(G)|-fw(G)+\gamma}$ perfect matchings.

Let $w \neq u$ be a neighbor of v and let w_1 and w_2 be the two other neighbors of w. Let x_i, y_i be the neighbors of w_i distinct from w for $i = 1, 2$. Let $G_1, \dots, G_4$ be the graphs obtained from G by splitting along the paths vww_1x_1, vww_1y_1, vww_2x_2 and vww_2y_2. Let G_i' be obtained from G_i by contracting irrelevant triangles for $i = 1, \dots, 4$. By Lemma 24(2) we have $|V(G_i')| \geq |V(G)| - 6$.

Suppose first that one of the resulting graphs, without loss of generality G_1', does not have a core. By Corollary 13, Lemma 11 and Lemma 24, we have

$$\alpha|V(G)| \leq \alpha(|V(G_1')| + 6) \leq fw(G_1') + 7\alpha - \beta_2 \leq fw(G_1) + 7\alpha - \beta_2 \leq$$
$$\leq fw(G) + 2\beta_1 + 7\alpha - \beta_2.$$

Therefore

$$\alpha|V(G)| - fw(G) + \gamma \leq \gamma + 2\beta_1 + 7\alpha - \beta_2 \leq 1$$

by **(8)** and the lemma follows from Lemma 23.

We now assume that all four graphs $G_1', \dots, G_4'$ have a core. By Lemma 24(4), every perfect matching containing e in G_i canonically extends to a perfect matching containing e in G. Let S be the sum of the number of perfect matchings of G_i containing e, for $i \in \{1, 2, 3, 4\}$. By induction hypothesis and Lemmas 11 and 24,

$$S \geq 4 \cdot 2^{\alpha(|V(G)|-6)-fw(G)-2\beta_1+\gamma}.$$

On the other hand, a perfect matching M of G containing e is the canonical extension of a perfect matching containing e in precisely three of the graphs G_i, $i \in \{1, 2, 3, 4\}$. For instance if $w_1y_1, ww_2 \in M$, then G_2 is the only graph (among the four) that does not have a perfect matching M' that canonically extends to M (see Figure 4). As a consequence, there are precisely $S/3$ perfect matchings containing e in G. Therefore,

$$m^\star(G) \geq \tfrac{4}{3} \cdot 2^{\alpha(|V(G)|-6)-fw(G)-2\beta_1+\gamma} \geq 2^{\alpha|V(G)|-fw(G)+\gamma},$$

by **(9)**, as desired.

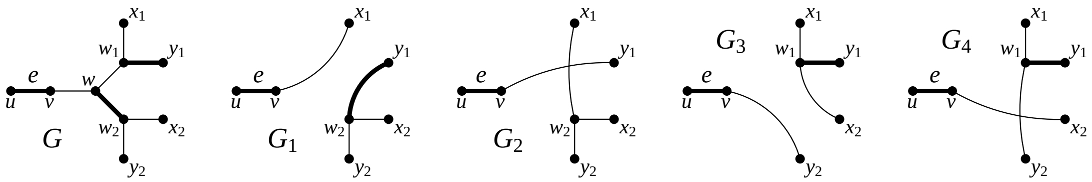

FIGURE 4. Perfect matchings in only three of the G_i's canonically extend to a given perfect matching of G containing e.

It remains to consider the case when G contains a cyclic edge-cut C of size at most 3. Suppose first that for such edge-cut C, both C-contractions H_1 and H_2 have a core. Then, by Lemma 8(3), $fw(G) \geq fw(H_1) + fw(H_2) - 2\beta_1$ and, by induction hypothesis, applied to H_1 and H_2 (after possibly contracting one irrelevant triangle in each) and Lemma 8,

$$m^\star(G) \geq m^\star(H_1)m^\star(H_2) \geq 2^{\alpha|V(G)|-4\alpha-fw(G)-2\beta_1+2\gamma} \geq 2^{\alpha|V(G)|-fw(G)+\gamma},$$

by **(10)**, as desired. Finally, if for every cyclic edge-cut C of size at most 3 only one C-contraction has a core, we apply Corollary 17 to G. Let (T, ϕ) be the resulting small-cut-decomposition of G. There exists a unique vertex $t \in V(T)$ such that the hub H of G at t is cyclically 4-edge-connected with $|V(H)| \geq 6$. Let $T_1, \dots, T_k$ be the components of $T - t$ and let $Z_i = \phi^{-1}(V(T_i))$. We apply Lemma 12 to $Z_1, \dots, Z_k$. Note that Lemma 12 is indeed applicable, as G is pruned, and therefore every triangle in G belongs to an elementary twig. Consequently, no edge-cut corresponding to an edge of (T, ϕ) separates exactly 3 vertices of G.

Let $\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_k$ be the foliages satisfying the lemma. Let $\mathcal{X}_0$ be the maximal foliage in H avoiding new vertices and edges created by contraction of the edge-cuts $\delta(Z_1), \dots, \delta(Z_k)$. Then $fw(\mathcal{X}_0) \geq fw(H) - k\beta_2$, as H contains no twigs (it is cyclically 4-edge-connected). Since $\mathcal{X}_0 \cup \mathcal{X}_1 \cup \ldots \cup \mathcal{X}_k$ is a foliage in G we have

$$fw(G) \geq fw(H) - k\beta_2 + \sum_{i=1}^{k} fw(\mathcal{X}_i) \geq fw(H) + \alpha \sum_{i=1}^{k} |Z_i|,$$

by the choice of $\mathcal{X}_1, \dots, \mathcal{X}_k$. It remains to observe that

$$m^\star(G) \geq m^\star(H) \geq 2^{\alpha|V(H)|-fw(H)+\gamma} \geq 2^{\alpha(|V(G)|-\sum_{i=1}^{k}|Z_i|)-fw(H)+\gamma} \geq$$
$$\geq 2^{\alpha|V(G)|-fw(G)+\gamma},$$

by the above. □

6. Concluding remarks

6.1. **Improving the bound.** We expect that the bound in Theorem 1 can be improved at the expense of more careful case analysis. In particular, it is possible to improve the bound on the length of the path in Corollary 22. We have chosen not to do so in an attempt to keep the argument as short and linear as possible.

In [2] it is shown that for some constant $c > 0$ and every integer n there exists a cubic bridgeless graph on at least n vertices with at most $c2^{n/17.285}$ perfect matchings.

6.2. **Number of perfect matchings in k-regular graphs.** In [11, Conjecture 8.1.8] the following generalization of the conjecture considered in this paper is stated. A graph is said to be *matching-covered* if every edge of it belongs to a perfect matching.

Conjecture 25. *For $k \geq 3$ there exist constants $c_1(k), c_2(k) > 0$ such that every k-regular matching covered graph contains at least*

$$c_2(k)c_1(k)^{|V(G)|}$$

perfect matchings. Furthermore, $c_1(k) \to \infty$ as $k \to \infty$.

While our proof does not seem to extend to the proof of this conjecture, the following weaker statement can be deduced from Theorem 1. We are grateful to Paul Seymour for suggesting the idea of the following proof.

Theorem 26. *Let G be a k-regular $(k-1)$-edge-connected graph on n vertices for some $k \geq 4$. Then*

$$\log_2 m(G) \geq (1 - \tfrac{1}{k})(1 - \tfrac{2}{k})\tfrac{n}{3656}.$$

Proof. Let $\mathbf{w}$ be an edge-weighting of G assigning weight $1/k$ to every edge. It is easy to deduce from Theorem 3 that $\mathbf{w} \in \mathcal{PMP}(G)$. Let $\mathbf{M_w}$ be a random variable in $\mathcal{M}(G)$ corresponding to $\mathbf{w}$. We choose a triple of perfect matchings of G as follows. Let $M_1 \in \mathcal{M}(G)$ be arbitrary. We have

$$\mathbb{E}[|\mathbf{M_w} \cap M_1|] = \frac{n}{2k}.$$

Therefore we can choose $M_2 \in \mathcal{M}(G)$ so that $|M_2 \cap M_1| \leq \frac{n}{2k}$. Let $Z \subseteq V(G)$ be the set of vertices not incident with an edge of $M_1 \cap M_2$. Then $|Z| \geq (1 - \frac{1}{k})\,n$. For each $v \in Z$ we have

$$\Pr[\mathbf{M_w} \cap \delta(\{v\}) \cap (M_1 \cup M_2) = \emptyset] = 1 - \tfrac{2}{k}.$$

Therefore the expected number of vertices whose three incident edges are in $\mathbf{M_w}$, M_1 and M_2 respectively, is at least $(1 - \frac{1}{k})(1 - \frac{2}{k})\,n$. It follows that we can choose $M_3 \in \mathcal{M}(G)$ so that the subgraph G' of G with $E(G') = M_1 \cup M_2 \cup M_3$ has at least $(1 - \frac{1}{k})(1 - \frac{2}{k})\,n$ vertices of degree three. Note that G' is by definition matching-covered. It follows that the only bridges in G' are edges joining pairs of vertices of degree one. Let G'' be obtained from G' by deleting vertices of degree one and replacing by an edge every maximal path in which all the internal vertices have degree two. The graph G'' is cubic and bridgeless and therefore by Theorem 1 we have

$$\log_2 m(G) > \log_2 m(G') \geq \log_2 m(G'') \geq \frac{1}{3656}|V(G'')| \geq (1-\tfrac{1}{k})(1-\tfrac{2}{k})\frac{n}{3656},$$

as desired. □

References

[1] M. Chudnovsky and P. Seymour, *Perfect matchings in planar cubic graphs*, Combinatorica, to appear.

[2] M. Cygan, M. Pilipczuk and R. Škrekovski, *A bound on the number of perfect matchings in klee-graphs*, preprint.

[3] J. Edmonds, *Maximum matching and a polyhedron with* $(0,1)$*-vertices*, J. Res. Nat. Bur. Standards B **69** (1965), 125–130.

[4] J. Edmonds, L. Lovász, W. R. Pulleyblank, *Brick decompositions and the matching rank of graphs*, Combinatorica **2** (1982), 247–274.

[5] L. Esperet, D. Král', F. Kardoš, *A superlinear bound on the number of perfect matchings in cubic bridgeless graphs*, manuscript (2010).

[6] L. Esperet, D. Král', P. Škoda, and R. Škrekovski, *An improved linear bound on the number of perfect matchings in cubic graphs*, European J. Combin., **31** (2010), 1316–1334.

[7] W. Imrich, *Explicit construction of regular graphs without short cycles*, Combinatorica **4** (1984), 53–59.

[8] F. Kardoš, D. Král', J. Miškuf, and J.-S. Sereni, *Fullerene graphs have exponentially many perfect matchings*, J. Math. Chem. **46** (2009), 443–447.

[9] D. Král', J.-S. Sereni, and M. Stiebitz, *A new lower bound on the number of perfect matchings in cubic graphs*, SIAM J. Discrete Math. **23** (2009), 1465–1483.

[10] M. Laurent and L. Schrijver, *On Leonid Gurvits' proof for permanents*, Amer. Math. Monthly **117(10)** (2010), 903–911.

[11] L. Lovász and M. D. Plummer, Matching theory, Elsevier Science, Amsterdam, 1986.

[12] D. Naddef, *Rank of maximum matchings in a graph*, Math. Programming **22** (1982), 52–70.

[13] J. Plesník, *Connectivity of regular graphs and the existence of 1-factors*, Mat. Časopis Sloven. Akad. Vied **22** (1972), 310–318.

[14] J. Petersen, *Die Theorie der regulären graphs*, Acta Math. **15** (1891), 193–220.

[15] A. Schrijver, *Counting* 1*-factors in regular bipartite graphs*, J. Combin. Theory Ser. B **72** (1998), 122–135.

[16] L. Valiant, *The complexity of computing the permanent*, Theor. Comput. Sci. **8** (1979), 189–201.

[17] M. Voorhoeve, *A lower bound for the permanents of certain* $(0,1)$*-matrices*, Nederl. Akad. Wetensch. Indag. Math. **41** (1979), 83–86.

Laboratoire G-SCOP (CNRS, Grenoble-INP), Grenoble, France
E-mail address: louis.esperet@g-scop.fr

Institute of Mathematics, Faculty of Science, Pavol Jozef Šafárik University, Košice, Slovakia
E-mail address: frantisek.kardos@upjs.sk

Department of Industrial Engineering and Operations Research, Columbia University, New-York, NY, USA
E-mail address: andrew.d.king@gmail.com

Department of Applied Mathematics and Institute for Theoretical Computer Science, Faculty of Mathematics and Physics, Charles University, Malostranské náměstí 25, 118 00 Prague, Czech Republic.
E-mail address: kral@kam.mff.cuni.cz

Department of Mathematics, Princeton University, Princeton, NJ, USA
E-mail address: snorin@math.princeton.edu