Graphs with two crossings are 5-choosable[*]

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Abstract

A graph G is k-choosable if G can be properly colored whenever every vertex has a list of at least k available colors. Thomassen's theorem states that every planar graph is 5-choosable. We extend the result by showing that every graph with at most two crossings is 5-choosable.

1 Introduction

All graphs considered in this paper are simple, i.e., without loops or parallel edges. We denote the set of vertices of a graph G by $V(G)$ and the set of edges by $E(G)$. The *crossing number* of G, denoted by $\mathrm{cr}(G)$, is the minimum possible number of crossings in a drawing of G in the plane.

Let G be a graph and C a set of colors. A *list assignment* is a function $L : V(G) \to 2^C$. We say that G is *L-colorable* if there exists a coloring $\varphi : V(G) \to C$ such that $\varphi(v) \in L(v)$ for every vertex $v \in V(G)$ and adjacent vertices are assigned different colors. We say that a graph G is *k-choosable* if G is L-colorable whenever L assigns at least k colors to each vertex.

The concept of list colorings and choosability was introduced by Vizing [8] and independently by Erdős et al. [3]. Clearly, if a graph is k-choosable, then it is also k-colorable. However, the opposite implication does not hold. For instance, there exist planar graphs that are not 4-choosable, see Voigt [9]. On the other hand, Thomassen [7] gave a strikingly beautiful proof that every planar graph is 5-choosable.

A general bound for choosability of graphs on surfaces, an analogue of Dirac's Map-Color Theorem, is known due to Böhme et al. [1]. However, only very few graphs have choosability close to this bound. DeVoss et al. [2] obtained more general result claiming that locally planar graphs are 5-choosable, which extended the result of Thomassen to non-planar graphs. Furthermore, Kawarabayashi and Mohar [5] proved that there are only finitely many minimal graphs that are not 5-choosable on any fixed surface.

Note that the genus of any graph is bounded by its crossing number, thus it might be possible to obtain more refined results by considering graphs with bounded crossing number. For example, for ordinary chromatic number it is easy to see that all graphs with crossing number at most two are 5-colorable. Oporowski and Zhao [6] improved this result by showing that K_6 is the only 6-critical graph with crossing number three, and conjectured that it is the only 6-critical graph with crossing number at most 5. This was eventually settled by Erman et al. [4], who proved that K_6 is the only 6-critical graph with crossing number at most 4 and gave an example of a 6-critical graph K_6-free graph with crossing number 5.

Similarly, the result of Thomassen can be easily used to derive that every graph with at most one crossing is 5-choosable. Erman et al. [4] posed a question if this is also true for graphs with two crossings. The main result of this paper is the positive answer to this question:

Theorem 1. *Every graph G with* $\mathrm{cr}(G) \leq 2$ *is 5-choosable.*

Inspired by the result of Erman et al. [4], we pose the following open problem:

Problem 1. *Is it true that every K_6-free graph G with $\mathrm{cr}(G) \leq 4$ is 5-choosable?*

Note that the Four Color Theorem is used heavily in the coloring case, thus it is not obvious that the results should generalize to the choosability case.

2 5-choosability of graphs with two crossings

In order to show 5-choosability of planar graphs, Thomassen [7] proved the following more general statement.

Theorem 2. *Let G be a plane graph, F a face of G and xy an edge incident with F. Then G is L-colorable for any list assignment L such that*

- *$|L(v)| \geq 5$ for $v \in V(G) \setminus V(F)$,*

- $|L(v)| \geq 3$ *for* $v \in V(F) \setminus \{x, y\}$,

- $|L(x)|, |L(y)| \geq 1$, *and*

- *if* $|L(x)| = |L(y)| = 1$, *then* $L(x) \neq L(y)$.

We frequently use the following observation:

Observation 3. *Let G be a connected graph with a list assignment L. Suppose that the subgraph $G[S]$ of G induced by a set $S \subset V(G)$ has an L-coloring ψ. Let $G' = G - S$ and for every $v \in V(G')$ let $L'(v) = L(v) \setminus \{\psi(u) : u \in N(v) \cap S\}$. If G' and L' satisfy the assumptions of Theorem 2 then ψ can be extended to a coloring of G.*

Instead of proving Theorem 1 directly, we prove a slightly stronger theorem.

Theorem 4. *Let G be a graph and L a list assignment such that either*

- $\mathrm{cr}(G) \leq 2$ *and* $|L(v)| \geq 5$ *for every* $v \in V(G)$, *or*

- $\mathrm{cr}(G) \leq 1$, *G contains a triangle T, $L(v) = 1$ for all $v \in V(T)$, $L(u) \neq L(v)$ if u and v are two distinct vertices of T and $|L(v)| \geq 5$ for all $v \in V(G) \setminus V(T)$.*

Then G is L-choosable.

Proof of Theorem 4. Let G and L be a counterexample with the smallest crossing number, subject to that the smallest number of vertices, and subject to that the largest number of edges. Observe that G is connected, as otherwise we can color each connected component of G separately. Furthermore, the minimum degree of $v \in V(G) \setminus V(T)$ is 5, as if v had degree at most 4, then an L-coloring of $G - v$ (which exists by the minimality of G) can be extended to an L-coloring of G. Moreover, we assume that $|L(v)| = 5$ for all $v \in V(G) \setminus V(T)$ as removing colors from lists does not turn G into an L-colorable graph.

Claim 1. $\mathrm{cr}(G) \geq 1$.

Proof. Suppose for contradiction that G is planar. By Theorem 2, it contains a precolored triangle $T = t_1 t_2 t_3$. Let $S = \{t_1\}$ and let ψ be a coloring of S such that $\psi(t_1) \in L(t_1)$. By applying Observation 3 we get an L-coloring of G. $\qquad\square$

We call crossings and T *dangerous configurations.* Since all graphs with crossing number one are 5-choosable by [4], we can assume that G has two dangerous configurations: either $\mathrm{cr}(G) = 2$ or $\mathrm{cr}(G) = 1$ and G contains T. Let us fix a drawing of G with the minimum number of crossings.

Claim 2. *If T exists, then no edge of T is crossed.*

Proof. Let $T = uvw$, and assume for contradiction that the edge uv is crossed by an edge xy. Let G_1 be the subgraph of G induced by the vertices drawn in the closed disk bounded by T, and let $G_2 = G - (V(G_1) \setminus V(T))$. By symmetry, assume that $x \neq w$ and $x \in V(G_1)$. There exists an L-coloring of G_2, and by Observation 3 applied with $S = V(G_2) \setminus \{v, w\}$, this coloring extends to an L-coloring of G. This is a contradiction. $\qquad \square$

Let C be a cycle in G. Let G_1 be the subgraph of G consisting of vertices and edges drawn in the *closed* disk bounded by C and G_2 the subgraph of G consisting of the vertices and edges drawn outside of the *open* disk bounded by C. Note that $G_1 \cap G_2 = C$. If no edge of C is crossed and $V(G_1) \neq V(C) \neq V(G_2)$, then we say that C is a *separating cycle.* We call G_1 and G_2 the *C-components.*

Claim 3. *G does not contain a separating triangle.*

Proof. Suppose for a contradiction that there is a separating triangle $C = x_1 x_2 x_3$, and let G_1 and G_2 be the C-components. If both dangerous configurations are in G_1, we first color G_1 by induction and then extend coloring of C to G_2 by Claim 1. Otherwise, without loss of generality, we assume that if $\mathrm{cr}(G) = 1$, then $T \subset G_1$. We first color G_1 and then extend the coloring of C to G_2, where C plays the role of the precolored triangle in G_2. $\qquad \square$

Similarly (by adding vertices to extend the cut to a triangle if necessary) one can prove the following.

Claim 4. *G is 2-connected and if $\{u, v\}$ is a cut in G, then uv is not a non-crossed edge.*

Furthermore, we can restrict separating 4-cycles.

Claim 5. *G does not contain a separating cycle C of length four with both dangerous configurations draw on the same side of C.*

Proof. Suppose for a contradiction that there is a separating cycle $C = x_1 x_2 x_3 x_4$ with C-components G_1 and G_2, where both dangerous configurations are in G_1. By Claim 3, we can assume that C is an induced cycle in

G_2. By the minimality of G, there exists an L-coloring ψ of G_1. Observation 3 used with $S = V(G_1) \setminus \{x_1, x_2\}$ implies that ψ can be extended to an L-coloring of G, which is a contradiction. $\square$

Claim 6. *No edge is crossed twice.*

Proof. Suppose for a contradiction that an edge $e = uv$ is crossed by edges e_1 and e_2. We distinguish two cases depending on the number of vertices incident with e_1 and e_2.

Suppose first that there exists a vertex w incident to both e_1 and e_2. Let $S = \{u, w\}$ and let ψ be an arbitrary L-coloring of S. Observation 3 implies that ψ can be extended to an L-coloring of G, which is a contradiction.

Therefore, no vertex is incident with both e_1 and e_2. Let $e_1 = w_1 z_1$ and $e_2 = w_2 z_2$. As G has the largest possible number of edges, we can assume that $C = u w_1 w_2 v z_2 z_1$ is a cycle of length 6 with both crossings drawn inside. By Claim 5, no vertex is drawn inside C. Let $S = \{u, w_1, w_2\}$ and let B be the set of common neighbors of the vertices of S.

Since every vertex of C has degree at least 5, Claims 3, 4 and 5 imply that $B \cap V(C) = \emptyset$. Suppose there exists $x \in B$. Claim 5 implies that triangles $uw_1 x$ and $xw_1 w_2$ bound faces. Hence w_1 has degree four, contradicting that the minimum degree of G is 5. So, $B = \emptyset$. We conclude that we can apply Observation 3 for an arbitrary L-coloring of $G[S]$ and obtain an L-coloring of G. $\square$

It turns out that we can restrict our attention only to the case that T exists.

Claim 7. $\mathrm{cr}(G) = 1$.

Proof. For contradiction, assume that $\mathrm{cr}(G) = 2$, and let edges $e = xx'$ and $f = yy'$ cross each other. Let $X = \{x, x', y, y'\}$. Let G' be the graph obtained from $G - \{e, f\}$ by adding a new vertex v adjacent to all vertices of X. Note that xy is an edge as G has the largest number of edges. Let L' be the list assignment such that $L'(x) \subseteq L(x)$ and $L'(y) \subseteq L(y)$ are distinct lists of size one, $L'(v) = \{c\}$ for a new color c that does not appear in any of the lists, $L'(x') = (L(x') \setminus L'(x)) \cup \{c\}$, $L'(y') = (L(y') \setminus L'(y)) \cup \{c\}$ and $L'(z) = L(z)$ for every $z \in V(G) \setminus X$. Since $\mathrm{cr}(G') < \mathrm{cr}(G)$, the graph G' (with xyv playing the role of the precolored triangle) has an L'-coloring φ. Note that $\varphi(x) \neq \varphi(x') \neq c$ and $\varphi(y) \neq \varphi(y') \neq c$, hence φ is also an L-coloring of G. $\square$

Let $X = \{v_1, v_2, v_3, v_4\}$, where $e = v_1 v_3$ and $f = v_2 v_4$ cross each other. Since G has the largest possible number of edges, the following claim holds.

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Claim 8. $G[X]$ *is a complete graph.*

So, $v_1v_2v_3v_4$ is a cycle of length four enumerated in the clockwise order. Let the precolored triangle T have vertices t_1, t_2 and t_3 in the clockwise order.

Claim 9. *Let $h = uv$ be equal to e or f. If a vertex w is adjacent to both u and v, then w belongs to X.*

Proof. Suppose for a contradiction that w does not belong to X and let $\{x, y\} = X \setminus \{u, v\}$. By symmetry between the cycles $wuxv$ and $wuyv$, we can assume that both dangerous configurations appear inside the closed disk bounded by $wuxv$, and by Claim 5 the cycle $wuxv$ is not separating. We conclude that x has degree at most four, which is a contradiction. $\square$

Claim 10. *G does not contain a separating cycle C of length four such that $|V(C) \cap V(X)| \geq 2$.*

Proof. Suppose for a contradiction that C is such a cycle. Note that no edge of C is crossed as C is separating. Let G_1 and G_2 be the C-components, where $X \subseteq V(G_2)$, and let u and v be two vertices in $V(C) \cap X$. Claim 5 implies that both G_1 and G_2 contain a dangerous configuration.

By Claim 8, u and v are adjacent in G_2. By Claim 9, uv is not a crossed edge. By Claim 3, we conclude that uv is an edge of C and that C is an induced cycle in G_2. By the minimality of G, there exists an L-coloring ψ of G_1. Observation 3 used with $S = (V(G_1) \setminus V(C)) \cup \{u, v\}$ implies that ψ can be extended to an L-coloring of G, which is a contradiction. $\square$

Claim 11. *$V(T)$ and X are disjoint.*

Proof. Let uv be a non-crossed edge such that $u \in V(T) \cap X$ and $v \in X$. Let $S = \{u, v\}$ and ψ be an arbitrary L-coloring of $G[S \cup V(T)]$. See Figure 1(a). Observe that $G - S$ is planar and all neighbors of S are incident with one face. Hence, we can apply Observation 3 and obtain an L-coloring of G, which is a contradiction. $\square$

Claim 12. *There is no edge tv such that $t \in V(T)$ and $v \in X$.*

Proof. Suppose without loss of generality that t_1v_1 is an edge. Let $S = \{t_1, v_1, v_2\}$, see Figure 1(b). By symmetry between v_2 and v_4 and by Claim 9, we can assume that $t_1v_4 \notin E(G)$.

Let us construct an L-coloring ψ of S. Let $\psi(t_1)$ be the unique color in $L(t_1)$ and choose the color of v_1 so that $\psi(v_1) \in L(v_1) \setminus (L(t_1) \cup L(t_2) \cup L(t_3))$. Now we need to choose $\psi(v_2)$ such that all vertices of $G \setminus S$ except for t_2 and t_3 have at least three colors left in their lists. Hence we only need to
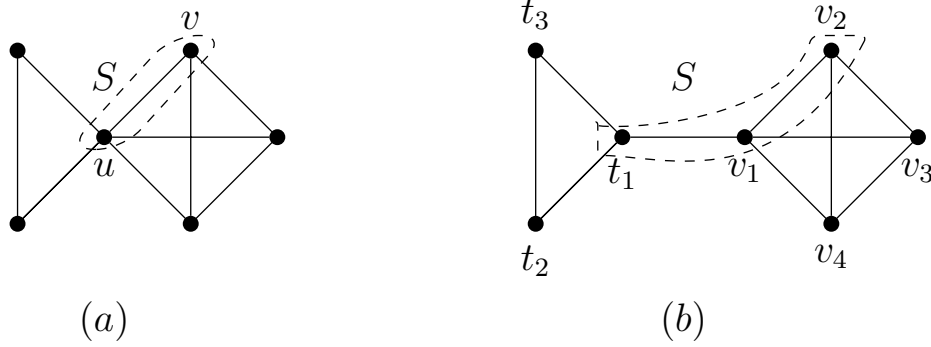
Figure 1: The configurations of Claims 11 and 12.

deal with vertices in $V(G) \setminus V(T)$ adjacent to all three of t_1, v_1 and v_2. We call such vertex y a *troublemaker* if $\psi(t_1), \psi(v_1) \in L(y)$. If there is no troublemaker, then we choose $\psi(v_2) \in L(v_2) \setminus (\{\psi(v_1), \psi(t_1)\} \cup L(t_2) \cup L(t_3))$ and use Observation 3 to obtain an L-coloring of G.

Since $t_1 v_4 \notin E(G)$, v_4 is not a troublemaker. Furthermore, if $t_1 v_3 \in E(G)$, then v_4 or v_2 would have degree at most four by Claim 5, which is a contradiction. Consequently, v_3 is not a troublemaker. Suppose that there is a troublemaker $y \in V(G) \setminus (X \cup V(T))$. By Claim 3, $t_1 v_1 y$ and $v_1 v_2 y$ are faces (hence, there is no other troublemaker) and t_1 is not adjacent to v_2. Claim 10 implies that v_2 is adjacent to neither t_2 nor t_3.

We choose $\psi(v_2)$ arbitrarily from $L(v_2) \setminus (L(y) \setminus \{\psi(t_1)\})$. Note that there is at least one choice for $\psi(v_2)$, since $\psi(t_1) \in L(y)$. Because $\psi(v_1) \in L(y)$, the resulting coloring of $G[S]$ is proper. Furthermore,

$$|L(y) \setminus \{\psi(t_1), \psi(v_1), \psi(v_2)\}| = 3,$$

hence Observation 3 applies. □

Let $P = p_1 \dots p_k$ be a path such that $p_1 \in V(T)$ and $p_{k-1}, p_k \in X$ and no edge of P is crossed, see Figure 2. Let the *score* of this path be $2k - b$, where

$$b = \begin{cases} 1 & \text{if } p_{k-2}p_k \in E(G) \\ 0 & \text{otherwise.} \end{cases}$$

Let $P \subseteq G$ be such a path with the smallest possible score. By Claim 12, we have $k \geq 4$. Let $\overline{P} = p_1 p_2 \dots p_{k-1}$, see Figure 2. The path $\overline{P}$ is induced, as otherwise P would contain a shorter subpath with a smaller score. Assume without loss of generality that $p_1 = t_1$.

Claim 13. *If $p_i, p_j \in V(\overline{P})$ are neighbors of a vertex $v \in V(G) \setminus V(P)$, then $|i - j| \leq 2$.*
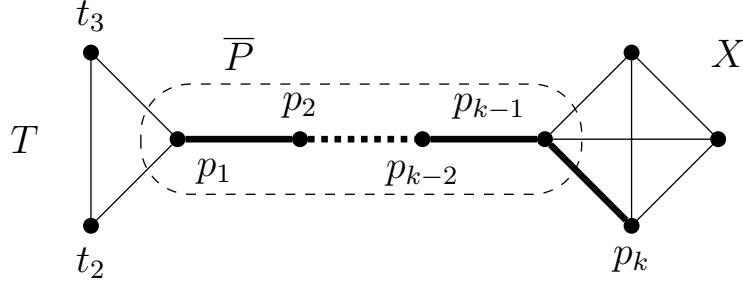
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Figure 2: Paths P and $\overline{P}$ connecting dangerous configurations.

Proof. Suppose that $i \geq j + 3$. Then the path $p_1 \ldots p_j v p_i \ldots p_k$ is shorter than P, contradicting the choice of P. □

Claim 14. *Every vertex $v \in V(G) \setminus V(P)$ has at most three neighbors in P.*

Proof. Suppose for contradiction that v has at least four neighbors in P. Claim 13 implies that v has four neighbors and they are $p_{k-3}, p_{k-2}, p_{k-1}$ and p_k. Since the path $p_1 \ldots p_{k-3} v p_k p_{k-1}$ does not have smaller score than P, it follows that p_k is adjacent to p_{k-2}. But that gives a contradiction with Claim 3, since either $p_{k-1} p_k v$ or $p_{k-2} p_{k-1} p_k$ is separating. □

Observe that t_2 and t_3 have at most two neighbors in P. Furthermore, consider a vertex $u \in X$; if u has three neighbors in P, then by the choice of P, these neighbors are p_k, p_{k-1} and p_{k-2}. By Claim 9, p_{k-2} is not adjacent to p_k. This contradicts the choice of P, since the path $p_1 \ldots p_{k-1} u$ has smaller score. Therefore, we have the following.

Claim 15. *If $v \in V(G) \setminus V(P)$ has three neighbors in P, then $v \notin V(T) \cup X$.*

For every vertex $v \in V(G) \setminus V(P)$ with three neighbors in P, let $g_P(v) = p_i$, where $p_i \in V(P)$ is the neighbor of v with the largest i. We define $g_P(v) = v$ if v has at most two neighbors in P. We write $g(v)$ instead of $g_P(v)$ for brevity when the path P is clear from the context.

Claim 16. *If u and v are distinct vertices of $V(G) \setminus V(P)$, then $g(u) \neq g(v)$.*

Proof. Let $p_g = g(u) = g(v)$ for two distinct vertices u and v. If $g \neq k$, then both u and v are adjacent to p_{g-2}, p_{g-1} and p_g by Claim 13. However, that contradicts Claim 3 or 5. Hence, we have $g = k$.

By the choice of P, all neighbors of u and v in P are contained in $\{p_{k-3}, p_{k-2}, p_{k-1}, p_k\}$. By Claim 3, u and v cannot both be adjacent to p_{k-1}, thus assume that say u is adjacent to p_{k-3}, p_{k-2} and p_k. By Claims 5 and
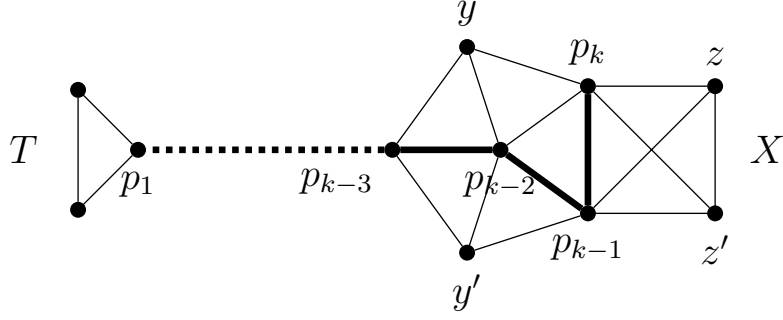
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Figure 3: Positions of T, X, y and y'.

10, the cycle $p_{k-2}p_{k-1}p_k u$ is not separating, hence by Claim 3, v is not adjacent to p_{k-1}. But then v is adjacent to p_{k-2} and the cycle $p_{k-2}p_{k-1}p_k v$ is separating, which is a contradiction. $\qquad\square$

Let $S = V(P)$. We now attempt to construct an L-coloring ψ of $G[S]$ so that the assumptions of Observation 3 are satisfied.

We will assign the colors to all vertices of P in order. We start with the unique choice $\psi(p_1) \in L(p_1)$ and color p_2 by a color $\psi(p_2) \in L(p_2) \setminus (L(t_1) \cup L(t_2) \cup L(t_3))$. Note that no other vertex of P has a neighbor in T. Suppose that we have already colored the vertices $p_1, \ldots, p_{j-1}$ and let $R_j = V(G) \setminus \{t_2, t_3, p_1, \ldots, p_{j-1}\}$. For a vertex $v \in R_j$, let $B_j(v) = L(v) \setminus \{\psi(p_i) : 1 \leq i \leq j-1, vp_i \in E(G)\}$. We choose the color $\psi(p_j) \in B_j(p_j)$ in such a way that $|B_{j+1}(v)| \geq 3$ for any $v \in R_{j+1}$. This coloring of P ensures that all vertices of $G - S$ other than t_2 and t_3 have at least three available colors; and since t_2 and t_3 are adjacent we can apply Observation 3 and obtain an L-coloring of G, which is a contradiction.

Let us now describe how $\psi(p_j)$ is chosen. Let y be a vertex such that $g(y) = p_j$ if such a vertex exists. Regardless of the choice of $\psi(p_j)$, for any vertex $v \in R_{j+1}$ other than y we have $|B_{j+1}(v)| \geq 3$, since v has at most two neighbors in $\{p_1, \ldots, p_j\}$ or it is not adjacent to p_j. The same holds for y if $|B_j(y)| \geq 4$, thus assume that $|B_j(y)| = 3$. If $B_j(p_j) \not\subseteq B_j(y)$, then we can choose $\psi(p_j) \in B_j(p_j) \setminus B_j(y)$. Therefore, since $|B_j(p_j)| \geq 3$, we can assume that $B_j(p_j) = B_j(y)$. Since $|B_j(p_j)| = 3$, p_j has two neighbors $p_i, p_l \in V(P)$ with $i < l < j$. Since the path P is induced, this is only possible if $j = k$ and p_k is adjacent to p_{k-2}. By Claim 3, y is not adjacent to p_{k-1}. Since y has exactly three neighbors in P, y is adjacent to p_{k-3}, p_{k-2} and p_k.

Consider now the path $P' = p_1 \ldots p_{k-2}p_k p_{k-1}$ instead of P. Note that P' has the same score as P, thus we conclude that there also exists a vertex y' adjacent to p_{k-1}, p_{k-2} and p_{k-3}. By Claim 3, $y p_{k-3}p_{k-2}$, $y p_{k-2}p_k$, $y' p_{k-3}p_{k-2}$ and $y' p_{k-2}p_{k-1}$ are faces, see Figure 3.
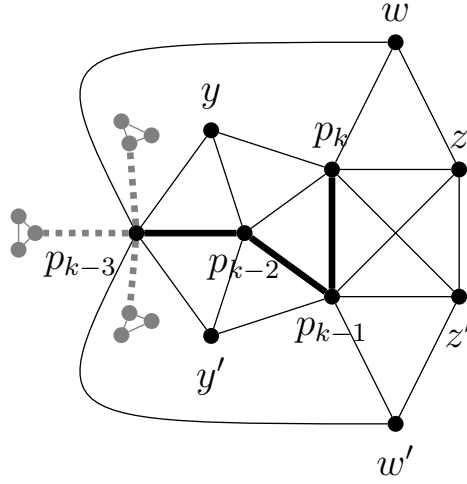
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Figure 4: Vertices w and w' and position of T.

Let us fix a coloring ψ of $P - \{p_{k-2}, p_{k-1}, p_k\}$ such that $|B_{k-2}(v)| \geq 3$ for every $v \in R_{k-2}$. By the preceding arguments, we can assume the following.

Claim 17. *There is no L-coloring of $G[V(T) \cup V(P - p_k)]$ such that $|B_k(v)| \geq 3$ for every $v \in R_k$ and $|B_k(y)| \geq 4$ or $B_k(p_k) \neq B_k(y)$.*

The only neighbors of p_{k-2} in $V(G) \setminus V(P)$ are y and y', hence there is no vertex $v \in V(G) \setminus V(P)$ with $g(v) = p_{k-2}$. Since ψ cannot be extended to a coloring contradicting Claim 17, we will show that:

Claim 18. $B_{k-2}(p_{k-2}) = B_{k-2}(y) = B_{k-2}(y')$ *and* $L(p_{k-1}) = L(p_k) = B_{k-2}(p_{k-2}) \cup \{c\}$ *for some color c.*

Proof. Since p_{k-3} is the only neighbor of y, y' and p_{k-2} in $P - \{p_{k-2}, p_{k-1}, p_k\}$, we have $|B_{k-2}(y)|, |B_{k-2}(y')|, |B_{k-2}(p_{k-2})| \geq 4$. If

$$|B_{k-2}(y)| = 5 \text{ or } B_{k-2}(p_{k-2}) \not\subseteq B_{k-2}(y),$$

we can choose $\psi(p_{k-2}) \in B_{k-2}(p_{k-2})$ so that $|B_{k-1}(y)| \geq 4$, and further extend ψ to a coloring of $P - p_k$ contradicting Claim 17. It follows that $|B_{k-2}(y)| = 4$ and $B_{k-2}(p_{k-2}) = B_{k-2}(y)$. Symmetrically, we obtain $B_{k-2}(p_{k-2}) = B_{k-2}(y')$ by considering P'.

Consider colors $c_1 \in L(p_{k-1}) \setminus B_{k-2}(p_{k-2})$ and $c_2 \in L(p_k) \setminus B_{k-2}(p_{k-2})$. If $c_1 \neq c_2$, then choose $\psi(p_{k-2}) \in B_{k-2}(p_{k-2})$ arbitrarily and set $\psi(p_{k-1}) = c_2$. Since $c_2 \in B_k(p_k) \setminus B_k(y)$, this coloring contradicts Claim 17. Therefore, $c_1 = c_2$, which implies that $L(p_{k-1}) = L(p_k) = B_{k-2}(p_{k-2}) \cup \{c_1\}$. $\square$

Let us now choose $\psi(p_{k-2}) \in B_{k-2}(p_{k-2})$ arbitrarily and set $\psi(p_{k-1}) = c$. Note that $B_k(y) = B_k(y') = B_k(p_k)$.

Let $X \setminus \{p_{k-1}, p_k\} = \{z, z'\}$, where z is joined to p_k by a non-crossed edge. By planarity, z' is not adjacent to p_{k-2}, thus $B_k(z') \geq 4$. Let $Q = p_1 \ldots p_{k-1} z'$. If no vertex $w' \notin \{y, y', p_k\}$ satisfies $g_Q(w') = z'$, then we can choose $\psi(z') \in B_k(z') \setminus B_k(y')$ and apply Observation 3 with $S = V(Q)$, obtaining an L-coloring of G. Therefore, we may assume that there exists such a vertex w'. Since w' has at least three neighbors in Q and it is not adjacent to p_{k-2} by planarity, the choice of P implies that w' is adjacent to p_{k-1} and p_{k-3}. Symmetrically, by considering the path $Q' = p_1 \ldots p_{k-2} p_k z$, we conclude that there exists a vertex $w \notin \{y, y', p_{k-1}\}$ adjacent to z, p_k and p_{k-3}. However, by planarity either $p_{k-3} p_{k-2} p_{k-1} w'$ or $p_{k-3} p_{k-2} p_k w$ contradicts Claim 5, see Figure 4 showing the possible positions of T with respect to these cycles. $\square$

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