Strong d-collapsibility

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Abstract

We introduce a notion of strong d-collapsibility. Using this notion, we simplify the proof of Matoušek and the author [MT08] showing that the nerve of a family of sets of size at most d is d-collapsible.

1 Introduction

Simplicial complexes and d-collapsibility. A finite *simplicial complex* K is a collection of subsets (called *faces* or *simplices*) of a finite set X which is downwards closed, i.e, if $\sigma \in \mathsf{K}$ and $\tau \subset \sigma$ then $\tau \in \mathsf{K}$. The *dimension* of a face $\sigma \in \mathsf{K}$ is defined to be the value $|\sigma| - 1$. The *dimension* of K is the maximum of the dimensions of faces contained in K. Zero-dimensional faces are called *vertices*. Often it is assumed that X is the set of vertices; in particular we will work with this assumption.

Wegner in his seminal 1975 paper [Weg75] introduced d-collapsible simplicial complexes. To define this notion, we first introduce an *elementary d-collapse*. Let K be a simplicial complex and let $\sigma, \tau \in \mathsf{K}$ be faces (simplices) such that

(i) $\dim \sigma \leq d - 1$,
(ii) τ is an inclusion-maximal face of K,
(iii) $\sigma \subseteq \tau$, and
(iv) τ is the *only* face of K satisfying (ii) and (iii).

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Then we say that σ is a *d-collapsible face* of K and that the simplicial complex $\mathsf{K}' := \mathsf{K} \setminus \{\eta \in \mathsf{K} : \sigma \subseteq \eta \subseteq \tau\}$ arises from K by an elementary d-collapse. If we want to emphasize σ, we write $\mathsf{K} \to^{\sigma} \mathsf{K}'$ (note that K' is uniquely determined by σ and K). A simplicial complex K is *d-collapsible* if there exists a sequence of elementary d-collapses that reduces K to the empty complex $\emptyset$.

The motivation of introducing d-collapsibility comes from combinatorial geometry as a tool for studying intersection patterns of convex sets. Our task in this short note is not to describe this interesting connection; however, we refer, e.g., to [Weg75, KM05, MT08] for more background.

A nerve and its d-collapsibility. Given a finite collection $\mathcal{C} = \{C_1, \dots, C_n\}$ of sets, the nerve $\mathsf{N}(\mathcal{C})$ of this collection is a simplicial complex where $\mathcal{C}$ is the (multi)set of its vertices and where its faces are collections $C_{i_1}, \dots, C_{i_k}$ of vertices such that $C_{i_1} \cap \dots \cap C_{i_k}$ is non-empty. We emphasize that it is allowed that $C_i = C_j$ for $i \neq j$; i.e., $\mathcal{C}$ is a multiset. In particular for such C_i and C_j there are two (twin) vertices in the nerve.

Matoušek and the author [MT08] studied, how far is the notion of d-collapsibility form its geometrical motivation. As one of the main tools they proved the following proposition.

Proposition 1. *Suppose that $\mathcal{C}$ is a collection of sets of size at most d. Then $\mathsf{N}(\mathcal{C})$ is d-collapsible.*

We will introduce a notion of strong d-collapsibility and using this notion we simplify the proof of Matoušek and the author. We also hope that this notion can be used in a different context as well.

Strong d-collapsibility.[1] Assume that η is a face of a complex K. The *link* of η in K is a simplicial complex defined by $\mathrm{lk}(\eta, \mathsf{K}) = \{\vartheta \in \mathsf{K} : \vartheta \cap \eta = \emptyset, \vartheta \cup \eta \in \mathsf{K}\}$.

Assume that v is a vertex of K such that $\mathrm{lk}(\{v\}, \mathsf{K})$ is $(d-1)$-collapsible. By an elementary strong d-collapse of K we mean the simplicial complex K' obtained by removing all the faces containing v, i.e. $\mathsf{K}' = \mathsf{K} - v = \{\vartheta \in \mathsf{K} : v \notin \vartheta\}$. If we want to emphasize v, we write $\mathsf{K} \Rightarrow^{v} \mathsf{K}'$. A simplicial complex is strongly d-collapsible if it can be vanished by a sequence of elementary strong d-collapses.[2]

[1] The introduction of this notion is motivated by strong collapsibility in topology.

[2] In an elementary strong d-collapse we could also use an inductive definition where $\mathrm{lk}(\{v\}, \mathsf{K})$ would be assumed to be strong $(d-1)$-collapsible and strong 0-collapsible would mean being a simplex. Thus we would get a similar (but perhaps different) notion of strong d-collapsibility. The forthcoming results would remain unchanged.

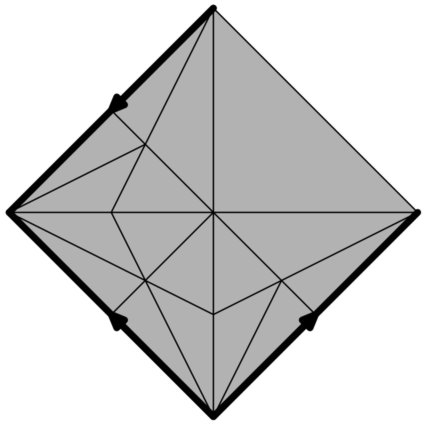

Figure 1: A complex which is 2-collapsible, but not strongly 2-collapsible.

We will prove the following results.

Proposition 2. *Let d be a non-negative integer. Assume that a simplicial complex K is strongly d-collapsible then it is d-collapsible as well.*

Theorem 3. *Let d be a positive integer. Suppose that $\mathcal{C}$ is a collection of sets of size at most d. Then $\mathsf{N}(\mathcal{C})$ is strongly d-collapsible.*

Proposition 1 is an obvious consequence of these two results.

2 Properties of Strong d-collapsibility

First, we prove Proposition 2.

Proof. It is sufficient to show that an elementary strong d-collapse $\mathsf{K} \Rightarrow^{v} \mathsf{K}'$ can be simulated by a sequence of elementary d-collapses. Let $\mathsf{L} = \mathrm{lk}(\{v\}, \mathsf{K})$. We know that L is $(d-1)$-collapsible. Let $\mathsf{L} \to^{\sigma_1} \mathsf{L}_2 \to^{\sigma_2} \cdots \to^{\sigma_k} \emptyset$ be a sequence of elementary d-collapses. Then it is routine to check that $\mathsf{K} \to^{\sigma_1 \cup \{v\}} \mathsf{K}_2 \to^{\sigma_2 \cup \{v\}} \cdots \to^{\sigma_k \cup \{v\}} \mathsf{K}'$ is a sequence of elementary d-collapses which indeed ends up with K'. (For this we remark that $\mathsf{K}_i = \mathsf{K}' \cup \{\vartheta \cup \{v\} : \vartheta \in \mathsf{L}_i\}$.)

□

We remark that there are complexes which are d-collapsible, but not strongly d-collapsible. An example of such a complex is drawn in Figure 2. The thick lines are identified according to the arrows. There are higher-dimensional analogues of this complex; see the construction of complex $\mathsf{C}(\rho)$ in [Tan08].

3 Strong d-collapsibility of a nerve

Here we prove Theorem 3.

Let a be a point which is not contained in the vertex set of a given complex K. The *cone* of K is a simplicial complex given by $a\mathsf{K} = \mathsf{K} \cup \{\sigma \cup \{a\} : \sigma \in \mathsf{K}\}$.

Lemma 4. *Suppose that* K *is* d*-collapsible then* $a\mathsf{K}$ *is* d*-collapsible as well.*

Proof. Let $\mathsf{K} \to^{\sigma_1} \mathsf{K}_2 \to^{\sigma_2} \cdots \to^{\sigma_k} \emptyset$ be a sequence of elementary d-collapses of K. Then $a\mathsf{K} \to^{\sigma_1} a\mathsf{K}_2 \to^{\sigma_2} \cdots \to^{\sigma_k} a\emptyset = \emptyset$ is a sequence of elementary d-collapses of $a\mathsf{K}$.[3]

□

Proof of Theorem 3. We proceed by induction on d and on the size of $\mathcal{C}$. Theorem 3 is surely true if $\mathcal{C}$ contains a single set or if $d = 1$.

Let $C_1 \in \mathcal{C}$ be a set of maximal size. We want to show that

$$\mathsf{N}(\mathcal{C}) \Rightarrow^{C_1} \mathsf{N}(\mathcal{C} \setminus \{C_1\}).$$

Then $\mathsf{N}(\mathcal{C} \setminus \{C_1\})$ is strongly d-collapsible by induction.

It is sufficient to check that $\operatorname{lk}(C_1, \mathsf{N}(\mathcal{C}))$ is $(d-1)$-collapsible. Let us denote $\mathcal{C}_{C_1} = \{C \cap C_1 \in \mathcal{C} : C \in \mathcal{C} \setminus \{C_1\}\}$. Then $\operatorname{lk}(C_1, \mathsf{N}(\mathcal{C})) = \mathsf{N}(\mathcal{C}_{C_1})$. If there is no set of size d in $\mathcal{C}_{C_1}$, then $\operatorname{lk}(C_1, \mathsf{N}(\mathcal{C}))$ is $(d-1)$-collapsible by induction and we are done.

For otherwise, let $\mathcal{D} = \{D_1, \dots, D_m\} \subseteq \mathcal{C}_{C_1}$ be the collection of all sets of size d in $\mathcal{C}_{C_1}$. For every $D \in \mathcal{D}$ we thus have $D = C_1$. It means that $\operatorname{lk}(C_1, \mathsf{N}(\mathcal{C})) = D_1 D_2 \dots D_m \mathsf{N}(\mathcal{C}_{C_1} \setminus \mathcal{D})$, where $D_1 D_2 \dots D_m$ stands for (iterated) cone with vertices $D_1, \dots, D_m$. By Lemma 4 and induction it follows that $\operatorname{lk}(C_1, \mathsf{N}(\mathcal{C}))$ is $(d-1)$-collapsible. □

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[3] Purely formally, one has to be a bit careful here and distinguish a simplicial complex $\{\emptyset\}$ containing a single empty face from $\emptyset$ containing no face.

References

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