DISTRIBUTIVITY OF THE NORMAL COMPLETION, AND ITS PRIESTLEY REPRESENTATION

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Abstract. The normal, or Dedekind-MacNeille, completion $\delta(L)$ of a distributive lattice L need not be distributive. However, $\delta(L)$ does contain a largest distributive sublattice $\beta(L)$ containing L, and $\delta(L)$ is distributive if and only if $\beta(L)$ is complete if and only if $\delta(L) = \beta(L)$. In light of these facts, it may come as a surprise to learn that $\beta(L)$ was developed (in [1]) for reasons having nothing to do with distributivity.

In fact, the cuts of $\beta(L)$ can be readily identified as those having the property we here term exactness. This provides a useful criterion for testing whether the normal completion of a given lattice is distributive. We illustrate the utility of this criterion by providing a simple demonstration that the normal completion of a Heyting algebra is distributive.

We prove these facts by simple arguments from first princples, and then bring out the geometry of the situation by developing the construct in Priestley spaces. While the elements of L appear as clopen up-sets of the (ordered) space, the elements of both extensions $\delta(L)$ and $\beta(L)$ are manifested as well defined more general types of open up-sets.

Introduction

It is well known that the normal completion $\delta(L)$ (also known as the Dedekind-MacNeille completion) does not preserve distributivity. A necessary and sufficient condition for the distributivity of $\delta(L)$ was

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developed in [2] by means of convergence and Cauchy structure techniques. Without appealing to these techniques, we highlight and reconfirm this result by easy arguments from first principles. The crucial construct is the *distributive extension* $\beta(L)$ [1].

Our purpose is to bring out the geometrical aspects of the situation by means of Priestley duality (see, e.g., [10, 11]). Both extensions β and δ emerge naturally as families of subsets of the Priestley space X of the lattice L in question. While the elements of L are represented by the clopen up-sets of X, the elements of $\delta(L)$ appear as the bi-regular open up-sets, which is to say the up-sets of X that are down-interiors of their up-closures. The exact cuts are then characterized by an easy formula concerning the two-sided interior.

The utility of the resulting criterion for testing the distributivity of $\delta(L)$ is exemplified by an easy demonstration that Heyting algebras have distributive completions: their cuts are clearly exact. (The distributivity of the completion of a Heyting algebra is known, and, in fact, more is true. The completion of a Heyting algebra is itself a Heyting algebra; see [7].) In the geometrical representation, the fact that Heyting algebras have distributive completions follows from the nature of their Priestley spaces, in which the two-sided interiors of up-sets coincide with their down-interiors.

For the standard facts about posets see, e.g., [5].

1. Preliminaries

1.1. Notation. In a poset $P = (P, \leq)$ we will denote, for a subset $A \subseteq P$, as usual,

$$\downarrow A = \{x \mid \exists a \in A,\ x \leq a\}, \qquad \uparrow A = \{x \mid \exists a \in A,\ x \geq a\},$$

and write $\downarrow a$ resp. $\uparrow a$ for $\downarrow \{a\}$ resp. $\uparrow \{a\}$. The set of all *lower* resp. *upper bounds* of $A \subseteq P$ will be denoted by

$$\mathfrak{l}(A) \quad \text{resp.} \quad \mathfrak{u}(A).$$

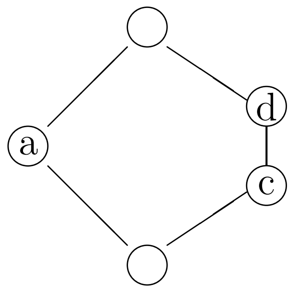
Thus, $\mathfrak{l}(A) = \bigcap\{\downarrow a \mid a \in A\}$ and $\mathfrak{u}(A) = \bigcap\{\uparrow a \mid a \in A\}$, while $\downarrow A = \bigcup\{\downarrow a \mid a \in A\}$ and $\uparrow A = \bigcup\{\uparrow a \mid a \in A\}$.

1.2. We will be mostly concerned with (bounded) *distributive lattices* L, but an important auxilliary role will be played by the *modular* ones, that is, the lattices satisfying the implication

$$a \leq c \quad \Rightarrow \quad a \vee (b \wedge c) = (a \vee b) \wedge c.$$

[1] We use the notation $\beta(L)$ in conformity with the notation of [2], see Section 3 below.

Recall that modular lattices are well known to be precisely those lattices that do not contain this subconfiguration.



The labelling will be used in some proofs below.

Also, we will be interested in *Heyting algebras*, that is, bounded lattices with an extra operation $\rightarrow$ satisfying

$$a \wedge b \leq c \quad \text{iff} \quad a \leq b \rightarrow c$$

and with the dually defined co-Heyting ones. Note that the adjunction between $- \wedge b$ and $b \rightarrow -$ makes Heyting algebras automatically distributive (and more).

1.3. We set

$$a \downarrow b = \{x \mid x \wedge b \leq a\} \quad \text{and} \quad a \uparrow b = \{x \mid x \vee a \leq b\}.$$

Remarks. 1. This is a special case of the operations $A \downarrow B$ and $A \uparrow B$ with subsets $A, B \subseteq L$ used, e.g., in [1, 2, 3]. For our purposes, the one-point sets will do.

2. Note that $a \downarrow b$ is an ideal and $a \uparrow b$ is a filter.

3. In case of a Heyting algebra we have $a \downarrow b = \ \downarrow(b \rightarrow a)$. Thus, in a general case, $a \downarrow b$ can be viewed as a "surrogate multivalued Heyting operation", with $a \uparrow b$ having a similarly co-Heyting connotation.

1.4. A *Priestley space* is a compact partially ordered topological space $(X, \leq, \tau)$ such that for any $x \not\leq y$ in X there is a clopen up-set U such that $x \in U$ and $y \notin U$.

In a Priestley space denote by $\tau_{\downarrow}$ the topology of all open down-sets, and by $\tau_{\uparrow}$ the topology of open up-sets. It is a standard fact that $\tau_D \cup \tau_U$ is a subbasis of τ; in fact, the system of all *clopen* down-sets generates τ, and likewise for the family of clopen up-sets. We will denote the closure operators of the three topologies by $c(A)$, $c_{\uparrow}(A)$, and $c_{\downarrow}(A)$, respectively, and the corresponding interior operators by $i(A)$, $i_{\uparrow}(A)$, and $i_{\downarrow}(A)$, respectively.

1.5. Denote by **PSp**. the category of Priestley spaces and monotone continuous maps (the *Priestley maps*) and by **DLat** the category of bounded distributive lattices and bounded lattice homomorphisms. Recall the *Priestley duality* ([10, 11]) constituted by the contravariant functors

$$\mathcal{L} : \mathbf{PSp} \to \mathbf{DLat}, \quad \mathcal{P} : \mathbf{DLat} \to \mathbf{PSp}$$

defined by

$$\mathcal{L}(X) = (\{U \mid U \subseteq X \text{ clopen up-set }\}, \subseteq), \quad \mathcal{L}(f)(U) = f^{-1}[U],$$
$$\mathcal{P}(L) = (\{x \mid x \text{ prime filter in } L\}, \subseteq, \tau), \quad \mathcal{P}(h)(x) = h^{-1}[x]$$

(τ is the topology of 2^L) with the natural equivalences

$$\rho_X = (x \mapsto \{U \mid x \in U\}) : X \to \mathcal{PL}(X),$$
$$\lambda_L = (a \mapsto \{x \mid a \in x\}) : L \to \mathcal{LP}(L).$$

We will often think of a distributive lattices L as represented in its Priestley space by identifying L with $\mathcal{LP}(L)$. Furthermore, we will obtain some extensions of L as lattices of more general open sets in $\mathcal{P}(L)$.

2. Cuts and the Dedekind-MacNeille completion $\delta(L)$

2.1. For subsets A, B of a poset P we write

$$A \leq B \text{ if } \forall a \in A\, \forall b \in B\; a \leq b, \quad \text{that is, if} \quad A \subseteq \mathfrak{l}(B) \text{ and } B \subseteq \mathfrak{u}(A).$$

If $A = \mathfrak{l}(B)$ and $B = \mathfrak{u}(A)$ we speak of the pair (A, B) as of *cut* in P. Thus a cut is obviously determined by either of its components. Note that for each element $a \in P$ we have the cut

$$(\downarrow a, \uparrow a).$$

We refer to these as the *principal cuts*.

We obviously have

$$A_1 \subseteq A_2 \;\Rightarrow\; \mathfrak{u}(A_1) \supseteq \mathfrak{u}(A_2) \quad \text{and} \quad B_1 \subseteq B_2 \;\Rightarrow\; \mathfrak{l}(B_1) \supseteq \mathfrak{l}(B_2)$$

and

$$A \subseteq \mathfrak{lu}(A) \quad \text{and} \quad B \subseteq \mathfrak{ul}(B);$$

Consequently, we easily infer that

$$\mathfrak{ulu}(A) = \mathfrak{u}(A) \quad \text{and} \quad \mathfrak{lul}(B) = \mathfrak{l}(B)$$

and hence the cuts are precisely the pairs of the form $(\mathfrak{lu}(A), \mathfrak{u}(A))$ resp. $(\mathfrak{l}(B), \mathfrak{ul}(B))$.

2.2. The Dedekind-MacNeille completion. For cuts (A_1, B_1) and (A_2, B_2) we write

$$(A_1, B_1) \leq (A_2, B_2) \quad \text{if} \quad A_1 \subseteq A_2 \text{ and } B_1 \supseteq B_2$$

Of course, either inclusions implies the other. Thus we obtain the poset of cuts

$$\delta(P),$$

the well-known *Dedekind-MacNeille completion* of P (see e.g. [4, 9]). Let us recall some of its properties.

(a) With the order above, $\delta(P)$ is a complete lattice.
(b) The mapping $\delta_P = (a \mapsto (\downarrow a, \uparrow a)) : P \to \delta(P)$ preserves all the existing suprema and all the existing infima.
(c) If $P \subseteq L \subseteq \delta(P)$, if L is a sublattice of $\delta(P)$, and if L is complete then $L = \delta(P)$.

2.3. In the sequel we will need concrete formulas for the meets and joins in $\delta(P)$. It is easy to check that they are as follows

$$(A_1, B_1) \wedge (A_2, B_2) = (A_1 \cap A_2, \mathfrak{u}(A_1 \cap A_2)),$$
$$(A_1, B_1) \vee (A_2, B_2) = (\mathfrak{l}(B_1 \cap B_2), B_1 \cap B_2).$$

2.4. A topological representation of $\delta(L)$.

Consider a bounded distributive lattice L with Priestley space $\mathcal{P}(L) = X$. Recall from 1.4 that, for a subset $U \subseteq X$, $c_\downarrow(U)$ designates the closure of U in the topology of open down-sets, so that it is the smallest closed up-set containing U. Since any closed up-set in X is the intersection of basic clopen up-sets, i.e., those of the form $\lambda_L(a)$, $a \in L$, we have

$$c_\downarrow(U) = \bigcap_{U \subseteq \lambda_L(b)} \lambda_L(b)$$

Likewise $i_\uparrow(U)$, the largest open up-set contained within U, is equal to $\bigcup_{\lambda_L(a) \subseteq U} \lambda_L(a)$.

We define a *cut in* X to be a nonempty subset $U \subseteq X$ such that

$$U = i_\uparrow c_\downarrow(U),$$

necessarily an open up-set. For example, each basic open set of the form $\lambda_L(a)$, $a \in L$, is a cut. We denote the set of cuts in X by

$$\overline{\mathcal{L}}(X) = \{U \mid U \text{ is a cut in } X\}$$

2.5. Lemma. *Let L be a bounded distributive lattice with Priestley space X. The maps*

$$\delta(L) \ni (A,B) \longrightarrow \bigcup_A \lambda_L(a) \in \overline{\mathcal{L}}(X)$$
$$(\{a \mid \lambda_L(a) \subseteq U\}, \{b \mid \lambda_L(b) \supseteq U\}) \longleftarrow U$$

are inverse order-preserving bijections. Consequently, $\overline{\mathcal{L}}(X)$ is a lattice, with

$$U_1 \wedge U_2 = U_1 \cap U_2 \quad \textit{and} \quad U_1 \vee U_2 = i_\uparrow c_\downarrow (U_1 \cup U_2).$$

Proof. First observe that, since λ_L is order-preserving,

$$c_\downarrow \left(\bigcup_{a\in A} \lambda_L(a) \right) = \bigcap_{b\in \mathfrak{u}(A)} \lambda_L(b) \quad \text{and} \quad i_\uparrow \left(\bigcap_{b\in B} \lambda_L(b) \right) = \bigcup_{a\in \mathfrak{l}(B)} \lambda_L(a)$$

for all $A, B \subseteq L$. It follows that $i_\uparrow c_\downarrow \bigcup_A \lambda_L(a) = \bigcup_{\mathfrak{lu}(A)} \lambda_L(a)$, so that $\bigcup_A \lambda_L(a)$ is a cut in X if (A, B) is a cut in L. On the other hand, if, for a subset $U \subseteq X$, we set $A = \{a \mid \lambda_L(a) \subseteq U\}$ and $B = \{b \mid \lambda_L(b) \supseteq U\}$, we have

$$i_\uparrow c_\downarrow (U) = i_\uparrow \left(\bigcap_B \lambda_L(b) \right) = \bigcup_{\mathfrak{l}(B)} \lambda_L(a).$$

Therefore, if U is a cut in X, i.e., if $U = i_\uparrow c_\downarrow(U)$, it follows that $\mathfrak{l}(B) \subseteq A$, and since $A \leq B$ it then follows that $A = \mathfrak{l}(B)$. And if $b \in \mathfrak{u}(A)$ then $\lambda_L(b) \supseteq \bigcup_A \lambda_L(a) = U$ and so $b \in B$. That is, (A, B) is a cut in L.

Suppose (A, B) is a cut in L and put $U = \bigcup_A \lambda_L(a)$. If $\lambda_L(c) \subseteq U$ for some $c \in L$ then, since A is upper-directed in L and $\lambda_L(c)$ is compact, it follows that $c \in A$, with the result that $A = \{c \mid \lambda_L(c) \subseteq U\}$. Likewise, if $\lambda_L(c) \supseteq U$ then $c \in \mathfrak{u}(A) = B$, so that $B = \{c \mid \lambda_L(c) \supseteq U\}$. On the other hand, if U is any cut in X then $U = \bigcup_{\lambda_L(a)\subseteq U} \lambda_L(a)$ for similar reasons. Finally, the order-preserving nature of the maps is evident. □

2.6. Theorem. *For a bounded distributive lattice L, the Dedekind-MacNeille completion $\delta_L : L \to \delta(L)$ is realized by the inclusion of the $\mathcal{LP}(L)$ into $\overline{\mathcal{L}}\mathcal{P}(L)$.*

3. Exact cuts and the distributive extension $\beta(L)$

It can be regarded as a defect of the Dedekind-MacNeille completion $\delta(L)$ of a distributive lattice that it may fail to be distributive. However, the completion $\delta(L)$ does have a largest distributive sublattice containing L. This sublattice, which we refer to as the *distributive* *extension* of L, and denote by $\beta(L)$, was developed originaly in [2]. It may come as a surprise that the motivation and development there had nothing to do with distributivity. It was the Cauchy completion of L with respect to an appropriate natural Cauchy structure, and therefore one spoke of a *completion* $\beta(L)$. This refers to a type of completeness different from order completeness, which we mean by completeness in this article. Hence we speak here, rather, of an *extension* then of a completion.

3.1. A cut (A, B) in a lattice L is said to be *exact* if there is no pair $c < d$ such that $A \subseteq c \downarrow d$ and $B \subseteq c \uparrow d$. The set of all exact cuts in L will be denoted by

$$\beta(L).$$

Generally the lattice L is not embedded into $\beta(L)$ by the map δ_L from 2.2, but we do have this.

3.1.1. Observation. *A lattice L is modular iff each principal cut $(\downarrow a, \uparrow a)$ is exact.*

(Recall the configuration from 1.2: obviously $\downarrow a \subseteq c \downarrow d$ and $\uparrow a \subseteq c \uparrow d$.)

3.2. Lemma. *Let (A_i, B_i), $i = 1, 2$, be exact cuts in a distributive lattice L. Let $c < d$. Then*

$$A_1 \cup A_2 \subseteq c \downarrow d \quad \Rightarrow \quad B_1 \cap B_2 \nsubseteq c \uparrow d$$

and

$$B_1 \cup B_2 \subseteq c \uparrow d \quad \Rightarrow \quad A_1 \cap A_2 \nsubseteq c \downarrow d.$$

Proof. We will prove the first implication, the proof of the second one is similar. Suppose $A_1 \cup A_2 \subseteq c \downarrow d$ so that, in particular, $A_1 \subseteq c \downarrow d$. Consequently, by exactness, $B_1 \nsubseteq c \uparrow d$, and there is a $b_1 \in B_1$ such that

$$b_1 \vee c \ngeq d.$$

Now we have

$$c \leq c' = (b_1 \vee c) \wedge d < d$$

and $A_2 \subseteq c \downarrow d \subseteq c' \downarrow d$, hence $B_2 \nsubseteq c' \uparrow d$ and there is a $b_2 \in B_2$ such that $b_2 \vee c' \ngeq d$. Set $b = b_1 \vee b_2$. Then

$$(b \vee c) \wedge d = (b_2 \vee (b_1 \vee c)) \wedge d = (b_2 \vee ((b_1 \vee c) \wedge d)) \wedge d = (b_2 \vee c') \wedge d < d$$

so that $b \vee c \ngeq d$ and since $b \in B_1 \cap B_2$ this shows that $B_1 \cap B_2 \nsubseteq c \uparrow d$ □

Since obviously, for cuts (A_i, B_i), $(B_1 \cup B_2) \subseteq \mathfrak{u}(A_1 \cap A_2)$ and $(A_1 \cup A_2) \subseteq \mathfrak{l}(B_1 \cap B_2)$, we easily infer the following important fact from the formulas in 2.3.

3.2.1. Corollary. *If L is a a distributive lattice then $\beta(L)$ is a sublattice of $\delta(L)$.*

3.3. Proposition. *Let L be a distributive lattice. Then $\beta(L)$ is distributive.*

Proof. We want to show that, for exact cuts,

$$(A, B)\vee((A_1, B_1)\wedge(A_2, B_2)) = ((A, B)\vee(A_1, B_1))\wedge((A, B)\vee(A_2, B_2))$$

which reduces to showing that

$$\mathfrak{l}(B \cap \mathfrak{u}(A_1 \cap A_2)) \supseteq \mathfrak{l}(B \cap B_1) \cap \mathfrak{l}(B \cap B_2).$$

Suppose for contradiction there is an $x \in \mathfrak{l}(B \cap B_1) \cap \mathfrak{l}(B \cap B_2)$ and a $y \in B \cap \mathfrak{u}(A_1 \cap A_2)$ such that $x \not\leq y$ and set $c = x \wedge y < x = d$. Since $x \in \mathfrak{l}(B \cap B_1)$ we have trivially $B \cap B_1 \subseteq c \uparrow d$ and hence by Lemma 3.2, $A \cup A_1 \not\subseteq c \downarrow d$, and since $A \subseteq c \downarrow d$ because $y \in B = \mathfrak{u}(A)$, we have $A_1 \not\subseteq c \downarrow d$ and we can choose an $a_1 \in A_1$ such that $a_1 \wedge d \not\leq c$. Set $d' = (a_1 \wedge d) \vee c$ to obtain $c < d' \leq d$. Now $B \cap B_2 \subseteq c \uparrow d \subseteq c \uparrow d'$ and we have, by the same argument as above, an $a_2 \in A_2$ such that $a_2 \wedge d' \not\leq c$. Now $a = a_1 \wedge a_2$ satisfies

$$(a \wedge d) \vee c = ((a_1 \wedge d) \wedge a_2) \vee c = d' \wedge (a_2 \vee c) = (d' \wedge a_2) \vee c > c.$$

Since $y \in \mathfrak{u}(A_1 \cap A_2)$ and $a \in A_1 \cap A_2$, this contradicts $(a \wedge d) \vee c \leq (y \wedge d) \vee c = c$ and we conclude that $\beta(L)$ is distributive. □

3.4. Theorem. *Let L be a bounded distributive lattice. Then the following statements are equivalent.*

(1) *$\delta(L)$ is distributive,*
(2) *$\delta(L)$ is modular,*
(3) *$\beta(L)$ is complete,*
(4) *$\beta(L) = \delta(L)$,*
(5) *if subsets $A \leq B$ in L satisfy $A \subseteq c \downarrow d$ and $B \subseteq c \uparrow d$ for some $c < d$ in L then $\mathfrak{l}(B) \not\subseteq A$ or $\mathfrak{u}(A) \not\subseteq B$,*
(6) *if subsets $A \leq B$ in L satisfy $A \subseteq c \downarrow d$ and $B \subseteq c \uparrow d$ for some $c < d$ in L then $\mathfrak{l}(B) \not\leq \mathfrak{u}(A)$.*

Proof. (5) and (6) are just reformulations of (4). Note that $A \leq B$ generates a unique cut iff $\mathfrak{l}(B) \not\leq \mathfrak{u}(A)$).

(4)⇒(3) since $\delta(L)$ is complete.

(3)⇒(4) by property (c) in 2.2.

(4)⇒(1) by 3.3, and (1)⇒(2) is trivial.

(1)⇒(4): Suppose there is a non-exact cut (A, B) witnessed by $A \subseteq c \downarrow b$ and $B \subseteq c \uparrow d$. Then the elements (A, B), $\delta_L(c)$ and $\delta_L(d)$ generate a pentagon from 1.2 violating the modularity. □

3.5. Note. In both 3.2 and 3.3 the distributivity was used rather inconspicuously. Namely, we have used the equalities

$$(x \vee (y \vee c)) \wedge d = (x \vee ((y \vee c) \wedge d)) \wedge d, \ (x \wedge (y \wedge d)) \vee c = (x \wedge ((y \wedge d) \vee c)) \vee c$$

This was essential, at least for 3.3; it may be of some interest to observe that, in view of 3.3, a modular lattice is distributive iff it satisfies one of these equations. On the other hand, Corollary 3.2.1 may perhaps be proved by other means.

3.6. A topological representation of $\beta(L)$.

Let U be a cut in a Priestley space X. We say that U is *exact* if U is dense in $ic_{\downarrow}(U)$, the two-sided interior of $c_{\downarrow}(U)$.

3.7. Lemma. *Let L be a bounded distributive lattice with Priestley space X. Then, in the correspondence of 2.5, the exact cuts in L correspond to the exact cuts in X.*

Proof. Let (A, B) be an inexact cut in L, say $A \subseteq c \downarrow d$ and $B \subseteq c \uparrow d$ for some $c < d$, and let $U = \bigcup_A \lambda_L(a)$ be the corresponding cut in X. Then $V = \lambda_L(d) \smallsetminus \lambda_L(c)$ is a nonempty clopen subset of X, and the fact that $\lambda_L(a) \cap \lambda_L(d) \subseteq \lambda_L(c)$ for all $a \in A$ implies that $U \cap V = \emptyset$, while the fact that $\lambda_L(b) \cup \lambda_L(c) \supseteq \lambda_L(d)$ for all $b \in B$ implies that $V \subseteq c_{\downarrow}(U)$. We have shown that U fails to be dense in $ic_{\downarrow}(U)$.

On the other hand, consider a cut U in X with corresponding cut (A, B) in L. If U is inexact then it is only because there is some nonempty basic open set $V \subseteq c_{\downarrow}(U) \smallsetminus U$. Such a set is of the form $V = \lambda_L(d) \smallsetminus \lambda_L(c)$ for some $c < d$. From the fact that $V \cap U = \emptyset$ it follows that $a \wedge d \leq c$ for all $a \in A$, and from the fact that $V \subseteq c_{\downarrow}(U)$ it follows that $b \vee c \geq d$ for all $b \in B$. That is, (A, B) is inexact in L. □

We denote the set of exact cuts in X by

$$\mathcal{L}^{\beta}(X) = \{U \mid U \text{is an exact cut in} X\}$$

It follows from 3.3 that $\mathcal{L}^{\beta}(X)$ forms a distributive sublattice of $\overline{\mathcal{L}}(X)$ containing (the image of) L. More precisely, we have this.

3.8. Theorem. *Let L be a bounded distributive lattice. The distributive extension $\beta_L : L \to \beta L$ is realized by the inclusion $\mathcal{L}\mathcal{P}(L) \to \mathcal{L}^{\beta}\mathcal{P}(L)$.*

4. An application: the Heyting case

4.1. It may seem that the criterion in 3.4.(6) is of little use, but it can at least help with the Heyting case (cf [7]).

Proposition. *Let L be a Heyting or a co-Heyting lattice. Then its Dedekind-MacNeille completion $\delta(L)$ is distributive.*

Proof. Let $c < d$ and $A \le B$ be such that $A \subseteq c \downarrow d$ and $B \subseteq c \uparrow d$. Then, for each $a \in A$, the fact that $a \wedge d \le c$ means that $a \le d \to c$, which is to say that $d \to c \in \mathfrak{u}(A)$. If we had $\mathfrak{u}(A) \subseteq B$ then $d \to c \in c \uparrow d$, i.e., $c \vee (d \to c) \ge d$. Since $c \le d \to c$ in any Heyting algebra, this yields $d \le d \to c$ and finally $d \le c$. The argument in the co-Heyting case is similar. □

4.2. A topological proof. We will finish by showing how the statement of 4.1 follows from the Priestley representation in view of 2.4 and 3.6.

First let us recall the well known characterization of the Priestley spaces dual to Heyting algebras (sometimes called H-spaces).

A Priestley space X is dual to a Heyting algebra iff for each open $U \subseteq X$ the set $\uparrow U$ is open.

Thus we obtain an

4.2.1. Observation. *In an H-space*

$$i_\uparrow(U) = i(U).$$

for every up-set U.

Proof. We have

$$\begin{aligned} i(U) &= \bigcup\{V \mid U \supseteq V \in \tau\} = \bigcup\{\uparrow V \mid U \supseteq V \in \tau\} \\ &\subseteq \bigcup\{W \mid U \supseteq W \in \tau_\uparrow\} = i_\uparrow(U) \subseteq i(U). \end{aligned}$$

□

Now we can prove the fact that

the completion $\delta(L)$ of a Heyting algebra L coincides with $\beta(L)$

as follows:

By 2.4, cuts in L are represented as open up-sets $U \subseteq \mathcal{P}(L)$ such that $U = i_\uparrow c_\downarrow(U)$. Hence, by 4.2.1, in an H-space they are represented as the open down-sets U with $U = ic_\downarrow(U)$. Now by 3.6, a (representation of a) cut U is *exact* if it is dense in $i(c_\downarrow(U))$, that is, if $c(U) \supseteq i(c_\downarrow(U))$. Now if U is any cut, $i(U) = i(i_\uparrow c_\downarrow(U)) = i(ic_\downarrow(U)) = i(c_\downarrow(U)) = i_\uparrow(c_\downarrow(U)) = U$.

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