On Ramsey properties of classes with forbidden trees

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Abstract

Let $\mathscr{F}$ be a set of relational trees and let $\mathrm{Forb_h}(\mathscr{F})$ be the class of all structures that admit no homomorphism from any tree in $\mathscr{F}$; all this happens over a fixed finite relational signature σ. There is a natural way to expand $\mathrm{Forb_h}(\mathscr{F})$ by unary relations to an amalgamation class. This expanded class, enhanced with a linear ordering, has the Ramsey property.

Keywords: forbidden substructure; amalgamation; Ramsey class; partite method
MSC2010: 05C55

1 Introduction

Ramsey's Theorem [16] states the following:

> *Given any r, n, and μ we can find an m_0 such that, if $m \geq m_0$ and the r-element subsets of any m-element set Γ are divided in any manner into μ mutually exclusive classes C_i ($i = 1, 2, \ldots, \mu$), then Γ must contain an n-element subset Δ such that all the r-element subsets of Δ belong to the same C_i.*

In this paper we study generalizations of Ramsey's Theorem in the context of the so-called *structural Ramsey theory*.

Relational structures. A *signature* σ is a set of relation symbols; each of the symbols has an associated *arity*; the arity of R is $\mathrm{ar}(R)$. A *σ-structure* A is a set of elements, called the *domain* of A, together with a relation R^A on the domain of arity $\mathrm{ar}(R)$ for every relation symbol $R \in \sigma$. An *ordered σ-structure* is a $(\sigma \cup \{\preceq\})$-structure A such that $\preceq^A$ is a linear ordering. A σ-structure A is a *substructure* of a σ-structure B if $\operatorname{dom} A \subseteq \operatorname{dom} B$ and for each k-ary $R \in \sigma$ we have $R^A = R^B \cap (\operatorname{dom} A)^k$. An *embedding* of A into B is a one-to-one mapping $f : \operatorname{dom} A \to \operatorname{dom} B$ such that for any $R \in \sigma$ and any tuple $\bar{x}$ we have $\bar{x} \in R^A$ iff $f(\bar{x}) \in R^B$, where f is applied on $\bar{x}$ component-wise. If $\sigma \subset \tau$, the *σ-reduct* of a τ-structure A is the σ-structure A^* obtained from A by leaving out all the relations R^A for $R \in \tau \setminus \sigma$. (In some literature a reduct is called a *shadow*.)

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Ramsey classes. For any structures A, B, let $\binom{B}{A}$ denote the set of all embeddings of A into B. The partition arrow $C \to (B)^A_r$ means that whenever $\binom{C}{A} = \mathscr{E}_1 \cup \mathscr{E}_2 \cup \cdots \cup \mathscr{E}_r$ (a *colouring* with r colours), then there exists $g \in \binom{C}{B}$ and $j \le r$ such that $\binom{g[B]}{A} \subseteq \mathscr{E}_j$. In this case we call g (or $g[B]$) a *monochromatic copy* of B in C.

Let $\mathscr{C}$ be a class of finite structures and let $A \in \mathscr{C}$. The class $\mathscr{C}$ has the A-*Ramsey property* if for any $B \in \mathscr{C}$ and any natural number r there exists $C \in \mathscr{C}$ such that $C \to (B)^A_r$. The class $\mathscr{C}$ is called a *Ramsey class* if it has the A-Ramsey property for all $A \in \mathscr{C}$.

The most notable result about Ramsey classes is most likely the following:

Theorem 1.1 (Nešetřil–Rödl [9])**.** *Let σ be a finite relational signature. Then the class of all finite ordered σ-structures is a Ramsey class.*

The presence of orderings is indeed essential; cf. the discussion in [6].

Classes with forbidden homomorphic images. Let A, B be σ-structures. A *homomorphism* of A to B is a mapping $f : \operatorname{dom} A \to \operatorname{dom} B$ such that for any $R \in \sigma$ and any $\bar{x} \in R^A$ we have $f(\bar{x}) \in R^B$.

The interest of this paper lies in classes of finite σ-structures that can be defined by forbidding the existence of a homomorphism from a given set of structures. More explicitly, for a set $\mathscr{F}$ of σ-structures let $\mathrm{Forb_h}(\mathscr{F})$ be the class of all finite σ-structures A such that whenever $F \in \mathscr{F}$, there exists no homomorphism of F to A. We also say that A *is $\mathscr{F}$-free.*

In general, such classes are not Ramsey classes. A Ramsey class of structures always has the *amalgamation property* (see [6]) but these classes will usually not possess it. Following Hubička–Nešetřil [4], however, there is a *canonical way* to add new relations to the signature σ in order to obtain the amalgamation property. Thus it is natural to ask whether this *expanded class*, enhanced with a linear ordering, is a Ramsey class.

Main result. It has recently been announced by Nešetřil [8] that the ordered expanded class is a Ramsey class if $\mathscr{F}$ is a **finite** set of finite connected σ-structures. Here a similar result is shown for **infinite** $\mathscr{F}$, but under the assumption that all its elements are (relational) **trees**. See next section for the definition of a relational tree.

Proof method. We use the *partite method* of Nešetřil and Rödl [10, 11, 13]. To prove the *partite lemma*, which is often proved by an application of the Hales–Jewett theorem (as in [12, 13, 14]), we apply induction. Our proof is inspired by one of Prömel and Voigt [15].

Conventions. 1. A tuple has a bar, so $\bar{x} = (x_1, x_2, \ldots, x_k)$ for some k. If M is the domain of some function f and $\bar{x} \in M^k$, then $f(\bar{x}) = (f(x_1), f(x_2), \ldots, f(x_k))$.

2. Instead of "substructure of X generated by M" I write "substructure of X induced by M" with the intended connotation that the domain of such a substructure is actually M.

3. For a $(\sigma \cup \tau)$-structure A, A^* almost always denotes the σ-reduct of A.

4. Usually $R \in \sigma$ and $S \in \tau$, but sometimes $R \in \sigma \cup \tau$.

2 Amalgamation and other constructions

Amalgamation. A class $\mathscr{C}$ of finite structures has the *joint-embedding property* if for any structures $A_1, A_2 \in \mathscr{C}$ there exists $B \in \mathscr{C}$ such that both A_1 and A_2 admit an embedding into B. A class $\mathscr{C}$ of finite structures has the *amalgamation property* if for any $A, B_1, B_2 \in \mathscr{C}$ and any embeddings $f_1 : A \to B_1$ and $f_2 : A \to B_2$ there exists $C \in \mathscr{C}$ and embeddings $g_1 : B_1 \to C$ and $g_2 : B_2 \to C$ such that $g_1 f_1 = g_2 f_2$. The amalgamation is *free* if $\operatorname{dom} C = g_1[\operatorname{dom} B_1] \cup g_2[\operatorname{dom} B_2]$ and $R^C = g_1[R^{B_1}] \cup g_2[R^{B_2}]$ for all $R \in \sigma$. If the latter is true only for $R \in \tau \subset \sigma$, the amalgamation is said to be *free with respect to* τ.

Let $\mathscr{F}$ be a possibly infinite set of finite connected σ-structures. The class $\mathrm{Forb_h}(\mathscr{F})$ is hereditary and closed under taking disjoint unions, hence it has the joint embedding property. We turn it into an amalgamation class by adding new relations.

Incidence graph. The *incidence graph* $\mathrm{Inc}(X)$ of a σ-structure X is the bipartite undirected multigraph whose vertex set is $\operatorname{dom} X \cup \bigcup\{R^X \times \{R\} : R \in \sigma\}$, and which contains for every $R \in \sigma$, every $\bar{x} \in R^X$, and every i, an edge joining $(\bar{x}, R)$ and x_i.

A σ-structure X is *connected* if $\mathrm{Inc}(X)$ is connected; X is a *tree* (or a σ*-tree*) if $\mathrm{Inc}(X)$ is a tree. (Thus in particular X is *not* a tree if some tuple of some relation of X contains the same element two or more times.)

Pieces. Without loss of generality let us assume that the domain of each $F \in \mathscr{F}$ is the set $\{1, 2, \ldots, |F|\}$. A *cut* of some $F \in \mathscr{F}$ is a set $C \subset \operatorname{dom} F$ such that $\mathrm{Inc}(F) \setminus C$ has at least two distinct connected components that contain vertices from $\operatorname{dom} F$; a *minimal cut* is a cut which is inclusion-minimal. Thus C is a (minimal) cut of a structure if and only if it is a (minimal) vertex cut of its Gaifman graph.

Let C be any minimal cut of F and let D be the vertex set of some connected component of $\mathrm{Inc}(F) \setminus C$ that contains a vertex from $\operatorname{dom} F$. A *piece* of F is $\mathfrak{M} = (M, (m_1, \ldots, m_k))$, where M is the substructure of F induced by $C \cup (D \cap \operatorname{dom} F)$ and $\{m_1, \ldots, m_k\} = C$ so that $m_1 < m_2 < \cdots < m_k$.

Remarks. 1. $\{m_1, \ldots, m_k\} = C$ is the set of all elements of M appearing in some tuple of F that is not a tuple of M.

2. A piece of F is a nonempty connected substructure of F, $M \neq F$, and $C \neq \operatorname{dom} M$.

3. For any given minimal cut, the corresponding pieces cover $\operatorname{dom} F$.

Expansion. Let τ contain a relation symbol $S_{\mathfrak{M}}$ for each piece $\mathfrak{M}$ of each $F \in \mathscr{F}$. Let $\tilde{\mathscr{C}}$ be the class of finite $(\sigma \cup \tau)$-structures such that A belongs to $\tilde{\mathscr{C}}$ if and only if the σ-reduct A^* of A is in $\mathrm{Forb_h}(\mathscr{F})$ and for any piece $\mathfrak{M} = (M, (m_1, \ldots, m_k))$ of some $F \in \mathscr{F}$ and any k-tuple $\bar{x} \in (\operatorname{dom} A)^k$ we have

$$\bar{x} \in S_{\mathfrak{M}}^A \qquad \Longleftrightarrow \qquad \exists f : M \to A^* \text{ with } f(m_i) = x_i \text{ for all } i. \tag{2.1}$$

Let $\mathscr{C}$ be the class of all substructures of the structures in $\tilde{\mathscr{C}}$. The class $\mathscr{C}$ is called the *expanded class* for $\mathrm{Forb_h}(\mathscr{F})$. The structures in $\tilde{\mathscr{C}}$ are called *canonical.* We can also say that *A is $\mathscr{F}$-free* if $A^* \in \mathrm{Forb_h}(\mathscr{F})$; so being $\mathscr{F}$-free is a necessary but not sufficient condition for membership in $\mathscr{C}$.

Theorem 2.1. *Let σ be a finite relational signature, let $\mathscr{F}$ be a set of finite connected σ-structures and let $\mathscr{C}$ be the expanded class for* $\mathrm{Forb_h}(\mathscr{F})$. *Then*
(1) the class of all σ-reducts of the structures in $\mathscr{C}$ is $\mathrm{Forb_h}(\mathscr{F})$*;*
(2) $\mathscr{C}$ is closed under isomorphism;
(3) $\mathscr{C}$ is closed under taking substructures;
(4) $\mathscr{C}$ has the amalgamation property (free with respect to σ).

This theorem was proved by Hubička and Nešetřil [4] for finite $\mathscr{F}$ but the proof for infinite $\mathscr{F}$ is analogous.

Remarks. 1. If all structures in $\mathscr{F}$ are *irreducible*, that is, any two elements lie in a common tuple, then there are no pieces because there are no cuts. Hence the theorem implies that the class $\mathrm{Forb_h}(\mathscr{F})$ has the amalgamation property (without any new relations).

2. If all structures in $\mathscr{F}$ are trees, then every minimal cut has size one. Thus all the relations in τ are unary. Every piece of a tree is a tree. Moreover, $\{x\}$ is a minimal cut of F if and only if x is an element of F that belongs to more than one tuple of the relations of F.

3. If all relations in τ are unary, then $\mathscr{C}$ has free amalgamation.

4. Every structure in $\mathscr{C}$ satisfies the right-to-left implication in (2.1).

5. If $\mathfrak{M} = (M, (m_1, \ldots, m_k))$ is a piece such that there is a homomorphism to M from some $F' \in \mathscr{F}$, them $S^A_{\mathfrak{M}} = \emptyset$ for any $A \in \mathscr{C}$.

Sum. For two σ-structures A, B, their *sum* $A+B$ is defined by

$$\operatorname{dom}(A+B) = (\{A\} \times \operatorname{dom} A) \cup (\{B\} \times \operatorname{dom} B),$$
$$R^{A+B} = (\{A\} \otimes R^A) \cup (\{B\} \otimes R^B),$$

where

$$\{X\} \otimes R^X = \big\{\big((X, x_1), (X, x_2), \ldots, (X, x_k)\big) \colon (x_1, x_2, \ldots, x_k) \in R^X\big\}.$$

The definition can be extended to arbitrary finite sums in the obvious way. We may also write $\coprod\{A_1, A_2, \ldots, A_k\}$ for $A_1 + A_2 + \cdots + A_k$. If all elements of $\mathscr{F}$ are connected, as we assume throughout this paper, then both $\mathrm{Forb_h}(\mathscr{F})$ and the expanded class $\mathscr{C}$ are closed under taking sums.

Factor structure. If A is a σ-structure and $\sim$ is an equivalence relation on $\operatorname{dom} A$, let the *factor structure* $A/{\sim}$ be defined on $\operatorname{dom} A/{\sim} = (\operatorname{dom} A)/{\sim}$ (the set of all equivalence classes of $\sim$) by letting $(X_1, X_2, \ldots, X_k) \in R^{A/\sim}$ if and only if there exist $x_1 \in X_1$, $x_2 \in X_2$, $\ldots$, $x_k \in X_k$ such that $(x_1, x_2, \ldots, x_k) \in R^A$.

Remark. If all structures in $\mathscr{F}$ are trees, then amalgamation in Theorem 2.1 can be proved by taking the factor structure $(B_1 + B_2)/{\sim}$, where $\sim$ is the minimal equivalence relation such that $(B_1, f_1(a)) \sim (B_2, f_2(a))$ for all $a \in \operatorname{dom} A$, with the obvious embeddings g_1, g_2.

Canonizing. Suppose $\mathscr{F}$ is a set of trees, and let $\mathscr{C}$ be the expanded class for $\mathrm{Forb_h}(\mathscr{F})$. Given a $(\sigma \cup \tau)$-structure A, we want to find a superstructure $\tilde{A}$ of A that satisfies the left-to-right implication of (2.1). This is possible assuming that

$$\textit{every one-element substructure of } A \textit{ is in } \mathscr{C}. \tag{2.2}$$

For every $x \in \operatorname{dom} A$, let A_x be the substructure of A induced by $\{x\}$. By assumption, for every x we have $A_x \in \mathscr{C}$; so there exists $\tilde{A}_x \in \tilde{\mathscr{C}}$ containing A_x. Let

$$A' = A + \coprod\{\tilde{A}_x \colon x \in \operatorname{dom} A\}$$

and let $\sim$ be the smallest equivalence relation on $\operatorname{dom} A'$ such that $(A, x) \sim (\tilde{A}_x, x)$ for all$x \in \operatorname{dom} A$. Let $\tilde{A} = A/\!\sim$.

By convention, we will still use x to denote the element $[(A, x)]_\sim$ of $\tilde{A}$.

Whenever $x \in S^{\tilde{A}}_{\mathfrak{M}}$, then there exists $f : M \to \tilde{A}_x$ such that $f(m) = x$, because $\tilde{A}_x \in \tilde{\mathscr{C}}$. Hence $\tilde{A}$ satisfies the left-to-right implication of (2.1). Moreover, every one-element substructure of $\tilde{A}$ is isomorphic to a substructure of some $\tilde{A}_x$, and so in $\mathscr{C}$.

Proving membership in $\mathscr{C}$. A *tuple trace* of some $(x_1, x_2, \dots, x_k) \in R^A$ is the structure T with $\operatorname{dom} T = \{1, 2, \dots, k\}$; $R^T = \{(1, 2, \dots, k)\}$; $\check{R}^T = \{j \colon x_j \in \check{R}^A\}$ for all unary $\check{R} \in \sigma$; $R'^T = \emptyset$ for any other $R' \in \sigma \setminus \{R\}$; $S^T = \{j \colon x_j \in S^A\}$ for $S \in \sigma$.

Lemma 2.2. *Suppose $\mathscr{F}$ is a set of finite σ-trees; let $\mathscr{C}$ be the expanded class for* $\mathrm{Forb_h}(\mathscr{F})$*. Let X be a $(\sigma \cup \tau)$-structure. Then $X \in \mathscr{C}$ if and only if each one-element substructure of X belongs to $\mathscr{C}$, and for any $R \in \sigma$ and any $\bar{x} \in R^X$, the tuple trace of $\bar{x}$ belongs to $\mathscr{C}$.*

Proof. By assumption, X satisfies (2.2); apply the canonizing procedure on X to get $\tilde{X}$. We have observed that $\tilde{X}$ satisfies the left-to-right implication of (2.1). Now we shall show that it also satisfies the right-to-left implication.

Let $\tilde{X}^*$ be the σ-reduct of $\tilde{X}$, let $\mathfrak{M} = (M, (m))$ be a piece of some $F \in \mathscr{F}$ and consider any homomorphism $f : M \to \tilde{X}^*$ such that $f(m) \in \operatorname{dom} X$. We want to show that $f(m) \in S^{\tilde{X}}_{\mathfrak{M}}$. For the sake of contradiction, assume that $f(m) \notin S^{\tilde{X}}_{\mathfrak{M}}$ and that $\mathfrak{M}$ is a minimal such piece, that is, we assume that whenever $N \subset M$ and $\mathfrak{N} = (N, (n))$ is a piece of F, then $f'(n) \in S^{\tilde{X}}_{\mathfrak{N}}$ for any homomorphism $f' : N \to \tilde{X}^*$. Because $\{m\}$ is a cut of the tree F, m belongs to a unique tuple $\bar{x}$ of M, $\bar{x} \in R^M$ for some $R \in \sigma$; $m = x_j$; $f(\bar{x}) \in R^X$. As M has more than one tuple, $\bar{x}$ contains at least one element $n \neq m$ such that $\{n\}$ is a minimal cut of F. Let $\mathfrak{N}_1 = (N_1, (n_1))$, $\mathfrak{N}_2 = (N_2, (n_2))$, ..., $\mathfrak{N}_\ell = (N_\ell, (n_\ell))$ be all the pieces of F corresponding to all minimal cuts $\{n_k\}$ such that $n_k = x_i$ for some $i \neq j$, and $m \notin \operatorname{dom} N_k$. Notice that each $N_k \subset M$; thus by minimality of the counterexample $f(n_k) \in S^{\tilde{X}}_{\mathfrak{N}_k}$ for each $k = 1, \dots, \ell$. But then the tuple trace of $f(\bar{x}) \in R^X$ is not in $\mathscr{C}$, a contradiction.

Next we show that $\tilde{X}$ is $\mathscr{F}$-free. Suppose there is some $F \in \mathscr{F}$ and a homomorphism $f : F \to \tilde{X}^*$. Then the image of f contains elements of X. If F has only one element, then the one-element substructure $f[F]$ of X is not in $\mathscr{C}$. If F has more than one element but it is irreducible (that is, if it contains exactly one tuple of a relation of arity more than one), then the tuple trace of $f[F]$ is not in $\mathscr{C}$, a contradiction. Hence there is a cut $\{m\}$ of F such that $f(m) \in \operatorname{dom} X$. Also, for any piece $\mathfrak{N} = (N, \{m\})$ of F the restriction $g = f \restriction N$ is a homomorphism $N \to \tilde{X}^*$ such that $g(m) = f(m)$. Thus $f(m) \in S^X_{\mathfrak{N}}$ for any such piece $\mathfrak{N}$. But then the 1-element substructure of X induced by $\{f(m)\}$ is a substructure of no canonical structure, hence it is not in $\mathscr{C}$: again a contradiction. We conclude that $\tilde{X}^* \in \operatorname{Forb_h}(\mathscr{F})$.

Therefore $\tilde{X} \in \tilde{\mathscr{C}}$, and so $X \in \mathscr{C}$.

The converse implication: If $X \in \mathscr{C}$, then each substructure of X is in $\mathscr{C}$ as well. Let T be the tuple trace of some $(x_1, \ldots, x_k) \in R^X$. Let A_i be the substructure of X induced by $\{x_i\}$; $i = 1, \ldots, k$. Since $A_i \in \mathscr{C}$, there exists $\tilde{A}_i \in \tilde{\mathscr{C}}$ that contains A_i as a substructure. Let $T' = T + A_1 + \cdots + A_k$ and let $\sim$ be the minimal equivalence relation on $\operatorname{dom} T'$ such that $(T, i) \sim (A_i, x_i)$. Let $\tilde{T} = T'/\sim$. It is not difficult to show that $\tilde{T} \in \tilde{\mathscr{C}}$ and therefore $T \in \mathscr{C}$. □

Note that the "tuple trace" is a necessary complication due to the context of arbitrary relational structures. If σ were the signature of digraphs (one binary relation), we could simply test all one- and two-element substructures of X.

3 Partite lemma

Orderings. An *ordered υ-structure* is a $(\upsilon \cup \{\preceq\})$-structure A such that the relation $\preceq^A$ is a linear ordering.

Definition 3.1. Let σ be a finite relational signature and let $\mathscr{F}$ be a set of finite connected σ-structures. The *ordered expanded class* for $\operatorname{Forb_h}(\mathscr{F})$ is the class $\vec{\mathscr{C}}$ of ordered $(\sigma \cup \tau)$-structures such that $A \in \vec{\mathscr{C}}$ if and only if $\preceq^A$ is a linear ordering and the $(\sigma \cup \tau)$-reduct of A is in the expanded class $\mathscr{C}$ for $\operatorname{Forb_h}(\mathscr{F})$.

Rectified structures. Let $A \in \vec{\mathscr{C}}$. An *A-rectified structure* is a pair (X, ι_X) such that $X \in \vec{\mathscr{C}}$, $\iota_X : \operatorname{dom} X \to \operatorname{dom} A$, $x \preceq^X x'$ implies that $\iota_X(x) \preceq^A \iota_X(x')$, and for any $R \in \sigma \cup \tau$ and any $\bar{x} \in (\operatorname{dom} X)^{\operatorname{ar}(R)}$ we have

$$\bar{x} \in R^X \qquad \Longleftrightarrow \qquad \iota_X \text{ is injective on } \bar{x} \text{ and } \iota_X(\bar{x}) \in R^A. \tag{3.1}$$

Observe that X is uniquely determined by A, $\operatorname{dom} X$ and ι_X via (3.1).

A mapping $e : \operatorname{dom} X \to \operatorname{dom} Y$ is an embedding of A-rectified structure (X, ι_X) into (Y, ι_Y) if $e : X \to Y$ is an embedding of $(\sigma \cup \tau \cup \{\preceq\})$-structures and $\iota_X = \iota_Y e$.

Note. (A, id_A) is always A-rectified; and for any A-rectified (X, ι_X), any mapping $e : \operatorname{dom} A \to \operatorname{dom} X$ such that $\iota_X e = \operatorname{id}_A$ is an embedding of A into X, as well as an embedding of (A, id_A) into (X, ι_X).

Lemma 3.2. *Let $\mathscr{F}$ be a set of finite connected σ-structures and let $\vec{\mathscr{C}}$ be the ordered expanded class for* $\operatorname{Forb_h}(\mathscr{F})$*; let $A \in \vec{\mathscr{C}}$. Let (B, ι_B) be A-rectified, $r \geq 1$. Then there exists A-rectified (E, ι_E) such that $(E, \iota_E) \to (B, \iota_B)^{(A, \operatorname{id}_A)}_r$.*

Proof. By induction on $|A|$. If $|A|=1$, take E to be the sum (disjoint union) of $r\cdot(|B|-1)+1$ copies of A with an arbitrary linear ordering $\preceq^E$; ι_E is constant.

If $|A|\geq 2$, assume that $\operatorname{dom} A=\{0,1,\ldots,n\}$. Let A' be the substructure of A induced by the subset $\{1,\ldots,n\}$; let B' be the substructure of B induced by $\iota_B^{-1}[\{1,\ldots,n\}]$, and $\iota_{B'}=\iota_B\restriction\operatorname{dom} B'$. Then $(B',\iota_{B'})$ is A'-rectified. Apply induction to get A'-rectified $(E',\iota_{E'})$ such that $(E',\iota_{E'})\to(B',\iota_{B'})^{(A',\iota_{A'})}_{r^k}$, where $k=r\cdot\big(|\iota_B^{-1}(0)|-1\big)+1$. Assuming that $\operatorname{dom} E'\cap\{1,2,\ldots,k\}=\emptyset$ let $\operatorname{dom} E=\operatorname{dom} E'\cup\{1,2,\ldots,k\}$ and define $\iota_E(x)=0$ if $x\in\{1,2,\ldots,k\}$ and $\iota_E(x)=\iota_{E'}(x)$ otherwise. Let all $(\sigma\cup\tau)$-relations of E be defined by (3.1); let $\preceq^E$ be an extension of $\preceq^{E'}$ that is preserved by ι_E. Thus E' is the substructure of (E,ι_E) on $\iota_E^{-1}[\{1,\ldots,n\}]$.

Next, to prove that $(E,\iota_E)\to(B,\iota_B)^{(A,\mathrm{id}_A)}_r$, consider any r-colouring χ of $\binom{(E,\iota_E)}{(A,\mathrm{id}_A)}$. Define $\chi':\binom{(E',\iota_{E'})}{(A',\iota_{A'})}\to\{1,\ldots,r\}^{\iota_E^{-1}(0)}$ by $\chi'(e')=\big(c\mapsto\chi(e'\cup(0\mapsto c))\big)$, that is, the vector of colours of all extensions of $e'\in\binom{(E',\iota_{E'})}{(A',\mathrm{id}_{A'})}$ to some $e\in\binom{(E,\iota_E)}{(A,\mathrm{id}_A)}$. By the definition of $(E',\iota_{E'})$, there is a monochromatic $g'\in\binom{(E',\iota_{E'})}{(B',\iota_{B'})}$. Hence for any fixed $c\in\iota_E^{-1}(0)$, the mapping $\varphi_c: h'\mapsto\chi((g'h')\cup(0\mapsto c))$ is constant on $\binom{(B',\iota_{B'})}{(A',\mathrm{id}_{A'})}$. Define $\psi:\iota_E^{-1}(0)\to\{1,\ldots,r\}$ by setting $\psi(c)$ to be the constant value of φ_c. Since $|\iota_E^{-1}(0)|=k>r\left(|\iota_B^{-1}(0)|-1\right)$, there exists a subset $M\subseteq\iota_E^{-1}(0)$ with $|M|=|\iota_B^{-1}(0)|$ such that ψ is constant on M. Define $g\in\binom{(E,\iota_E)}{(B,\iota_B)}$ to be an extension of g' by the $\preceq$-preserving bijection of $\iota_B^{-1}(0)$ and M. Then g is monochromatic.

Finally, to show that (E,ι_E) is A-rectified we need only to check that $E\in\vec{\mathscr{C}}$. First, the σ-reduct E^* of E is $\mathscr{F}$-free, for if there were a homomorphism $f:F\to E^*$ of some $F\in\mathscr{F}$, then $\iota_E f$ would be a homomorphism $F\to A^*$ – but A is $\mathscr{F}$-free. Moreover, because $A\in\vec{\mathscr{C}}$, A is a substructure of a canonical $\tilde{A}$. Let $\operatorname{dom}\tilde{E}=\operatorname{dom} E\cup(\operatorname{dom}\tilde{A}\setminus\operatorname{dom} A)$ (assuming $\operatorname{dom} E$ and $\operatorname{dom}\tilde{A}$ are disjoint) and let the relations of $\tilde{E}$ be defined by (3.1), with $\iota_{\tilde{E}}=\iota_E\cup\mathrm{id}_{\operatorname{dom}\tilde{E}\setminus\operatorname{dom} E}$. Clearly $\tilde{E}$ is canonical and E is a substructure of $\tilde{E}$. ☐

4 Main result

Recall Definition 3.1 of the ordered expanded class for $\mathrm{Forb_h}(\mathscr{F})$.

Theorem 4.1. *Let σ be a finite relational signature and let $\mathscr{F}$ be a set of finite σ-trees. Then the ordered expanded class for* $\mathrm{Forb_h}(\mathscr{F})$ *has the Ramsey property.*

The remainder of this section is devoted to the proof of this theorem.

Partite structures. Let P be an ordered σ-structure and let $\vec{\mathscr{C}}$ be the ordered expanded class for $\mathrm{Forb_h}(\mathscr{F})$. A *$P$-partite $\vec{\mathscr{C}}$-structure* is a pair (A,ι_A) where $A\in\vec{\mathscr{C}}$ and $\iota_A:\operatorname{dom} A\to\operatorname{dom} P$ is a homomorphism of the $(\sigma\cup\{\preceq\})$-reduct A^* of A to P that is injective on any tuple of the relation R^A for any $R\in\sigma$, and such that the restriction of ι_A to any one-element substructure of A^* is an embedding of this one-element $(\sigma\cup\{\preceq\})$-structure into P. A P-partite $\vec{\mathscr{C}}$-structure (A,ι_A) is *transversal* if ι_A is an embedding of A^* to P.

A mapping $e:\operatorname{dom} A\to\operatorname{dom} B$ is an embedding of a P-partite $\vec{\mathscr{C}}$-structure (A,ι_A) into (B,ι_B) if $e:A\to B$ is an embedding of $(\sigma\cup\tau\cup\{\preceq\})$-structures and $\iota_A=\iota_B e$.

Lemma 4.2 ("rectification")**.** *Let $\vec{\mathscr{C}}$ be the ordered expanded class for* $\mathrm{Forb}_{\mathrm{h}}(\mathscr{F})$*, where $\mathscr{F}$ is a set of finite σ-trees. Let (C,ι_C) be a P-partite $\vec{\mathscr{C}}$-structure for some σ-structure P. If (D,ι_D) is defined by setting*

$$\begin{aligned}
&\operatorname{dom} D = \operatorname{dom} C,\\
&\iota_D = \iota_C,\\
&S^D = S^C \text{ for } S \in \tau,\\
&\preceq^D = \preceq^C,\\
&\text{for } R \in \sigma,\ \bar{x} \in R^D \iff \iota_D \text{ is injective on } \bar{x}, \text{ and}\\
&\qquad\qquad\qquad\qquad \exists \bar{y} \in R^C : \iota_C(\bar{y}) = \iota_D(\bar{x}) \text{ and } \forall i, \forall S \in \tau : x_i \in S^D \Leftrightarrow y_i \in S^C,
\end{aligned} \tag{4.1}$$

then (D,ι_D) is a P-partite $\vec{\mathscr{C}}$-structure.

Proof. It is straightforward that ι_D is a homomorphism of the reduct D^* to P because ι_C is a homomorphism of C^* to P. By definition, ι_D is injective on any tuple of any σ-relation of D, and every one-element substructure of D is isomorphic to the corresponding one-element substructure of C.

To show that $D \in \vec{\mathscr{C}}$, first apply the "only if" direction of Lemma 2.2 to prove that the tuple trace of any $\bar{y} \in R^D$ is in $\mathscr{C}$. Then observe that the tuple trace of any $\bar{y} \in R^D$ is equal to the tuple trace of some $\bar{x} \in R^C$. Also, any one-element substructure of D is isomorphic to some one-element substructure of C. Finally apply the "if" direction of Lemma 2.2. □

Observe that the P-partite $\mathscr{C}$-structure (D,ι_D) from Lemma 4.2 is *rectified* in the following sense:

$$\begin{aligned}
&\text{For any } R \in \sigma \text{ and any } \bar{y} \in R^D, \text{ if } \bar{x} \text{ is a tuple such that } \iota_D(\bar{x}) = \iota_D(\bar{y}),\\
&\qquad \iota_D \text{ is injective on } \bar{x}, \text{ and } y_i \in S^D \Leftrightarrow x_i \in S^D \text{ for any } i \text{ and any } S \in \tau, \text{ then } \bar{x} \in R^D.
\end{aligned} \tag{4.2}$$

Note that if (C,ι_C) satisfies (4.2) and (D,ι_D) is defined by (4.1), then $(D,\iota_D) = (C,\iota_C)$. An important special case: if (C,ι_C) is transversal.

Lemma 4.3. *Let (D,ι_D) be a P-partite $\vec{\mathscr{C}}$-structure satisfying* (4.2)*, and let (A,ι_A) be a transversal P-partite $\vec{\mathscr{C}}$-structure. Suppose there is an embedding of (A,ι_A) into (D,ι_D). Define*

$$\operatorname{dom} B = \left\{ x \in \operatorname{dom} D \colon \iota_D(x) \in \iota_A[\operatorname{dom} A] \text{ and for any } S \in \tau : x \in S^D \Leftrightarrow \iota_A^{-1}(\iota_D(x)) \in S^A \right\} \tag{4.3}$$

and let B be the substructure of D induced by $\operatorname{dom} B$*. Set $\iota_B = \iota_A^{-1}(\iota_D \restriction \operatorname{dom} B)$. Then (B,ι_B) is A-rectified.*

Proof. First, $B \in \vec{\mathscr{C}}$ because it is a substructure of $D \in \vec{\mathscr{C}}$. Since (D,ι_D) is P-partite, ι_D is injective on any tuple of any relation of B, and so is ι_B. Because there exists an embedding of (A,ι_A) into (D,ι_D), it follows from (4.2) that a mapping $e : \operatorname{dom} A \to \operatorname{dom} D$ such that $\iota_A = \iota_D e$ is an embedding of (A,ι_A) into (D,ι_D) if and only if for any $a \in \operatorname{dom} A$ and any $S \in \tau$ we have $a \in S^A \Leftrightarrow e(a) \in S^D$. Therefore (B,ι_B) satisfies (3.1). □

Proof of Theorem 4.1. Let $\mathscr{F}$ be a set of finite σ-trees and let $\mathscr{C}$ be the expanded class and $\vec{\mathscr{C}}$ the ordered expanded class for $\mathrm{Forb_h}(\mathscr{F})$. Consider $A, B \in \vec{\mathscr{C}}$ and a positive integer r. We construct $C \in \vec{\mathscr{C}}$ such that $C \to (B)^A_r$.

Let A^*, B^* be the $(\sigma \cup \{\preceq\})$-reducts of A, B, respectively. By Theorem1.1 there exists an ordered σ-structure P such that $P \to (B^*)^{A^*}_r$. Define (C_0, ι_{C_0}) by

$\operatorname{dom} C_0 = \binom{P}{B^*} \times \operatorname{dom} B$,

for any k-ary $R \in \sigma \cup \tau$:

$$R^{C_0} = \left\{((f,x_1),(f,x_2),\ldots,(f,x_k)) : f \in \tbinom{P}{B^*} \text{ and } (x_1,x_2,\ldots,x_k) \in R^B\right\},$$

$\iota_{C_0} : \operatorname{dom} C_0 \to \operatorname{dom} P$ is defined by $\iota_{C_0} : (f,x) \mapsto f(x)$,

$\preceq^{C_0}$ is any linear ordering that is preserved by ι_{C_0}.

Thus C_0 is isomorphic to a sum of structures, and each of the summands is isomorphic to B. See Figure 1. Observe that (C_0, ι_{C_0}) is a P-partite $\vec{\mathscr{C}}$-structure.

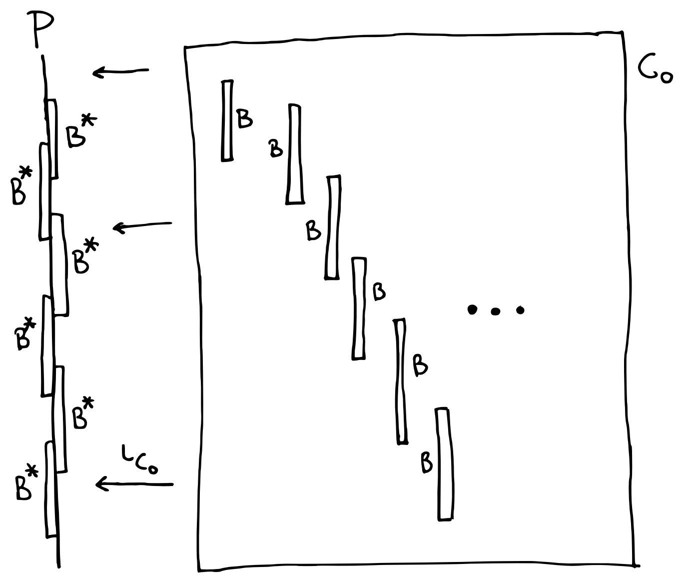

Figure 1: C_0.

If (D_0, ι_{D_0}) is obtained from (C_0, ι_{C_0}) by (4.1), then each of the basic embeddings $x \mapsto (f,x)$ of B to C_0 is also an embedding of B to D_0.

Fix some numbering of $\binom{P}{A^*} = \{e_1, \ldots, e_N\}$. We will inductively construct P-partite $\vec{\mathscr{C}}$-structures (C_1, ι_{C_1}), …, (C_N, ι_{C_N}).

Let $k \in \{1, \ldots, N\}$ and suppose $(C_{k-1}, \iota_{C_{k-1}})$ has been constructed. If there is no P-partite embedding of (A, e_k) into $(C_{k-1}, \iota_{C_{k-1}})$, let $(C_k, \iota_k) = (C_{k-1}, \iota_{C_{k-1}})$. Otherwise let $(D_{k-1}, \iota_{D_{k-1}})$ be defined from $(C_{k-1}, \iota_{C_{k-1}})$ by (4.1). Let (B_k, ι_{B_k}) be obtained from $(D_{k-1}, \iota_{D_{k-1}})$ as in Lemma 4.3, using (A, e_k) in place of (A, ι_A). Then (B_k, ι_{B_k}) is A-rectified and we can apply the Partite Lemma,

Lemma 3.2, in order to get A-rectified (E_k, ι_{E_k}) such that $(E_k, \iota_{E_k}) \to (B_k, \iota_{B_k})^{(A,\mathrm{id}_A)}_r$ (w.r.t. embeddings of A-rectified structures). Therefore $(E_k, e_k\iota_{E_k}) \to (B_k, e_k\iota_{B_k})^{(A,e_k)}_r$ (w.r.t. embeddings of P-partite structures). Set

$$\operatorname{dom} C_k = \operatorname{dom} E_k \cup \left(\binom{(E_k,\iota_{E_k})}{(B_k,\iota_{B_k})} \times (\operatorname{dom} D_{k-1} \setminus \operatorname{dom} B_k) \right).$$

Define $\lambda_k : \binom{(E_k,\iota_{E_k})}{(B_k,\iota_{B_k})} \times \operatorname{dom} D_{k-1} \to \operatorname{dom} C_k$ by

$$\lambda_k : (g,x) \mapsto \begin{cases} g(x) & \text{if } x \in \operatorname{dom} B_k, \\ (g,x) & \text{otherwise.} \end{cases}$$

For any ℓ-ary $R \in \sigma \cup \tau$, let

$$R^{C_k} = \left\{ \left(\lambda_k(g,x_1), \dots, \lambda_k(g,x_\ell)\right) : g \in \binom{(E_k,\iota_{E_k})}{(B_k,\iota_{B_k})},\ (x_1,\dots,x_\ell) \in R^{D_{k-1}} \right\}.$$

Furthermore define $\iota_{C_k} : \operatorname{dom} C_k \to \operatorname{dom} P$ by

$$\iota_{C_k} : y \mapsto e_k\iota_{E_k}(y) \quad \text{if } y \in \operatorname{dom} E_k,$$
$$\iota_{C_k} : (g,x) \mapsto \iota_{D_{k-1}}(x) \quad \text{otherwise.}$$

Finally, let $\preceq^{C_k}$ be a linear ordering such that $y \preceq^{C_k} y'$ if $y \preceq^{E_k} y'$, $\lambda_k(g,x) \preceq^{C_k} \lambda_k(g,x')$ if $x \preceq^{D_{k-1}} x'$, and $z \preceq^{C_k} z'$ if $\iota_{C_k}(z) \preceq^P \iota_{C_k}(z')$. See Figure 2.

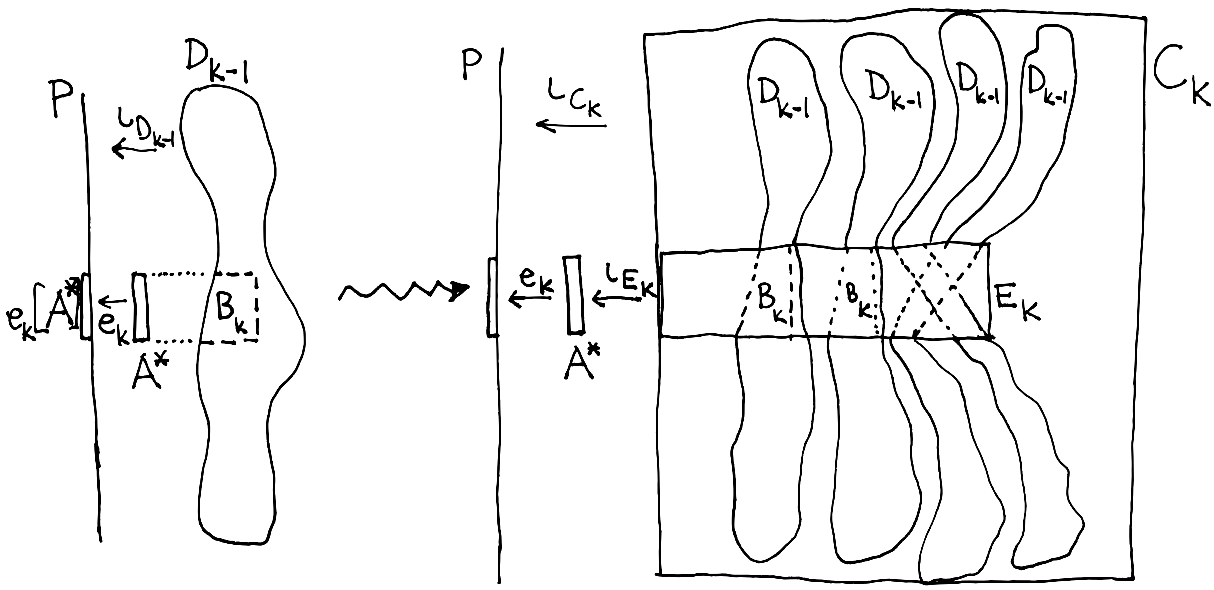

Figure 2: C_k.

Note that for a fixed g, the mapping $\lambda_k(g,-) : x \mapsto \lambda_k(g,x)$ is an embedding of $(D_{k-1}, \iota_{D_{k-1}})$ to (C_k, ι_{C_k}). By definition of D_{k-1}, $\lambda_k(g,-)$ is an injective homomorphism of $(C_{k-1}, \iota_{C_{k-1}})$ to (C_k, ι_{C_k}). The inclusion mapping is an embedding of E_k to C_k because (E_k, ι_{E_k}) is A-rectified.

Now we claim that (C_k, ι_{C_k}) is a P-partite $\vec{\mathscr{C}}$-structure. First, for $R \in \sigma \cup \tau$, if $\bar{x} \in R^{C_k}$, then $\bar{x} = \lambda_k(g, \bar{y})$ for some $(g, \bar{y})$. Since $\iota_{D_{k-1}}$ is injective on $\bar{y}$ and preserves it if $R \in \sigma$, ι_{C_k} is injective on $\bar{x}$ and preserves it if $R \in \sigma$. Next, $\preceq^{C_k}$ is preserved by ι_{C_k} by definition. The tuple trace of any tuple of any relation of C_k is the tuple trace of some tuple of the corresponding relation of D_{k-1}, hence in $\vec{\mathscr{C}}$. By Lemma 2.2, $C_k \in \vec{\mathscr{C}}$.

Let $C = C_N$. We show that $C \to (B)^A_r$. Consider any colouring $\chi : \binom{C}{A} \to \{1,\dots,r\}$. By downward induction we exhibit injective homomorphisms $h_i : (C_{i-1}, \iota_{C_{i-1}}) \to (C_i, \iota_{C_i})$ for $i = N, N-1, \dots, 1$ that have certain monochromatic properties.

Suppose h_i is known for $i = N, \dots, k+1$ (possibly for no i yet). If $(C_k, \iota_{C_k}) = (C_{k-1}, \iota_{C_{k-1}})$, let h_k be the identity mapping. Otherwise define the colouring $\chi_k : \binom{(E_k,\iota_{E_k})}{(A,\mathrm{id}_A)} \to \{1,\dots,r\}$ by setting $\chi_k(q) = \chi(h_N h_{N-1} \cdots h_{k+1} q)$. (Observe that the composed mapping is indeed an embedding.) Since $(E_k, \iota_{E_k}) \to (B_k, \iota_{B_k})^{(A,\mathrm{id}_A)}_r$, there exists a χ_k-monochromatic embedding $g_k : (B_k, \iota_{B_k}) \to (E_k, \iota_{E_k})$. Let $h_k = \lambda(g_k, -)$.

Let $h = h_N h_{N-1} \cdots h_1 : (C_0, \iota_{C_0}) \to (C_N, \iota_{C_N})$. Consider any $e_j \in \binom{P}{A^*}$. Any embedding d of A to C_0 such that $\iota_{C_0} d = e_j$ is also a P-partite embedding of (A, e_j) to (C_0, ι_{C_0}). Moreover, hd is a P-partite embedding of (A, e_j) to (C_N, ι_{C_N}). By definition of h_j, all such embeddings take the same colour under χ. Thus we define $\chi_0 : \binom{P}{A^*} \to \{1,\dots,r\}$ by $\chi_0(e_j) = \chi(hd)$ if there exists $d \in \binom{C_0}{A}$ such that $\iota_{C_0} d = e_j$, and arbitrarily otherwise. By definition of P there exists χ_0-monochromatic $f \in \binom{P}{B^*}$. Let $c : B \to C_0$ be the embedding given by $c : x \mapsto (f, x)$.

Conclude the proof by observing that hc is a χ-monochromatic embedding of B to C: It is an embedding because h is a composition of embeddings of $(D_{k-1}, \iota_{D_{k-1}})$ to (D_k, ι_{D_k}) and the copy of B given by $h_k h_{k-1} \cdots h_1 c[B]$ remains intact during the "rectification" – application of Lemma 4.2. □

5 Comments

Universal structures. If $\mathscr{F}$ is a set of finite connected σ-structures, then the expanded class for $\mathrm{Forb_h}(\mathscr{F})$ has a Fraïssé limit U. The σ-reduct U^* of U is a universal structure for $\mathrm{Forb_h}(\mathscr{F})$. For finite $\mathscr{F}$ this universal structure is ω-categorical; the existence of such a universal ω-categorical structure (and much more) was proved by Cherlin, Shelah and Shi [3]. If $\mathscr{F}$ is infinite, U^* is no longer necessarily ω-categorical; however, it is model-complete.

Extreme amenability. By a theorem of Kechris, Pestov and Todorčević [5], the automorphism group of a Ramsey structure is extremely amenable. Thus Theorem 4.1 provides a continuum of examples of structures with an extremely amenable automorphism group: take $\mathscr{F}'$ to be an infinite antichain of σ-trees; then the Fraïssé limit of the expanded class for $\mathrm{Forb_h}(\mathscr{F})$ provides such an example for any subset $\mathscr{F}$ of $\mathscr{F}'$.

Problem. It would be interesting to classify all sets $\mathscr{F}$ of σ-structures for which the corresponding ordered expanded class for $\mathrm{Forb_h}(\mathscr{F})$ is a Ramsey class. In particular, is it the case for any set $\mathscr{F}$ of connected finite σ-structures? Some possible applications of such new results are hinted at in [1].

Limits of the partite method. Nešetřil [8] asked whether one can prove all Ramsey classes by a variant of the partite (amalgamation) construction. This is certainly a question worth considering. It is not very satisfactory that the definition of a partite structure is rather different each time: compare [2, 7, 10, 11, 12, 13, 14]. Also, the partite lemma is sometimes proved by induction (such as here and in [2, 15]), sometimes by an application of the Hales–Jewett theorem (such as in [12, 13, 14]).

Acknowledgements

I thank Manuel Bodirsky, Jarik Nešetřil and Claude Tardif for many stimulating discussions, intriguing questions and useful comments.

I am fully supported by a fellowship from the Swiss National Science Foundation. Part of this work was done at Laboratoire d'Informatique (LIX), CNRS UMR 7161, Ecole Polytechnique, Paris, where I was supported by ERC Starting Grant 'CSP-Complexity' no. 257039 of Manuel Bodirsky. Another part was done when I was visiting Fields Institute in Toronto within the Summer Thematic Program on the Mathematics of Constraint Satisfaction. Finally, a part was done at Coffee&Company in Kingston, Ontario. I also acknowledge the support of the Institute for Theoretical Computer Science (ITI), project 1M0545 of the Ministry of Education of the Czech Republic.

References

[1] A. Atserias and M. Weyer. Decidable relationships between consistency notions for constraint satisfaction problems. In E. Grädel and R. Kahle, editors, *Computer Science Logic*, volume 5771 of *Lecture Notes in Comput. Sci.*, pages 102–116, 2009.

[2] J. Böttcher and J. Foniok. Ramsey properties of permutations. arXiv:1103.5686v1, submitted.

[3] G. Cherlin, S. Shelah, and N. Shi. Universal graphs with forbidden subgraphs and algebraic closure. *Adv. in Appl. Math.*, 22(4):454–491, 1999.

[4] J. Hubička and J. Nešetřil. Homomorphism and embedding universal structures for restricted classes. arXiv:0909.4939, 2009.

[5] A. S. Kechris, V. G. Pestov, and S. Todorčević. Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups. *Geom. Funct. Anal.*, 15(1):106–189, 2005.

[6] J. Nešetřil. Ramsey classes and homogeneous structures. *Combin. Probab. Comput.*, 14(1–2):171–189, 2005.

[7] J. Nešetřil. Metric spaces are Ramsey. *European J. Combin.*, 28(1):457–468, 2007.

[8] J. Nešetřil. First order definable Ramsey classes. Workshop on Homogeneous Structures, July 2011. School of Mathematics, University of Leeds.

[9] J. Nešetřil and V. Rödl. Partitions of finite relational and set systems. *J. Combinatorial Theory Ser. A*, 22(3):289–312, 1977.

[10] J. Nešetřil and V. Rödl. Simple proof of the existence of restricted Ramsey graphs by means of a partite construction. *Combinatorica*, 1(2):199–202, 1981.

[11] J. Nešetřil and V. Rödl. Two proofs of the Ramsey property of the class of finite hypergraphs. *European J. Combin.*, 3(4):347–352, 1982.

[12] J. Nešetřil and V. Rödl. Strong Ramsey theorems for Steiner systems. *Trans. Amer. Math. Soc.*, 303(1):183–192, 1987.

[13] J. Nešetřil and V. Rödl. The partite construction and Ramsey set systems. *Discrete Math.*, 75(1–3):327–334, 1989.

[14] J. Nešetřil. Ramsey theory. In R. L. Graham, M. Grötschel, and L. Lovász, editors, *Handbook of Combinatorics*, volume 2, chapter 25, pages 1331–1403. Elsevier, 1995.

[15] H. J. Prömel and B. Voigt. A short proof of the restricted Ramsey theorem for finite set systems. *J. Combin. Theory Ser. A*, 52(2):313–320, 1989.

[16] F. P. Ramsey. On a problem of formal logic. *Proc. London Math. Soc. (2)*, 30(1):264–286, 1930.