Contractors for flows

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Abstract

We answer a question raised by Lovász and B. Szegedy [Contractors and connectors in graph algebras, *J. Graph Theory* 60:1 (2009)] asking for a contractor for the graph parameter counting the number of B-flows of a graph, where B is a subset of a finite Abelian group closed under inverses. We prove our main result using the duality between flows and tensions and finite Fourier analysis.

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1. Introduction

In their paper [11] Lovász and B. Szegedy introduce the notion of connectors and contractors in quantum graph algebras. Connectors and contractors for a graph parameter allow expressions for the parameter of a graph obtained by the connection of two vertices by an edge or the contraction of an edge (either operation possibly producing multiple edges or loops, respectively) in terms of values of the parameter on graphs obtained by operations that preserve simplicity. For example, a connector for the flow polynomial is any path of length two or more. Contractors for graph parameters can be thought of as a generalization of the deletion-contraction identity for the chromatic polynomial, flow polynomial, and other specializations of the Tutte polynomial.

In [11, Section 2.3] Lovász and B. Szegedy comment that "there does not seem to be a simple explicit construction for a contractor" for the number of B-flows, where B is a subset of a finite Abelian group closed under inverses. In Section 4 we give an explicit construction of such contractors.

We prove our result using the Fourier transform on finite Abelian groups. Towards this end, in Section 2 we introduce homomorphism functions, flows and tensions of a graph, contractors in graph algebras, and then describe those properties of the Fourier transform that will be used to enable us to reach the main result (Theorem 3.10, and its special case for B-flows, Theorem 4.1). Using the Fourier transform we can express any weighted function on the set of tensions of a graph (effectively a homomorphism function to an edge-weighted graph) as a related weighted function on the set of flows of the graph. This in particular allows the number of B-flows to be written as the number of homomorphisms to an edge-weighted graph (Lemma 2.3). Although the duality relationship between flows and tensions is well known, there have not been many useful applications of this correspondence. In this paper it allows us to transform a special type of instance of a problem we do not in general know yet how to solve (Problem 5.1 in Section 5) into one that we can solve.

For graphs G and H, let $\hom(G, H)$ denote the number of homomorphisms from G to H. The definition can be extended to edge-weighted graphs H, as explained in Section 2.1. In Section 3 we consider the problem of finding a contractor for the homomorphism function $\hom(\,\cdot\,, H)$. The minimal polynomial of the adjacency matrix of H can be used to construct a contractor for $\hom(\,\cdot\,, H)$ if the adjacency matrix of H does not have eigenvalue 0 (Theorem 3.3). Theorem 3.4 gives a contractor for $\hom(\,\cdot\,, H)$ when the adjacency matrix of H has no diagonal entry equal to an off-diagonal entry. The relevant edge-weighted graph H that gives the number of B-flows turns out to have eigenvalue 0, and when B is a strict subset of non-zero group values there are off-diagonal zero entries as well as zeroes on the diagonal of the adjacency matrix of H. However, the special structure of H as an edge-weighted Cayley graph allows us to obtain a contractor by using the minimal polynomial of the graph whose edge weights are obtained by taking the Fourier transform of the edge weights of H (Theorem 3.10). In Section 4 we apply the theory developed in Sections 2 and 3 to construct a contractor for the number of B-flows.

In Section 5 we conclude by outlining some possible directions for future research. A sequel to this paper, which includes the application of our results to Jaeger's B-flow problems [9], is in preparation [5].

2. Preliminaries

2.1. Homomorphism functions

Given two graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$, a *homomorphism* from G to H, written as $\psi : G \to H$, is a mapping $\psi : V(G) \to V(H)$ such that $\psi(u)\psi(v) \in E(H)$ whenever $uv \in E(G)$. The graph parameter $\hom(G, H)$ counting the number of homomorphisms from G to H can be generalized to edge-weighted graphs H as follows. (We do not need to consider the more general case where H has vertex weights as well as edge weights, as is done in [11], since the relevant graphs H for counting B-flows do not require them.)

Let H be a graph with a real weight $\beta_H(ij)$ associated with each edge $ij \in E(H)$ and let G be an unweighted graph.

The homomorphism function $\hom(G, H)$ is defined by

$$\hom(G, H) = \sum_{\psi: V(G) \to V(H)} \prod_{uv \in E(G)} \beta_H(\psi(u), \psi(v)).$$

If we fix $U \subseteq V(G)$ and define for every function $\phi: U \to V(H)$ the weight

$$\hom_\phi(G, H) = \sum_{\substack{\psi: V(G) \to V(H) \\ \psi_{|U} = \phi}} \prod_{uv \in E(G)} \beta_H(\psi(u), \psi(v)),$$

then it is easy to see that we have the decomposition

$$\hom(G, H) = \sum_{\phi: U \to V(H)} \hom_\phi(G, H).$$

2.2. Flows and tensions

Let $G = (V, E)$ be a graph with an arbitrary fixed orientation of its edges. Denote by $c(G)$ the number of connected components of G, and by $r(G) = |V| - c(G)$ its rank. Let D be the $V \times E$ incidence matrix of G, with (v, e)-entry given by

$$D_{v,e} = \begin{cases} +1 & v \text{ is the head of } e, \\ -1 & v \text{ is the tail of } e, \\ 0 & v \text{ is not an endpoint of } e, \text{ or } e \text{ is a loop on } v. \end{cases}$$

Suppose Γ is a finite commutative ring with unity, and let $\Gamma^V = \{f : V \to \Gamma\}$ and $\Gamma^E = \{g : E \to \Gamma\}$.

The linear transformation $D : \Gamma^E \to \Gamma^V$ of Γ-modules has kernel equal to the set of Γ-*flows* of G. This is to say a function $g : E \to \Gamma$ is a Γ-flow if and only if, for each vertex $v \in V$,

$$\sum_{\substack{e = uv \\ v \text{ head of } e}} g(e) - \sum_{\substack{e = uv \\ v \text{ tail of } e}} g(e) = 0.$$

If $B = -B$ is a subset of Γ then a B-*flow* of G is a Γ-flow taking values in B. (We have followed Jaeger's [9] usage of the term B-flow; in [11] Lovász and B. Szegedy use the letter S instead of B, and so speak of S-flows rather than B-flows.) In particular, when $B = \Gamma \setminus \{0\}$ a B-flow is a *nowhere-zero Γ-flow* of G. It is well known that the number of nowhere-zero Γ-flows of G depends only on $|\Gamma|$ and is given by the evaluation of the *flow polynomial* $F(G; z)$ at $z = |\Gamma|$.

The transpose linear transformation $D^\top : \Gamma^V \to \Gamma^E$ has as image the set of Γ-*tensions* of G. A Γ-tension $g : E \to \Gamma$ arises from a vertex Γ-colouring $f : V \to \Gamma$ of G, by setting $g(e) = f(v) - f(u)$ for each edge e with tail u and head v.

A B-*tension* of G is a Γ-tension of G taking values in B. In particular, when $B = \Gamma \setminus \{0\}$ a B-tension arises from $|\Gamma|^{c(G)}$ proper vertex Γ-colourings of G. The number of proper vertex Γ-colourings is given by the evaluation of the *chromatic polynomial* $P(G; z)$ at $z = |\Gamma|$.

To define Γ-flows and Γ-tensions all that is needed is the structure of Γ as an additive Abelian group. The set of Γ-flows forms an additive subgroup of Γ^E isomorphic to $\Gamma^{|E|-r(G)}$ and the set of Γ-tensions a subgroup of Γ^E isomorphic to $\Gamma^{r(G)}$. As we shall see in Section 2.4, the extra ring structure on Γ allows us to relate the sets of tensions and flows to each other as orthogonal complements.

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2.3. Contractors

For fixed positive integer k, a *k-labelled graph* (G, λ) is a finite graph G (without loops but possibly multiple edges) together with a function $\lambda : [k] \to V(G)$. Lovász and Szegedy [11] restrict λ to be an injective function, but we shall follow Schrijver [12] and allow vertices to have multiple labels. A k-labelled graph (G, λ) is *simple* if G has no parallel edges, the function λ is injective and its image is an independent set of vertices.

A k-labelled *quantum graph* is a formal linear combination of k-labelled graphs with coefficients in $\mathbb{R}$. Let $\mathcal{G}_k$ denote the set of k-labelled quantum graphs, $\mathcal{G}_k^0$ the subset of k-labelled quantum graphs whose labelled vertices are independent, and $\mathcal{G}_k^{\mathrm{simp}}$ the subset of simple k-labelled quantum graphs.

We let K_n denote the complete graph on n vertices and $\overline{K}_n$ its complement, consisting of n isolated vertices.

For two k-labelled graphs X and Y, the *product XY* is defined [11] by taking the disjoint union of X and Y and then identifying vertices which share the same label. The product is associative and commutative, and extends linearly to $\mathcal{G}_k$. The identity for this multiplication on $\mathcal{G}_k$ is $(\overline{K}_k, \lambda)$ where λ is injective, i.e., k independent vertices each with one label.

In the sequel we shall just consider the case of 2-labelled graphs, i.e., $k = 2$.

A further binary operation on 2-labelled graphs is the *concatenation $X \circ Y$* of X and Y, defined by identifying the vertex of X labelled 2 with the vertex labelled 1 in Y and unlabelling this merged vertex. Concatentation is associative but not commutative. It is extended linearly to $\mathcal{G}_2$. The identity for this multiplication on $\mathcal{G}_2$ is K_1, with both labels 1 and 2 on the single vertex.

Let P_ℓ, $\ell \geq 1$, denote the 2-labelled path on $\ell + 1$ vertices with one endpoint labelled 1 and the other endpoint labelled 2. In this notation, $P_1 = K_2$. We have $P_\ell \circ P_m = P_{\ell+m}$, and $P_\ell P_m = C_{\ell+m}$, where the cycle has labels spaced ℓ edges apart in one direction, m edges apart in the other. If X is a 2-labelled graph then XP_1 is the 2-labelled graph with an edge added joining the vertices labelled 1 and 2. In particular, if X is a simple 2-labelled graph then XP_1 joins non-adjacent vertices with an edge.

The graph K_1 with vertex carrying both labels 1 and 2 when applied as a product with X (giving the graph XK_1) identifies the vertices labelled 1 and 2 in X (or does nothing if both labels are on a single vertex in X). The graph XK_1 is equal to the graph XP_1 with the edge that joins 1 and 2 (the edge not belonging to X) contracted. If vertices 1 and 2 are adjacent in X then XP_1 has parallel edges joining 1 and 2, and XK_1 has a loop for each edge between 1 and 2 in X.

The graph $\overline{K}_2$ with one vertex labelled 1 and the other labelled 2 is the identity for the product of 2-labelled graphs.

A *series-parallel* 2-labelled graph is obtained from P_1 by a sequence of products (parallel extensions) and concatenations (series extensions).

Definition 2.1. [11] Let h be a graph parameter. A 2-labelled quantum graph $Z \in \mathcal{G}_2$ is a *contractor* for h if for all $X \in \mathcal{G}_2^0$ we have

$$h(XZ) = h(XK_1).$$

Informally, attaching a contractor Z at two non-adjacent vertices acts like identifying these vertices as far as the value of the graph parameter h is concerned.

Example 2.2. The deletion-contraction relation for the chromatic polynomial takes the form $P(XP_1; z) = P(X; z) - P(XK_1; z)$ where X is a 2-labelled graph. When the labelled vertices are adjacent in X, so that XK_1 has a loop, this relation reduces to $P(XP_1; z) = P(X; z)$. Extending the chromatic polynomial linearly to $\mathcal{G}_2$, we then have $P(XK_1; z) = P(X(\overline{K}_2 - P_1); z)$ for $X \in \mathcal{G}_2$, and in particular for $X \in \mathcal{G}_2^0$. Thus, the quantum graph $\overline{K}_2 - P_1$ is a contractor for the chromatic polynomial.

2.4. B-flows by homomorphisms using Fourier analysis

In this section we develop the connection between flows and tensions in the context of Fourier analysis. Our main theorem in this section is Lemma 2.3, which expresses the number of B-flows in terms of homomorphisms to a weighted Cayley graph. In order to reach this result we have to briefly introduce the relevant concepts from finite Fourier analysis, for more on which see e.g. [13]. The duality between flows and tensions is well known, especially in statistical physics, see e.g. [2], with an early example being Van der Waerden's Eulerian subgraph expansion of the Ising model partition function [14].

Let $\Gamma \cong \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_r}$ be a finite additive Abelian group of order $n = n_1 n_2 \cdots n_r$. A *character* of Γ is a (group) homomorphism $\chi : \Gamma \to \mathbb{C}^\times$, where $\mathbb{C}^\times$ is the multiplicative group of the field of complex numbers. The set of characters of Γ form a group $\widehat{\Gamma}$ under pointwise multiplication which is isomorphic to Γ. For each $x \in \Gamma$, let χ_x denote the image of x under a fixed isomorphism $\Gamma \to \widehat{\Gamma}$. In particular, the trivial character χ_0 is defined by $\chi_0(y) = 1$ for all $y \in \Gamma$, and $\chi_{-x}(y) = \overline{\chi_x(y)}$, the bar denoting complex conjugation.

Denote by $\mathbb{C}^\Gamma$ the vector space over $\mathbb{C}$ of all functions from Γ to $\mathbb{C}$, which has Hermitian inner product defined for $\alpha, \beta \in \mathbb{C}^\Gamma$ by

$$\langle \alpha, \beta \rangle = \sum_{x \in \Gamma} \alpha(x) \overline{\beta(x)}.$$

The vector space $\mathbb{C}^\Gamma$ has the additional structure of an algebra under either of the following two definitions of multiplication:

(i) the *pointwise product* $\alpha \cdot \beta$ of $\alpha, \beta \in \mathbb{C}^\Gamma$, defined for $x \in \Gamma$ by $\alpha \cdot \beta(x) = \alpha(x)\beta(x)$,

(ii) the *convolution* $\alpha * \beta$ of $\alpha, \beta \in \mathbb{C}^\Gamma$, defined for $x \in \Gamma$ by

$$\alpha * \beta(x) = \sum_{y \in \Gamma} \alpha(y)\beta(x - y).$$

The set $\{\delta_x : x \in \Gamma\}$ of indicator functions defined by

$$\delta_x(y) = \begin{cases} 1 & x = y, \\ 0 & x \neq y, \end{cases}$$

form an orthonormal basis for $\mathbb{C}^\Gamma$, with $\langle \delta_x, \delta_y \rangle = \delta_x(y)$. We extend the indicator function notation to subsets B of Γ, defining $\delta_B = \sum_{x \in B} \delta_x$.

The characters of Γ are also orthogonal in this inner product space, with $\langle \chi_x, \chi_y \rangle = |\Gamma| \delta_x(y)$.

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For $\alpha \in \mathbb{C}^\Gamma$, the *Fourier transform* $\widehat{\alpha} \in \mathbb{C}^\Gamma$ is defined for $y \in \Gamma$ by

$$\widehat{\alpha}(y) = \langle \alpha, \chi_y \rangle = \sum_{x \in \Gamma} \alpha(x) \chi_y(-x). \tag{2.1}$$

The Fourier transform maps the basis of indicator functions to the basis of characters: $\widehat{\delta_y} = \chi_{-y}$. The *Fourier inversion formula* $\widehat{\widehat{\alpha}}(x) = |\Gamma| \alpha(-x)$, gives the inverse transform

$$\alpha(x) = \frac{1}{|\Gamma|} \langle \widehat{\alpha}, \chi_{-x} \rangle = \frac{1}{|\Gamma|} \sum_{y \in \Gamma} \widehat{\alpha}(y) \chi_x(y). \tag{2.2}$$

Note that $\widehat{\delta_\Gamma} = |\Gamma| \delta_0$ and $\widehat{\delta_0} = \delta_\Gamma$, since $\langle \delta_\Gamma, \chi_y \rangle = \langle \chi_0, \chi_y \rangle = |\Gamma| \delta_0(y)$.

The Fourier transform gives an isomorphism of the algebra $\mathbb{C}^\Gamma$ with multiplication pointwise product with the algebra $\mathbb{C}^\Gamma$ with multiplication convolution: for $y \in \Gamma$,

$$\widehat{\alpha * \beta}(y) = \widehat{\alpha} \cdot \widehat{\beta}(y), \tag{2.3}$$

$$\widehat{\alpha \cdot \beta}(y) = \frac{1}{|\Gamma|} \widehat{\alpha} * \widehat{\beta}(y). \tag{2.4}$$

We now give Γ the further structure of a ring by defining multiplication of elements $x = (x_1.x_2, \ldots, x_r)$ and $y = (y_1, y_2, \ldots, y_r)$ of Γ componentwise, setting $xy = (x_1 y_1, x_2 y_2, \ldots, x_r y_r)$.

A character $\chi \in \widehat{\Gamma}$ is a *generating character* for Γ if $\chi_x(y) = \chi(xy)$ for each character $\chi_x \in \widehat{\Gamma}$. A generating character of the ring $\mathbb{Z}_n$ is defined by $\chi(z) = \omega^z$ where ω is any primitive nth root of unity. If χ_1 is a generating character for Γ_1 and χ_2 a generating character for Γ_2 then $\chi = \chi_1 \oplus \chi_2$ defined by $\chi(x_1, x_2) = \chi_1(x_1)\chi_2(x_2)$ is a generating character for $\Gamma_1 \oplus \Gamma_2$.

Consider Γ^m, the m-fold direct sum of Γ, and suppose ψ is a generating character for Γ. Then χ defined by $\chi(x_1, \ldots, x_m) = \psi(x_1) \cdots \psi(x_m)$ for $(x_1, \ldots, x_m) \in \Gamma^m$ is a generating character for Γ^m. The *Euclidean inner product* (dot product) is defined for $x = (x_1, \ldots, x_m), y = (y_1, \ldots, y_m) \in \Gamma^m$ by $x \cdot y = x_1 y_1 + \cdots + x_m y_m$. Since $\psi(x_1) \cdots \psi(x_m) = \psi(x_1 + \cdots x_m)$, it follows that $\chi_x(y) = \chi(xy) = \psi(x \cdot y)$ for $x, y \in \Gamma^m$. The *orthogonal* to a subset C of Γ^m (with respect to the Euclidean inner product) is defined by $C^\perp = \{y \in \Gamma : x \cdot y = 0 \text{ for all } y \in C\}$.

For a subgroup C of Γ, the *annihilator* of C is defined by $C^\sharp = \{x \in R : \chi_x(y) = 1 \text{ for all } y \in C\}$. The annihilator $C^\sharp$ is a subgroup of Γ isomorphic to Γ/C. When considering a subgroup C of Γ^m, we have $C^\perp \subseteq C^\sharp$, since $\chi_x(y) = \psi(x \cdot y) = \psi(0) = 1$ for all $y \in C$ whenever $x \in C^\perp$, where ψ is a suitably defined generating character of Γ. For our choice of multiplicative structure on Γ the reverse inclusion holds too, although we do not need to prove this fact here. (The equality $C^\perp = C^\sharp$ holds generally for $C \subseteq \Gamma^m$ when Γ is a Frobenius ring - see for example Section 7 of [15].)

A key property of the Fourier transform is that it takes indicators of subgroups to (scalar multiples of) indicators of their annihilators:

$$\widehat{\delta}_C(y) = \sum_{x \in C} \chi_x(y) = |C| \delta_{C^\sharp}(y). \tag{2.5}$$

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Following from (2.2), (2.3) and (2.5) is the *Poisson summation formula*: for every $\alpha \in \mathbb{C}^{\Gamma}$,

$$\sum_{x \in C} \alpha(x + z) = \frac{1}{|C^{\sharp}|} \sum_{x \in C^{\sharp}} \widehat{\alpha}(x) \chi_z(x). \tag{2.6}$$

We now apply the machinery we have set up to relate flows and tensions of a graph.

Let D be the incidence matrix of a graph G, as defined in Section 2.2. Recall that the kernel $\ker D$ of the map $D : \Gamma^E \to \Gamma^V$ consists of the Γ-flows of G, forming an additive group isomorphic to $\Gamma^{|E|-r(G)}$. Likewise, the image $\operatorname{im} D^{\top}$ of the adjoint map $D^{\top} : \Gamma^V \to \Gamma^E$ consists of the Γ-tensions of G, isomorphic as an additive group to $\Gamma^{r(G)}$.

Put $C = \ker D \subseteq \Gamma^E$. We have $C^{\sharp} \supseteq C^{\perp} \supseteq \operatorname{im} D^{\top}$, the latter inclusion since $x \cdot D^{\top} y = Dx \cdot y$, so that whenever $x \in \ker D$ we have $x \cdot D^{\top} y = 0$. With $|\operatorname{im} D^{\top}| = |\Gamma|^{r(G)} = |\Gamma^E/C| = |C^{\sharp}|$, we have $|C^{\sharp}| = |C^{\perp}|$. Hence $C^{\sharp} = C^{\perp}$.

From equation (2.6) it follows that, for any $\alpha \in \mathbb{C}^{\Gamma}$,

$$\sum_{x \in \ker D} \alpha(x) = \frac{1}{|\Gamma|^{r(G)}} \sum_{y \in \operatorname{im} D^{\top}} \widehat{\alpha}(y). \tag{2.7}$$

Lemma 2.3. *Let $B = -B$ be a subset of an additive Abelian group Γ. Let $H = \operatorname{Cayley}(\Gamma, B)$ be the graph on vertex set Γ with an edge joining vertices i and j if and only if $j - i \in B$, and let $\widehat{H}$ be the edge-weighted Cayley graph on vertex set Γ with edge ij having weight $\widehat{\delta}_B(j - i)$.*

Then the number of B-tensions of a graph $G = (V, E)$ is equal to $|\Gamma|^{-c(G)} \operatorname{hom}(G, H)$ and the number of B-flows of G is equal to $|\Gamma|^{-|V|} \operatorname{hom}(G, \widehat{H})$.

Proof. A function $f : V \to \Gamma$ is a homomorphism contributing to $\operatorname{hom}(G, H)$ if and only if $f(v) - f(u) \in B$ for each edge $e = uv \in E$, which is to say that $g(e) := f(v) - f(u)$ is a B-tension of G. For a given B-tension $g : E \to \Gamma$ there are $|\Gamma|^{c(G)}$ vertex colourings $f : V \to \Gamma$ with $Df = g$. This proves that the number of B-tensions is $|\Gamma|^{-c(G)} \operatorname{hom}(G, H)$.

To prove that $|\Gamma|^{-|V|} \operatorname{hom}(G, \widehat{H})$ is the number of B-flows, take $\alpha = \delta_{B^E}$ in (2.7), which has Fourier transform for $g : E \to \Gamma$ given by

$$\widehat{\delta}_{B^E}(g) = \prod_{e \in E} \widehat{\delta}_B(g(e)),$$

which yields

$$|\ker D \cap B^E| = \frac{1}{|\Gamma|^{|V|}} \sum_{f : V \to \Gamma} \prod_{uv \in E} \widehat{\delta_B}(f(v) - f(u)).$$

$\square$

3. Contractors for homomorphism functions

After the preliminaries of Section 2 we now turn to the general problem of finding a contractor for $\operatorname{hom}(\,\cdot\,, H)$. (In Section 4 we then consider the special case of finding a contractor for the number of B-flows.)

Let H be an edge-weighted graph with adjacency matrix $A = (\beta(ij))$. To every $X \in \mathcal{G}_2$ assign a $V(H) \times V(H)$-matrix $M(X) = M_H(X)$ with (i, j)-entry equal to

$\text{hom}_\phi(X, H)$ where $\phi(1) = i$ and $\phi(2) = j$. As H is fixed we shall usually leave the dependence on H implicit and write $M(X)$ instead of $M_H(X)$. It is easy to see that the function $M : \mathcal{G}_2 \to \mathbb{C}^{V(H) \times V(H)}$ is linear and satisfies

$$M(X \circ Y) = M(X)M(Y), \quad M(XY) = M(X) \circ M(Y), \tag{3.1}$$

where (for matrices) $\circ$ is the entrywise (Hadamard, or Schur) product.

Let P_k denote the path on k edges with endpoints labelled 1 and 2. Observe that $M(P_1) = (\beta(ij))$ is the adjacency matrix of H.

The graph P_1^k (k-fold product of P_1 with itself) is a *thick edge*, consisting of two vertices joined by k parallel edges, with one vertex labelled 1 and the other vertex labelled 2. The graph $P_1^{\circ k}$ (k-fold concatenation of P_1 with itself) is equal to P_k.

Lemma 3.1. *If H has adjacency matrix A then*

$$M(P_k) = A^k, \quad M(P_1^k) = A^{\circ k},$$

where $A^{\circ k}$ denotes the k-fold Hadamard product of A with itself and A^k the k-fold ordinary matrix product of A with itself.

Proof. Since $M(P_1) = A$, $P_k = P_1^{\circ k}$, by the identities (3.1) we have $M(P_k) = M(P_1^{\circ k}) = M(P_1)^k = A^k$ and $M(P_1^k) = M(P_1)^{\circ k} = A^{\circ k}$. $\qquad\square$

Lemma 3.1 includes for $k = 0$ the fact that, for $P_0 = K_1$ a single vertex with labels 1 and 2, we have $M(K_1) = I$, where I is the identity matrix (for ordinary matrix multiplication). With $P_1^0 = \overline{K}_2$, we also have $M(\overline{K}_2) = J$, where $\overline{K}_2$ comprises two isolated vertices, one labelled 1 the other 2, and J is the identity matrix for entrywise matrix multiplication (the all-one matrix).

A fundamental lemma (based on the identity (3.1)) for finding contractors for homomorphism functions is the following:

Lemma 3.2. [11] *A 2-labelled quantum graph Z is a contractor for $\text{hom}(\,. \,, H)$ if and only if $M(Z) = I$.*

Using this lemma we can write down a a quantum path contractor for $\text{hom}(\cdot, H)$ when the adjacency matrix of H does not have 0 as an eigenvalue. (Compare [11, Section 4.1].)

Theorem 3.3. *Suppose the adjacency matrix A of H has minimal polynomial*

$$p_A(t) = p_0 + p_1 t + \cdots + p_\ell t^\ell.$$

If $p_0 \neq 0$ (i.e, if 0 is not an eigenvalue of A), then a contractor for $\text{hom}(\cdot, H)$ is given by

$$Z = -\frac{1}{p_0} [p_1 P_1 + p_2 P_2 + \cdots + p_\ell P_\ell].$$

Proof. If $p_0 \neq 0$ then $I = -\frac{1}{p_0} \sum_{k=1}^\ell p_k A^k$. Hence, using Lemmas 3.1 and 3.2, if $Z = -\frac{1}{p_0} \sum_{k=1}^\ell p_k P_k$ then $M(Z) = I$, and Z is a contractor for $\text{hom}(\cdot, H)$. $\qquad\square$

Similarly, by straightforward polynomial interpolation we have a quantum thick edge contractor for $\text{hom}(\cdot, H)$ when the adjacency matrix of H satisfies a suitable condition:

Theorem 3.4. *Suppose the adjacency matrix $A = (\beta(ij))$ of H has the property that*

$$\{\beta(ij) : i, j \in V(H), i \neq j\} \cap \{\beta(ii) : i \in V(H)\} = \emptyset,$$

and that $|\{\beta(ij) : i, j \in V(H)\}| = \ell$.

Then there exist constants $r_0, r_1, \ldots, r_\ell$ (computable by Lagrange interpolation) such that

$$Z = r_0 \overline{K}_2 + r_1 P_1 + r_2 P_1^2 + \cdots + r_\ell P_1^\ell,$$

is a contractor for $\hom(\,\cdot\,, H)$.

We are left with the problem of constructing a contractor for $\hom(\,\cdot\,, H)$ when the adjacency matrix of H has 0 as an eigenvalue, or when at least one of its diagonal entries also appears amongst its off-diagonal entries. When H is a weighted Cayley graph we are able to find such a contractor, and the remainder of this section is devoted to showing how.

We begin with the following result of Hoffmann, for which see e.g. [3, Chapter 3].

Theorem 3.5. *Let A be the adjacency matrix of a graph H. The all-one matrix J is a polynomial in A if and only if H is connected and A has eigenvector $\mathbf{1}$ (i.e., H is regular).*

Corollary 3.6. *Suppose that the adjacency matrix A of a connected graph H on n vertices has eigenvector $\mathbf{1}$ with eigenvalue λ_1, that its distinct eigenvalues are $\lambda_1 > \lambda_2 > \cdots > \lambda_\ell$, and that*

$$q(t) = \prod_{2 \leq k \leq \ell} (t - \lambda_k).$$

Then

$$J = \frac{n}{q(\lambda_1)} q(A).$$

Proof. We have $q(A) = rJ$ for some $r \neq 0$ and $q(A)$ has eigenvalues $q(\lambda_1)$ and $q(\lambda_k) = 0$ for $2 \leq k \leq \ell$. Since the only non-zero eigenvalue of rJ is rn it follows that $rn = q(\lambda_1)$. $\square$

The polynomial $q(t)$ in Corollary 3.6 is equal to $p_A(t)/(t - \lambda_1)$, where $p_A(t)$ is the minimal polynomial of A. (Since A is symmetric it is diagonalizable and so each of its Jordan blocks has size 1, which implies the multiplicity of each root of the minimal polynomial of A is equal to one.)

Suppose now that A has rows and columns indexed by an additive Abelian group Γ of order n and has (i, j) entry equal to $(\beta(i - j))$ for all $i, j \in \Gamma$, where $\beta : \Gamma \to \mathbb{C}$ satisfies $\beta(-i) = \beta(i)$. Thus A is the adjacency matrix of a weighted Cayley graph on Γ.

Recall the definition of characters and the Fourier transform from Section 2.4. The matrix A has eigenvectors $(\chi_i(j))_{j \in \Gamma}$ for each $i \in \Gamma$ with corresponding eigenvalue $\widehat{\beta}(i)$. Define $\widehat{A}$ to be the matrix with (i, j)-entry the Fourier transform $\widehat{\beta}(i - j)$. For example, $\widehat{I} = J$ and $\widehat{J} = nI$.

Let us call a matrix Γ-*circulant* if, like the matrix A, it takes the form $(\alpha(i - j))$ for some function $\alpha \in \mathbb{C}^\Gamma$. (When Γ is cyclic these are circulant matrices as ordinarily defined.)

Lemma 3.7. *The map $A \mapsto \widehat{A}$ is an algebra isomorphism of the subalgebra of Γ-circulant matrices under addition and ordinary matrix multiplication with the subalgebra of Γ-circulant matrices under addition and entrywise product:*

$$\widehat{A_1 A_2} = \widehat{A}_1 \circ \widehat{A}_2, \quad \widehat{A_1 \circ A_2} = \frac{1}{n}\widehat{A}_1 \widehat{A}_2.$$

In particular

$$\widehat{A^k} = \widehat{A}^{\circ k}, \quad \widehat{A^{\circ k}} = \frac{1}{n^{k-1}}\widehat{A}^k.$$

Proof. This is a consequence of the identities (2.3) and (2.4) for the Fourier transform of convolutions and pointwise products. Since the Fourier transform is linear it follows that $A \to \widehat{A}$ is an algebra isomorphism. $\qquad\square$

Lemma 3.8. *Given $q(A) = q_0 I + q_1 A + q_1 A^2 + \cdots + q_{\ell-1} A^{\ell-1}$, define*

$$q^\circ(A) = q_0 J + q_1 A + q_2 A^{\circ 2} + \cdots + q_{\ell-1} A^{\circ(\ell-1)}.$$

Then $\widehat{q(A)} = q^\circ(\widehat{A})$ and $\widehat{q^\circ(A)} = q(\frac{1}{n}\widehat{A})$.

Proof. This follows from Lemma 3.7. $\qquad\square$

Recall that for $Z \in \mathcal{G}_2$ to be a contractor for $\hom(\,\cdot\,, H)$ we need $M(Z) = I$. The following lemma is the key to obtaining such a contractor when H has a Γ-circulant adjacency matrix A.

Lemma 3.9. *Suppose A has eigenvector $\mathbf{1}$ with eigenvalue λ_1, minimum polynomial $p_A(t)$ and that $q(t) = p_A(t)/(t - \lambda_1)$. Then*

$$I = \frac{1}{q(\lambda_1)}q^\circ(\widehat{A}).$$

Proof. By Corollary 3.6 we have $J = \frac{n}{q(\lambda_1)}q(A)$ and by Lemma 3.8, $\widehat{q(A)} = q^\circ(\widehat{A})$. Since $\widehat{J} = nI$, the result follows. $\qquad\square$

Our main theorem of this section now results:

Theorem 3.10. *Suppose H is a connected graph with adjacency matrix a Γ-circulant matrix A and that $\widehat{H}$ has adjacency matrix $\widehat{A}$. Suppose further that A has eigenvector $\mathbf{1}$ with eigenvalue λ_1, has minimal polynomial $p_A(t)$, and that*

$$\frac{p_A(t)}{t - \lambda_1} = q(t) = q_0 + q_1 t + \cdots + q_{\ell-1} t^{\ell-1}.$$

Then a contractor for $\hom(\,\cdot\,, \widehat{H})$ is given by

$$Z = \frac{1}{q(\lambda_1)} \sum_{0 \le k \le \ell-1} q_k P_1^k.$$

Proof. By Lemma 3.9 we have $I = \frac{1}{q(\lambda_1)}q^\circ(\widehat{A})$. The matrix $\widehat{A}$ is the adjacency matrix of the edge-weighted graph $\widehat{H}$, and $M_{\widehat{H}}(X)$ the $V(\widehat{H}) \times V(\widehat{H})$ matrix whose (i,j)-entry is $\hom_\phi(X, \widehat{H})$, where $\phi(1) = i, \phi(2) = j$. Since $M_{\widehat{H}}(P_1^k) = \widehat{A}^{\circ k}$, the result follows by Lemma 3.2. $\qquad\square$

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4. A contractor for the number of B-flows

In this section we apply Theorem 3.10 to the special case where $H = \mathrm{Cayley}(\Gamma, B)$ for additive Abelian group Γ, and where $B \subseteq \Gamma \setminus 0$ satisfies $-B = B$.

Let δ_B be the indicator function of B. The adjacency matrix A of H has (i,j)-entry $\delta_B(i-j)$ and eigenvalues $\widehat{\delta}_B(i)$ for $i \in \Gamma$. The minimal polynomial $p_A(t)$ of A has degree ℓ equal to the number of distinct values of $\widehat{\delta}_B(c)$ for $c \in \Gamma$. The largest eigenvalue of A is $\widehat{\delta}_B(0) = |B|$, belonging to the eigenvector $\mathbf{1}$, and as in Theorem 3.10 we set

$$\frac{p_A(t)}{t - |B|} = q(t) = q_0 + q_1 t + \cdots + q_{\ell-1} t^{\ell-1}. \tag{4.1}$$

We reach the main theorem of this paper, giving a contractor for B-flows.

Theorem 4.1. *Let Γ be an additive Abelian group of order n and suppose $H = \mathrm{Cayley}(\Gamma, B)$ is a connected graph with adjacency matrix A.*
Let $\widehat{H}$ be the edge-weighted graph on vertex set Γ with adjacency matrix $\widehat{A}$.
Then, with the polynomial $q(t)$ defined as in Equation (4.1) above,

(i) a contractor for the number of B-flows is given by

$$Z = \frac{n}{q(|B|)} \sum_{0 \le k \le \ell-1} q_k P_1^k;$$

(ii) a contractor for $n^{c(G)}$ times the number of B-tensions of G is given by

$$Z = \frac{1}{q_0} \left[\frac{q(|B|)}{n} \overline{K}_2 - \sum_{1 \le k \le \ell-1} q_k P_k \right].$$

Proof. Part (i) is a corollary of Theorem 3.10, which gives

$$\frac{1}{q(|B|)} \sum_{0 \le k \le \ell-1} q_k P_1^k,$$

as a contractor for $\hom(\cdot, \widehat{H})$. The number of B-flows of G is by Lemma 2.3 equal to $n^{-|V(G)|}\hom(G, \widehat{H})$, and this implies the expression for the contractor as given in the theorem statement.

In order to show (ii), by Corollary 3.6 we have

$$J = \frac{n}{q(|B|)} \sum_{0 \le k \le \ell-1} q_k A^k.$$

Since $q_0 \ne 0$, we then obtain

$$I = \frac{1}{q_0} \left[\frac{q(|B|)}{n} J - \sum_{1 \le k \le \ell-1} q_k A^k \right],$$

and this implies $M_H(Z) = I$, where Z is the quantum graph given in the theorem statement, i.e., Z is a contractor for $\hom(\cdot, H)$. By Lemma 2.3, the number of B-tensions of G is equal to $n^{-c(G)}\hom(G, H)$. $\qquad\square$

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If B does not contain a set of generators for Γ then the Γ-circulant graph $H = \text{Cayley}(\Gamma, B)$ is not connected. The connected components of $\text{Cayley}(\Gamma, B)$ correspond to the cosets of the subgroup generated by B. Theorem 4.1 is easily extended to this disconnected case as follows:

Theorem 4.2. *Let Γ be an additive Abelian group of order n and suppose $H = \text{Cayley}(\Gamma, B)$ has adjacency matrix A, and that $q(t)$ is defined as in Equation (4.1) above.*

Suppose H has r isomorphic connected components, i.e., its adjacency matrix A is permutation-equivalent to a matrix of the form $I \otimes A_1$ for $r \times r$ identity matrix I and some $n/r \times n/r$ matrix A_1. Then, a contractor for the number of B-flows is given by

$$Z = \frac{n}{rq(|B|)} \sum_{0 \le k \le \ell-1} q_k P_1^k.$$

Proof. If B generates a subgroup Γ_1 of index r in Γ then $\text{Cayley}(\Gamma, B)$ has r components, each isomorphic to a connected Cayley graph $H_1 = \text{Cayley}(\Gamma_1, B)$ of order $n_1 = |\Gamma_1| = n/r$. Then $\text{hom}(G, H) = r^{c(G)}\text{hom}(G, H_1)$ is equal to the number of B-tensions of G multiplied by $n^{c(G)}$, and $r^{c(G)}n_1^{-|V(G)|}\text{hom}(G, \widehat{H}_1)$ is equal to the number of B-flows of G. If H_1 has adjacency matrix A_1 then H has adjacency matrix $I \otimes A_1$, where I is the $r \times r$ identity matrix. Let $q_1(t) = p_{A_1}(t)/(t - |B|)$, where $p_{A_1}(t)$ is the minimal polynomial of A_1. Then $q_1(t) = q(t)$, since the eigenvalues of A_1 coincide with those of A (only the eigenspaces differ). By Corollary 3.6, $J = \frac{n_1}{q_1(|B|)}q_1(A_1)$, whence $I = \frac{n_1}{q(|B|)}q^\circ(\widehat{A_1})$ by Corollary 3.9. The result now follows in the same fashion as for the proof of Theorem 4.1. $\square$

5. Conclusion

In this paper we provide a new and constructive proof for the existence of a contractor for B-flows.

Theorem 3.10 provides a general construction of a contractor for a homomorphism function to an edge-weighted Cayley graph. Specifically, it gives a contractor for $\text{hom}(\,.\,, \widehat{H})$ when H is an edge-weighted Cayley graph of a finite Abelian group Γ, and $\widehat{H}$ is the edge-weighted Cayley graph with weights given by the Fourier transform of the weights in H; Theorem 4.1 applies this to the special case of B-flows, where $H = \text{Cayley}(\Gamma, B)$.

We leave for development in another paper [5] the application of this result for choices of Abelian group and subset B that are of interest because of their relevance to long-standing open problems in graph theory. In [9] Jaeger reformulates the Cycle Double Cover Conjecture, Fulkerson's Conjecture, the Petersen Flow Conjecture and related conjectures as B-flow problems. Also in the paper [5] we consider the example $B = \{-1, +1\}$ in the additive group $\mathbb{Z}_n$ of integers modulo n, because the number of B-flows here is equal to the number of Eulerian orientations of a (general) graph G when the vertex degrees of G all belong to $\{0, 1, \ldots, n-1, n+1\}$. As shown in [10], the number of Eulerian orientations of a graph cannot be expressed in the form $\text{hom}(\,\cdot\,, H)$ for a finite edge-weighted graph H, and therefore does not have a contractor [11, Example 2.6]. The interesting question arises of whether there is a "limiting contractor" for Eulerian orientations by taking contractors for $\{\pm 1\}$-flows modulo n as $n \to \infty$.

Theorem 3.3 gives an algorithm that gives a quantum path contractor for $\hom(\,\cdot\,, H)$ when H has a non-singular adjacency matrix A, which involves finding the minimal polynomial of A. Theorem 3.4 gives an algorithm that finds a quantum thick edge as a contractor for $\hom(\,\cdot\,, H)$ when the adjacency matrix of H has no diagonal entry equal to an off-diagonal entry, which involves using Lagrange interpolation. Left open by these theorems, and only partially covered by Theorems 4.1 and 4.2, is the following:

Problem 5.1. Give an algorithm to find a contractor for $\hom(\,\cdot\,, H)$ when H is an edge-weighted connected graph with adjacency matrix A (that has 0 as an eigenvalue and some diagonal entry equal to an off-diagonal entry).

By Theorem 1.4 of [11], in Problem 5.1 it suffices to restrict the search to quantum series-parallel graphs.

Recall that the quantum graph Z is a contractor for the graph parameter $\hom(\,\cdot\,, H)$ if $\hom_\phi(G(Z - K_1), H) = 0$ for each $G \in \mathcal{G}_2$ and $\phi : [2] \to V(G)$. This equality has (with a switch of homomorphisms from the left and from the right) a similar form to the inequality in the following problem:

$$\text{Given a } k\text{-labelled quantum graph } Y, \text{ is it the case that} \\ \hom_\phi(Y, G) \geq 0 \text{ for all } k\text{-labelled graphs } G \text{ and maps } \phi : [k] \to V(G)? \tag{5.1}$$

Lovász's Seventeenth Problem asks whether such a positive quantum graph Y can be expressed as the sum of squares of labelled quantum graphs (with multiplication being the disjoint union of labelled graphs followed by identification of like-labelled vertices). This has been recently answered in the negative by Hatami and Norine [7]: Problem (5.1) is in general undecidable. It would be interesting to know what the complexity of this problem is when Y is restricted to a subclass of quantum graphs, such as series-parallel graphs, and likewise the computational complexity of Problem 5.1.

Finally, we highlight a particular case of Problem 5.1, since it would be interesting to compare its solution to the three-term deletion–contraction–elimination recursion for the polynomial of Averbouch–Godlin–Makowsky [1].

Problem 5.2. Find a contractor for the graph parameter $\hom(G, K^1_{q-p} + K^y_p)$, where K^y_p denotes the complete graph on p vertices with a loop of weight y on each vertex (and similarly for K^1_{q-p}), and the $+$ denotes graph join. (This homomorphism function is equal to $\xi_G(q, y-1, (p-q)(y-1))$ in the notation of [1]).

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