On the number of B-flows of a graph

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Abstract

We exhibit explicit constructions of contractors for the graph parameter counting the number of B-flows of a graph, where B is a subset of a finite Abelian group closed under inverses. These constructions are of great interest because of their relevance to the family of B-flow conjectures formulated by Tutte, Fulkerson, Jaeger, and others.

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1. Introduction

This paper is a sequel to our previous article [3], where we answer a question raised by Lovász and B. Szegedy in [9] asking for a simple explicit construction for a contractor for the number of B-flows, where B is a subset of a finite Abelian group Γ closed under inverses. Here we present some applications of our result for choices of Abelian group Γ and subset B that are of interest because of their relevance to long-standing open problems in graph theory.

Contractors for graph parameters were introduced by Lovász and B. Szegedy in [9] as a generalization of the deletion-contraction identity for the Tutte polynomial and, in particular, for the chromatic polynomial and flow polynomial. Contractors were defined in the context of graph algebras, introduced by Freedman, Lovász and Schrijver in [2] to study homomorphism functions. Using the notion of a contractor, Lovász and B. Szegedy give a new characterization of those graph parameters that can be realized as the number of homomorphisms to an edge-weighted graph (also known as a partition function).

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Our main result in [3] uses the duality between flows and tensions in the context of finite Fourier analysis to provide a new and constructive proof for the existence of a contractor for the graph parameter counting the number of B-flows of a graph. This contractor is a linear combination of digons (forming a "quantum digon"). Dually, a contractor for the graph parameter counting the number of B-tensions of a graph is a linear combination of paths (forming a "quantum path").

Our proof in [3] is based on the following ideas that will also be of relevance here. First, the number of B-tensions of a graph can be written in terms of homomorphims to an edge-weighted Cayley graph H on vertex set Γ. Second, using the Fourier transform on Γ, the sum of values of a function on the set of tensions of a graph is equal to the sum of values of a related function on the set of flows. This allows us to also express the number of B-flows in terms of the number of homomorphisms to an edge-weighted Cayley graph $\widehat{H}$ on Γ, which is related to the graph H by taking the Fourier transform of its edge weights. Finally, we apply the general characterization by Lovász and Szegedy [9] of contractors for graph parameters counting the number of homomorphisms to an edge-weighted graph to our case of the edge-weighted Cayley graph $\widehat{H}$. This yields an explicit expression for a contractor for the number of B-flows of a graph.

In this paper we present a series of concrete instances of contractors for B-flows. These are chosen on account of their relevance to such problems as the Cycle Double Cover Conjecture, Fulkerson's Conjecture, the Petersen Flow Conjecture and related conjectures which were reformulated by Jaeger [7] as B-flow problems. Also we consider the example $B = \{-1, +1\}$ in the additive group $\mathbb{Z}_n$ of integers modulo n, because in this case the number of B-flows of a graph G with vertex degrees belonging to $\{0, 1, \ldots, n-1, n+1\}$ is equal to the number of Eulerian orientations of G. As shown in [8], the number of Eulerian orientations of a graph cannot be expressed as the number of homomorphisms to a finite edge-weighted graph H, and therefore does not have a contractor [9, Example 2.6]. The interesting question arises of whether there is a "limiting contractor" for Eulerian orientations by taking contractors for $\{-1, +1\}$-flows modulo n as n tends to infinity.

The explicit constructions of contractors presented in this paper are contained in Section 3. We use a number of definitions and results from [3], which are introduced in Section 2.

2. Preliminaries and known results

2.1. Homomorphism functions

Given two graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$, a *homomorphism* from G to H, written as $\psi : G \to H$, is a mapping $\psi : V(G) \to V(H)$ such that $\psi(u)\psi(v) \in E(H)$ whenever $uv \in E(G)$. The graph parameter $\hom(G, H)$ counting the number of homomorphisms from G to H can be generalized to edge-weighted graphs H as follows.

Let H be a graph with a real weight $\beta_H(ij)$ associated with each edge $ij \in E(H)$ and let G be an unweighted graph. The homomorphism function $\hom(G, H)$ is defined by

$$\hom(G, H) = \sum_{\psi:V(G)\to V(H)} \prod_{uv\in E(G)} \beta_H(\psi(u), \psi(v)).$$

Note that if we fix $U \subseteq V(G)$ and define for every function $\phi : U \to V(H)$ the weight

$$\hom_\phi(G,H) = \sum_{\substack{\psi:V(G)\to V(H)\\ \psi_{|U}=\phi}} \prod_{uv\in E(G)} \beta_H(\psi(u),\psi(v)),$$

then $\hom(G,H) = \sum_{\phi:U\to V(H)} \hom_\phi(G,H)$.

2.2. Flows and tensions

Let $G=(V,E)$ be a graph with an arbitrary fixed orientation of its edges. Denote by $c(G)$ the number of connected components of G, and let Γ be an additive Abelian group.

A Γ-*flow* of G is a function $g : E \to \Gamma$ such that for each vertex $v \in V$,

$$\sum_{\substack{e=uv\\ v \text{ head of } e}} g(e) - \sum_{\substack{e=uv\\ v \text{ tail of } e}} g(e) = 0.$$

If $B=-B$ is a subset of Γ then a B-*flow* of G is a Γ-flow taking values in B. In particular, when $B=\Gamma\setminus\{0\}$ a B-flow is called a *nowhere-zero* Γ-*flow* of G.

A Γ-*tension* $g : E \to \Gamma$ arises from a vertex Γ-colouring $f : V \to \Gamma$ of G, by setting $g(e) = f(v) - f(u)$ for each edge e with tail u and head v. A B-*tension* of G is a Γ-tension of G taking values in B.

Note that to define Γ-flows and Γ-tensions all that is needed is the structure of Γ as an additive Abelian group. In [3], we impose the extra ring structure on Γ in order to connect flows and tensions in the context of finite Fourier analysis. Here we only present the definitions needed to state this connection.

Let Γ be a finite additive Abelian group of order n. A *character* of Γ is a (group) homomorphism $\chi : \Gamma \to \mathbb{C}^\times$, where $\mathbb{C}^\times$ is the multiplicative group of the field of complex numbers. The set of characters of Γ form a group $\widehat{\Gamma}$ under pointwise multiplication which is isomorphic to Γ. For each $x \in \Gamma$, let χ_x denote the image of x under a fixed isomorphism $\Gamma \to \widehat{\Gamma}$. In particular, the trivial character χ_0 is defined by $\chi_0(y) = 1$ for all $y \in \Gamma$, and $\chi_{-x}(y) = \overline{\chi_x(y)}$, the bar denoting complex conjugation.

Denote by $\mathbb{C}^\Gamma$ the vector space over $\mathbb{C}$ of all functions from Γ to $\mathbb{C}$. This is an inner product space with Hermitian inner product defined for $\alpha, \beta \in \mathbb{C}^\Gamma$ by

$$\langle \alpha, \beta\rangle = \sum_{x\in\Gamma} \alpha(x)\overline{\beta(x)}.$$

For $\alpha \in \mathbb{C}^\Gamma$, the *Fourier transform* $\widehat{\alpha} \in \mathbb{C}^\Gamma$ is defined for $y \in \Gamma$ by

$$\widehat{\alpha}(y) = \langle \alpha, \chi_y\rangle = \sum_{x\in\Gamma} \alpha(x)\chi_y(-x). \tag{1}$$

The set $\{\delta_x : x \in \Gamma\}$ of indicator functions defined by

$$\delta_x(y) = \begin{cases} 1 & x = y, \\ 0 & x \neq y, \end{cases}$$

form an orthonormal basis for $\mathbb{C}^\Gamma$, with $\langle \delta_x, \delta_y \rangle = \delta_x(y)$. We extend the indicator function notation to subsets B of Γ, defining $\delta_B = \sum_{x \in B} \delta_x$.

Lemma 2.1. [3] *Let $B = -B$ be a subset of an additive Abelian group Γ. Let $H = \mathrm{Cayley}(\Gamma, B)$ be the graph on vertex set Γ with an edge joining vertices i and j if and only if $j - i \in B$, and let $\widehat{H}$ be the edge-weighted Cayley graph on vertex set Γ with edge ij having weight $\widehat{\delta}_B(j-i)$.*

Then the number of B-tensions of a graph $G = (V, E)$ is equal to $|\Gamma|^{-c(G)}\hom(G, H)$ and the number of B-flows of G is equal to $|\Gamma|^{-|V|}\hom(G, \widehat{H})$.

2.3. Contractors

A 2-*labelled graph* G is a finite graph (without loops but possibly having multiple edges) in which two vertices are labelled by numbers 1 and 2 (in [2] this notion is defined in general for k-labelled graphs, $k \geq 0$). Lovász and Szegedy [9] (see also [2]) restrict labels to be on distinct vertices, but we shall follow Schrijver [10] and allow a vertex to have both labels. When refering to 2-labelled graphs, we denote by K_n the 2-labelled complete graph on $n \geq 2$ vertices with labels located at two distinct vertices, and $\overline{K}_n$ its complement, consisting of n isolated vertices (two of them labelled). Let K_1 be a single vertex carrying both labels 1 and 2.

A 2-labelled *quantum graph* is a formal linear combination of 2-labelled graphs with coefficients in $\mathbb{R}$. Let $\mathcal{G}_2$ denote the set of 2-labelled quantum graphs and $\mathcal{G}_2^0$ the subset of 2-labelled quantum graphs whose labelled vertices are independent.

For two 2-labelled graphs X and Y, the *product* XY is defined by taking the disjoint union of X and Y and then identifying vertices which share the same label [9]. The product is associative and commutative, and extends linearly to $\mathcal{G}_2$. The identity for this multiplication on $\mathcal{G}_2$ is $\overline{K}_2$.

Definition 2.2. [9] *Let h be a graph parameter. A 2-labelled quantum graph $Z \in \mathcal{G}_2$ is a* contractor *for h if for all $X \in \mathcal{G}_2^0$ we have*

$$h(XZ) = h(XK_1).$$

The graph K_1 when applied as a product with X (giving the graph XK_1) identifies the vertices labelled 1 and 2 in X. Thus, informally, attaching a contractor Z at two non-adjacent vertices acts like identifying these vertices as far as the value of the graph parameter h is concerned.

Lemma 2.4 below is a fundamental tool for finding contractors for homomorphism functions. It uses the following matrix associated to every 2-labelled graph.

Let H be an edge-weighted graph with adjacency matrix $A = (\beta(ij))$. To every $X \in \mathcal{G}_2$ assign a $V(H) \times V(H)$-matrix $M(X) = M_H(X)$ with (i,j)-entry equal to $\hom_\phi(X, H)$ where $\phi(1) = i$ and $\phi(2) = j$. For instance, $M(K_1) = I$ where I is the identity matrix and $M(\overline{K}_2) = J$ where J is the all-one matrix.

Lemma 2.3. [3] *If H has adjacency matrix A then $M(P_k) = A^k$ where P_k denotes the path on k edges with endpoints labelled 1 and 2.*

Lemma 2.4. [9] *A 2-labelled quantum graph Z is a contractor for $\hom(\, . \, , H)$ if and only if $M(Z) = I$.*

Example 2.5. *Suppose H is a strongly regular graph with parameters (n, k, λ, μ), i.e., a k-regular simple graph on n vertices such that any two adjacent (resp. non-adjacent) vertices of H have exactly λ (resp. μ) common neighbours. The adjacency matrix A of H satisfies*

$$A^2 + (\mu - \lambda)A + (\mu - k)I = \mu J.$$

In this case Lemmas 2.3 and 2.4 give the following contractor Z for $\hom(\,\cdot\,, H)$*,*

$$(\mu - k)Z = \mu \overline{K}_2 - (\mu - \lambda)P_1 - P_2.$$

Compare [4, Proposition 8]*, giving a recursive method for computing* $\hom(G, H)$ *when G is series-parallel.*

Suppose now that the adjacency matrix A of H has rows and columns indexed by an additive Abelian group Γ of order n and has (i, j)-entry equal to $(\beta(i - j))$, where $\beta \in \mathbb{C}^\Gamma$ satisfies $\beta(-i) = \beta(i)$. Thus A is the adjacency matrix of an edge-weighted Cayley graph on Γ. The matrix A has eigenvectors $(\chi_i(j))_{j\in\Gamma}$ for each $i \in \Gamma$ with corresponding eigenvalue $\widehat{\beta}(i)$. Define $\widehat{A}$ to be the matrix with (i, j)-entry the Fourier transform $\widehat{\beta}(i - j)$. Let us call a matrix Γ-*circulant* if, like the matrix A, it takes the form $(\alpha(i - j))$ for some function $\alpha \in \mathbb{C}^\Gamma$.

Theorem 2.6. [3] *Suppose H is a connected graph with adjacency matrix a Γ-circulant matrix A and let $\widehat{H}$ be the edge-weighted graph on vertex set Γ with adjacency matrix $\widehat{A}$. Suppose further that A has eigenvector $\mathbf{1}$ with eigenvalue λ_1, has minimum polynomial $p_A(t)$, and that $\frac{p_A(t)}{t-\lambda_1} = q(t) = q_0 + q_1 t + \cdots + q_{\ell-1}t^{\ell-1}$. Then a contractor for* $\hom(\,\cdot\,, \widehat{H})$ *is given by*

$$Z = \frac{1}{q(\lambda_1)} \sum_{0\le k\le \ell-1} q_k P_1^k.$$

In [3], we obtain an explicit construction for a contractor for the number of B-flows by applying Lemma 2.1 and Theorem 2.6 to $H = \mathrm{Cayley}(\Gamma, B)$ (for additive Abelian group Γ, and where $B \subseteq \Gamma \setminus 0$ satisfies $-B = B$). In this case, the adjacency matrix A of H has (i, j)-entry $\delta_B(i - j)$ and eigenvalues $\widehat{\delta_B}(i)$ for $i \in \Gamma$. The minimum polynomial $p_A(t)$ of A has degree ℓ equal to the number of distinct values of $\widehat{\delta_B}(c)$ for $c \in \Gamma$. The largest eigenvalue of A is $\widehat{\delta_B}(0) = |B|$, belonging to the eigenvector $\mathbf{1}$, and

$$\frac{p_A(t)}{t - |B|} = q(t) = q_0 + q_1 t + \cdots + q_{\ell-1}t^{\ell-1}. \tag{2}$$

Theorem 2.7. [3] *Let Γ be an additive Abelian group of order n and suppose $H =$* $\mathrm{Cayley}(\Gamma, B)$ *is a connected graph with adjacency matrix A. Let $\widehat{H}$ be the edge-weighted graph on vertex set Γ with adjacency matrix $\widehat{A}$. Then, with the polynomial $q(t)$ defined as in Equation* (2) *above,*

(i) a contractor for the number of B-flows is given by

$$Z = \frac{n}{q(|B|)} \sum_{0\le k\le \ell-1} q_k P_1^k;$$

(ii) a contractor for $n^{c(G)}$ times the number of B-tensions of G is given by

$$Z = \frac{1}{q_0}\left[\frac{q(|B|)}{n}\overline{K}_2 - \sum_{1\le k\le \ell-1} q_k P_k\right].$$

Theorem 2.8. [3] *Let Γ be an additive Abelian group of order n and suppose $H =$ $\mathrm{Cayley}(\Gamma, B)$ has adjacency matrix A, and that $q(t)$ is defined as in Equation (2) above. Suppose H has r isomorphic connected components, i.e., its adjacency matrix A is permutation-equivalent to a matrix of the form $I \otimes A_1$ for $r \times r$ identity matrix I and some $n/r \times n/r$ matrix A_1. Then, a contractor for the number of B-flows is given by*

$$Z = \frac{n}{rq(|B|)} \sum_{0\le k\le \ell-1} q_k P_1^k.$$

3. Explicit constructions of contractors for B-flows

In this section we apply Theorems 2.7 and 2.8 to obtain concrete instances of contractors for the graph parameter counting the number of B-flows of a graph. In order to do this we need to find the ℓ distinct eigenvalues of the adjacency matrix of $\mathrm{Cayley}(\Gamma, B)$ (the values $\widehat{\delta_B}(x)$ for $x \in \Gamma$), discard the largest one ($|B|$), and then the kth elementary symmetric functions of the remaining $\ell - 1$ eigenvalues form the coefficients in the expression for a contractor as a linear combination of graphs $P_1^{\ell-1-k}$ (two vertices joined by $\ell - 1 - k$ edges).

We begin with two easy applications of Theorem 2.7: proper vertex colourings and nowhere-zero flows.

3.1. Proper vertex colourings and nowhere-zero flows

When the Abelian group Γ is order n and $B = \Gamma \setminus \{0\}$, the graph $H = \mathrm{Cayley}(\Gamma, B)$ is isomorphic to the complete graph K_n. The adjacency matrix of H is $A = J - I$ and has minimum polynomial $p_A(t) = (t+1)(t-n+1)$. Then $q(t) = p_A(t)/(t-n+1) = 1+t$ which gives $q(|B|) = n$ and $q_0 = q_1 = 1$.

It is well known that homomorphisms of a graph G to K_n are just the n-colourings of G (see [5]). By Theorem 2.7(ii), the quantum graph $\overline{K}_2 - P_1$ is a contractor for $n^{c(G)}$ times the number of B-tensions of G which is exactly the same as counting homomorphisms from G to K_n. Therefore, $\overline{K}_2 - P_1$ is a contractor for the number of proper vertex colourings (which amounts to the deletion–contraction identity for the chromatic polynomial).

Theorem 2.7(i) gives the quantum graph $\overline{K}_2 + P_1$ as a contractor for the number of nowhere-zero Γ-flows (that this is the case amounts to the deletion–contraction identity for the flow polynomial).

Note that these contractors are also obtained applying Lemmas 2.1, 2.3 and 2.4 since $A = J - I$, $\hat{A} = nI - J$ and $M(\overline{K}_2) = J$ (for proper vertex colourings only Lemmas 2.3 and 2.4 are needed).

3.2. Cycle double covers

A *cycle double cover* of a graph G is a family of cycles of G such that every edge appears in exactly two cycles of this family. (As usual in this paper, by a cycle is meant an even subgraph.) A cycle double cover is said to be *d-colourable* ($d \geq 2$) if its cycles can be coloured with d colours in such a way that not only is each edge contained in exactly two cycles but these cycles have different colours.

The Cycle Double Cover Conjecture says that every bridgeless graph has a cycle double cover. This conjecture was reformulated by Jaeger in [7] in terms of B-flows as follows.

Let Δ be an additive group of m elements, and Δ^d the d-fold Cartesian product. The *Hamming weight* $|x|$ of a given element $x = (x_1, \ldots, x_d) \in \Delta^d$ is defined by $|x| = \#\{i : x_i \neq 0\}$. The sets $S_r^d = \{x \in \Delta^d : |x| = r\}$ are called *shells* (of radius r).

We consider $m = 2$ ($\Delta = \mathbb{F}_2$). Let $F_{S_r^d}(G)$ denote the number of S_r^d-flows of G, where S_r^d is the shell of radius r in $\mathbb{F}_2^d$, i.e., the set of vectors in $\mathbb{F}_2^d$ with exactly r non-zero coordinates.

Conjecture 3.1. (Flow version of the double cover conjecture) [7]. *For every bridgeless graph there exists $d \geq 2$ such that $F_{S_2^d}(G) \neq 0$.*

Jaeger also mentions the stronger 5-colourable Cycle Double Cover Conjecture, i.e., that $F_{S_2^5}(G) \neq 0$ for every bridgeless graph G.

We now proceed to exhibit contractors for the number of d-colourable cycle double covers. To do this we need to compute the values $\widehat{\delta_{S_2^d}}(x)$ for $x \in \mathbb{F}_2^d$. In general, the values $\widehat{\delta_{S_r^d}}(x)$ for $x \in \Delta^d$ are given (see e.g. [6]) by evaluations of the *Krawtchouk polynomial* $K_r(w; d, m)$ of degree r, defined for $0 \leq r, w \leq d$ by

$$K_r(w; d, m) = [z^r](1 + (m-1)z)^{d-w}(1-z)^w = \sum_{0 \leq i \leq r} (-1)^i (m-1)^{r-i} \binom{w}{i} \binom{d-w}{r-i}.$$

In particular, $K_0(w; d, m) = 1$, $K_1(w; d, m) = (m-1)d - mw$ and $K_d(w; d, m) = (-1)^w (m-1)^{d-w}$. The Fourier transform of the indicator function $\delta_{S_r^d}$ of a shell of radius r is given by

$$\widehat{\delta_{S_r^d}}(x) = K_r(|x|; d, m).$$

When $m = 2$ ($\Delta = \mathbb{F}_2$),

$$K_r(w; d, 2) = \sum_{0 \leq i \leq r} (-1)^i \binom{w}{i} \binom{d-w}{r-i}.$$

As a warm-up we begin with a simple illustration of Theorem 2.8, which includes the case of 2-colourable cycle double covers on setting $d = 2$.

The set $S_d^d \subseteq \mathbb{F}_2^d$ consists of just a single element, namely the all-one vector of Hamming weight d. The graph $H = \mathrm{Cayley}(\mathbb{F}_2^d, S_d^d)$ comprises 2^{d-1} copies of K_2. Here $\widehat{\delta_{S_d^d}}(x) = K_d(|x|; d, 2) = (-1)^{|x|}$, from which we find $q(t) = t + 1$, and a contractor for the number of S_d^d-flows is given by Theorem 2.8 as

$$Z = \frac{2^d}{2^{d-1}q(1)}\left[P_1^1 + P_1^0\right] = P_1 + \overline{K}_2.$$

This, as it should be, is also a contractor for the number of nowhere-zero $\mathbb{F}_2$-flows.

3.2.1. 3-colourable cycle double covers

Let $S_2^3 \subseteq \mathbb{F}_2^3$. Then $H = \mathrm{Cayley}(\mathbb{F}_2^3, S_2^3)$ consists of 2 connected components isomorphic to K_4 (hence there are S_2^3-tensions of G if and only if there are $\mathbb{F}_4$-tensions of G). The adjacency matrix of H has eigenvalues $\widehat{\delta}_{S_2^3}(x) \in \{3, -1\}$, giving $q(t) = t + 1$. Thus, Theorem 2.8 gives the following contractor for the number of S_2^3-flows,

$$Z = \frac{8}{2 \cdot q(3)}[P_1^1 + P_1^0] = P_1 + \overline{K}_2.$$

Again, this is as it should be as S_2^3-flows are in one-to-one correspondence with nowhere-zero $\mathbb{F}_4$-flows (truncate the elements of $S_2^3 \subseteq \mathbb{F}_2^3$ in the last position; conversely, to a nowhere-zero $\mathbb{F}_2^2$-flow add a parity check digit to make a S_2^3-flow).

3.2.2. 4-colourable cycle double covers

Let $S_2^4 \subseteq \mathbb{F}_2^4$. Then $H = \mathrm{Cayley}(\mathbb{F}_2^4, S_2^4)$ consists of 2 isomorphic connected components, each a regular graph of degree $\binom{4}{2} = 6$ on 8 vertices (i.e., K_8 minus a perfect matching). The adjacency matrix of H has eigenvalues $\widehat{\delta}_{S_2^4}(x) \in \{6, 0, -2\}$, giving $q(t) = t(t+2)$. Then, by Theorem 2.8, a contractor for the number of S_2^4-flows is given by

$$Z = \frac{16}{2 \cdot q(6)}[P_1^2 + 2P_1^1] = \frac{1}{6}[C_2 + 2P_1].$$

3.2.3. d-colourable cycle double covers

We now consider the general case of d-colourable cycle double covers. In this case we have

$$\begin{aligned}\widehat{\delta_{S_2^d}}(x) &= K_2(|x|; d, 2) \\ &= \binom{d-|x|}{2} - |x|(d-|x|) + \binom{|x|}{2} \\ &= \binom{d}{2} + 2|x|(|x|-d).\end{aligned}$$

The adjacency matrix A of the Cayley graph $H = \mathrm{Cayley}(\mathbb{F}_2^d, S_2^d)$ has $\lceil \frac{d+1}{2} \rceil$ distinct eigenvalues, namely $\binom{d}{2} + 2w(w-d)$ for $0 \leq w \leq \lfloor \frac{d}{2} \rfloor$. The largest eigenvalue $\binom{d}{2} = |S_2^d|$ belongs to the eigenvector $\mathbf{1}$. In the notation of Theorem 2.8,

$$q(t) = \prod_{1 \leq w \leq \lfloor d/2 \rfloor} \left[t - \binom{d}{2} + 2w(d-w)\right],$$

and

$$q(\binom{d}{2}) = \begin{cases} 2^{d/2-1}d! & d \text{ even,} \\ 2^{(d-1)/2}(d-1)! & d \text{ odd.} \end{cases}$$

Since the elements of S_2^d generate precisely the even-weight vectors in $\mathbb{F}_2^d$, the graph $H = \mathrm{Cayley}(\mathbb{F}_2^d, S_2^d)$ has 2 connected components. This allows us to explicitly construct a contractor for $F_{S_2^d}(G)$, as we have done for $d \in \{2, 3, 4\}$ and we now do for the case $d = 5$.

3.2.4. 5-colourable cycle double covers

Let A be the adjacency matrix of $\mathrm{Cayley}(\mathbb{F}_2^5, S_2^5)$, which has 2 connected components, eigenvalues $10, 2, -2$, and $q(t) = (t-2)(t+2) = t^2 - 4$, i.e., $q_0 = -4, q_1 = 0, q_2 = 1$, and $q(10) = 96$. We also have $n = 2^5 = 32$ and so Theorem 2.8 gives the following contractor for $F_{S_2^5}(G)$,

$$Z = \frac{32}{2 \cdot q(10)}[P_1^2 - 4P_1^0] = \frac{1}{6}\left[C_2 - 4\overline{K}_2\right].$$

Thus we have $F_{S_2^5}(XK_1) = F_{S_2^5}(XZ) = \frac{1}{6}F_{S_2^5}(XC_2) - \frac{4}{6}F_{S_2^5}(X\overline{K}_2)$ for every $X \in \mathcal{G}_2^0$. If we write $G = XP_1$ and let $e \in E(G)$ be the edge with endpoints labelled 1 and 2, then $X\overline{K}_2 = X = G\backslash e$ and $XK_1 = G/e$. Suppose we write $G\|e$ for the operation that inserts an extra edge parallel to $e \in E(G)$. Then $XC_2 = XP_1^2 = G\|e$. In this notation,

$$F_{S_2^5}(G\|e) = 4F_{S_2^5}(G\backslash e) + 6F_{S_2^5}(G/e). \tag{3}$$

3.3. Fulkerson flows

Fulkerson's Conjecture is that in every bridgeless cubic graph G there is a family of six perfect matchings such that each edge appears in exactly two of them. Jaeger [7, Theorem 6.1] shows that Fulkerson's Conjecture is equivalent to the assertion that a bridgeless cubic graph has a S_4^6-flow, where $S_4^6 \subseteq \mathbb{F}_2^6$ comprises those vectors containing exactly four 1s. We follow [1] and call these types of flow "Fulkerson flows".

Let A be the adjacency matrix of $H = \mathrm{Cayley}(\mathbb{F}_2^6, S_4^6)$, which has 2 connected components, eigenvalues $15, -5, -1, 3$, and $q(t) = (t+5)(t+1)(t-3) = t^3 + 3t^2 - 13t - 15$, i.e., $q_0 = -15, q_1 = -13, q_2 = 3$, $q_3 = 1$ and $q(15) = 20 \cdot 16 \cdot 12$, $n = 2^6$. Hence, by Theorem 2.8 a contractor for $F_{S_4^6}(G)$ is given by

$$Z = \frac{64}{2 \cdot q(15)}[P_1^3 + 3P_1^2 - 13P_1^1 - 15P_1^0] = \frac{1}{120}\left[P_1^3 + 3C_2 - 13P_1 - 15\overline{K}_2\right].$$

3.4. Petersen flows

The Petersen Flow Conjecture of Jaeger [7] states that every bridgeless graph has a Petersen flow. As explained in e.g. [1], this conjecture implies the 5-colourable cycle double conjecture and other long-standing conjectures. Petersen flows are defined as follows.

Take $\Gamma = \mathbb{F}_2^6$. Let $C_1, \ldots, C_6$ be a basis of the cycle space of the Petersen graph $P = (V, E)$ and let $g_1, \ldots, g_6$ be the corresponding indicator vectors in $\mathbb{F}_2^E$ (where $|E| = 15$), which we shall regard as the rows of a matrix $(g_{i,e})_{1 \le i \le 6, e \in E}$. Let $B = \{(g_{1,e}, \ldots, g_{6,e})^T :$

$e \in E\} \subseteq \mathbb{F}_2^6$. Then $|B| = 15$ since for every pair of distinct edges there is a cycle in the basis which contains exactly one of them.

The indicator function δ_B has Fourier transform

$$\begin{aligned}\widehat{\delta_B}(a_1, \ldots, a_6) &= \sum_{e \in E} (-1)^{a_1 g_{1,e} + \cdots + a_6 g_{6,e}} \\ &= \#\{e : a_1 g_{1,e} + \cdots + a_6 g_{6,e} = 0\} - \#\{e : a_1 g_{1,e} + \cdots + a_6 g_{6,e} = 1\} \\ &= |E| - 2|C|,\end{aligned}$$

where C is the cycle with indicator vector $a_1 g_1 + \cdots + a_6 g_6$. Since a cycle of the Petersen graph has length $0, 5, 6, 8$ or 9, it follows that $H = \mathrm{Cayley}(\mathbb{F}_2^6, B)$ has eigenvalues $15, 5, 3, -1$ and -3. Thus

$$q(t) = (t+1)(t^2 - 9)(t - 5) = t^4 - 4t^3 - 14t^2 + 36t + 45,$$

and $n = 64, q(15) = 34560$. Since B contains a set of generators for $\mathbb{F}_2^6$, the graph $\mathrm{Cayley}(\mathbb{F}_2^6, B)$ is connected. Hence Theorem 2.7 gives a contractor Z for the number of B-flows,

$$Z = \frac{64}{34560} \left[P_1^4 - 4P_1^3 - 14P_1^2 + 36P_1 + 45\overline{K}_2 \right],$$

the fraction simplifying to $\frac{1}{540}$.

This construction of a contractor for Petersen flows extends more generally to B-flows when B is defined in a similar way via a basis for the cycle space of a graph $Q = (V, E)$.

Suppose Q is a graph of nullity $k = |E| - |V| + c(Q)$, having cycle space basis vectors $g_1, \ldots, g_k \in \mathbb{F}_2^E$, rows of the matrix $(g_{i,e})_{1 \le i \le k, e \in E}$. Let $B = \{(g_{1,e}, \ldots, g_{k,e})^T : e \in E\} \subseteq \mathbb{F}_2^k$, i.e., B is the set of columns of the matrix $(g_{i,e})$.

Claim: If Q is edge-3-connected then the vectors $(g_{1,e}, \ldots, g_{k,e})$, $e \in E$, are distinct, so that $|B| = |E|$.

Proof of Claim: Suppose $e, f \in E(Q)$ are such that $(g_{1,e}, \ldots, g_{k,e}) = (g_{1,f}, \ldots, g_{k,f})$. Suppose $Q - e - f$ is connected and let $e = uv$. Then there is a path joining u and v in $Q - e - f$, which together with the edge e forms a cycle in Q not containing f, contrary to hypothesis. Therefore $Q - e - f$ must be disconnected, yet this contradicts the fact that Q is edge 3-connected. Hence $(g_{1,e}, \ldots, g_{k,e}) \neq (g_{1,f}, \ldots, g_{k,f})$ when $e \neq f$. □

By a calculation similar to that carried out above for the Petersen graph, when Q is edge-3-connected the eigenvalues of $\mathrm{Cayley}(\mathbb{F}_2^k, B)$ are equal to $|E| - 2|C|$ for cycles C of Q.

Since the row rank of the matrix $(g_{i,e})$ is equal to k, the column rank of $(g_{i,e})$ is equal to k. Therefore $\mathrm{Cayley}(\mathbb{F}_2^k, B)$ has just 1 component, the set B spanning $\mathbb{F}_2^k$.

If $0 < c_1 < \ldots < c_r$ are the sizes of cycles in Q, we have

$$q(t) = \prod_{1 \le j \le r} (t - |E| + 2c_j),$$

and $q(|B|) = 2^r \prod_{1 \le j \le r} c_j$. By Theorem 2.7, a contractor for the number of B-flows is given by

$$Z = \frac{2^{k-r}}{c_1 c_2 \cdots c_r} \sum_{0 \le j \le r} q_j P_1^j,$$

where

$$q_{r-i} = (-1)^i \sum_{\substack{S \subseteq \{1,\ldots,r\} \\ |S|=i}} \prod_{s \in S} (|E| - 2c_s)$$

3.5. Nowhere-zero $n, 1$-flows

We finish our selection of explicit constructions of contractors with a sequence of graph parameters whose limit is the graph parameter counting Eulerian orientations. The number of Eulerian orientations is mentioned by Lovász and Szegedy [8] as an example of a graph parameter not expressible as the number of homomorphisms to a finite edge-weighted graph but which is expressible via graph limits as a homomorphism into a measurable function on the unit square (what might be called a "graphon parameter").

Let us take $\Gamma = \mathbb{Z}_n$, the integers modulo n, and $B = \{-1, +1\}$. The set of B-flows of a graph G is in one-one correspondence with the set of orientations of G in which at each vertex the indegree is congruent to the outdegree modulo n (also known as a *nowhere-zero $n, 1$-flow*). When $n = 3$, B-flows are nowhere-zero $\mathbb{Z}_3$-flows and as we have seen earlier a contractor is given by $\overline{K}_2 + P_1$. In case G is a 4-regular graph, B-flows of G correspond to Eulerian orientations of G. More generally, for arbitrary n, B-flows of a graph G all of whose vertex degrees belong to $\{0, 1, 2, \ldots, n-1, n+1\}$ correspond to Eulerian orientations of G.

The indicator function of B has Fourier transform $\widehat{\delta_B}(a) = \zeta^a + \zeta^{-a}$, where $\zeta = e^{2\pi i/n}$. When $n = 4$ these eigenvalues are $2, 0, -1$ and we find that $q(t) = t(t+2)$. By Theorem 2.7, $\frac{1}{2}P_1^2 + P_1$ is a contractor for the number of B-flows modulo 4. When $n = 5$ we have

$$q(t) = (t - \zeta - \zeta^{-1})(t - \zeta^2 - \zeta^{-2}) = t^2 + t - 1,$$

and Theorem 2.7 implies that $P_2 + P_1 - \overline{K}_2$ is a contractor for the number of B-flows modulo 5.

After similar calculations we obtain contractors for the number of nowhere-zero $n, 1$-flows for $n \le 9$, as displayed in Table 1.

Table 1:

n	$q(t)$	Contractor for nowhere-zero $n, 1$-flows
3	$1 + t$	$\overline{K}_2 + P_1$
4	$2t + t^2$	$P_1 + \frac{1}{2}P_1^2$
5	$-1 + t + t^2$	$-\overline{K}_2 + P_1 + P_1^2$
6	$-2 - t + 2t^2 + t^3$	$-\overline{K}_2 - \frac{1}{2}P_1 + P_1^2 + \frac{1}{2}P_1^3$
7	$-1 - 2t + t^2 + t^3$	$-\overline{K}_2 - 2P_1 + P_1^2 + P_1^3$
8	$-4t - 2t^2 + 2t^3 + t^4$	$-2P_1 - P_1^2 + P_1^3 + \frac{1}{2}P_1^4$
9	$1 - 2t - 3t^2 + t^3 + t^4$	$\overline{K}_2 - 2P_1 - 3P_1^2 + P_1^3 + P_1^4$

Generally, for B-flows modulo n we have

$$q(t) = \prod_{a\in\{1,\dots,\lfloor n/2\rfloor\}} (t-\zeta^a-\zeta^{-a}).$$

When n is odd,

$$q(2) = \prod_{a\in\{1,\dots,\frac{n-1}{2}\}} (1-\zeta^a)(1-\zeta^{-a}) = \prod_{a\in\{1,\dots,n-1\}} (1-\zeta^a) = n,$$

and when n is even $q(2)=2n$.

Since

$$\prod_{a\in\{1,\dots,n-1\}} (t-\zeta^a) = t^{n-1}+\cdots+t+1,$$

we have

$$\sum_{\substack{A\subseteq\{1,\dots,n-1\}\\ |A|=k}} \zeta^{\sum_{a\in A} a} = (-1)^k. \tag{4}$$

What we would like is a similar closed-form expression for the coefficients of $q(t)$. This will yield an explicit expression for a contractor for the number of nowhere-zero $n,1$-flows.

First we derive a formula for the coefficients of $q(t)$ for odd values of n. Afterwards we shall indicate the result for even n, which can be obtained by a similar method.

So let us take n odd. The coefficient of $t^{\frac{n-1}{2}-k}$ in

$$q(t) = \prod_{a\in\{1,\dots,\frac{n-1}{2}\}} (t-\zeta^a-\zeta^{-a})$$

is given by

$$[t^{\frac{n-1}{2}-k}]q(t) = (-1)^k \sum_{(\epsilon_1,\dots,\epsilon_k)\in\{\pm1\}^k} \ \sum_{1\le a_1<\cdots<a_k\le\frac{n-1}{2}} \zeta^{\epsilon_1a_1+\cdots+\epsilon_ka_k}.$$

If we set

$$c_k = \sum_{\substack{A\subseteq\{1,\dots,n-1\}\\ |A|=k,\, |A\cap -A|=0}} \zeta^{\sum_{a\in A} a},$$

where $-A=\{-a : a\in A\}$, then

$$[t^{\frac{n-1}{2}-k}]q(t) = (-1)^k c_k.$$

We have $c_0=1$, and $c_1=-1$ by Equation (4).

Claim: For $0\le k\le n-1$ and $\max\{0,k-\frac{n-1}{2}\}\le \ell\le k/2$,

$$\sum_{\substack{A\subseteq\{1,\dots,n-1\}\\ |A|=k,\, |A\cap -A|=2\ell}} \zeta^{\sum_{a\in A} a} = \binom{\frac{n-1}{2}-k+2\ell}{\ell} \sum_{\substack{B\subseteq\{1,\dots,n-1\}\\ |B|=k-2\ell,\, |B\cap -B|=0}} \zeta^{\sum_{b\in B} b}. \tag{5}$$

Proof of Claim: Each subset $B \subseteq \{1, \ldots, n-1\}$ with $|B| = k - 2\ell$ and $|B \cap -B| = 0$ extends to $\binom{\frac{n-1}{2}-k+2\ell}{\ell}$ sets A with $|A| = k$ and $|A \cap -A| = 2\ell$, and $A \setminus -A = B$. This is because there are $n - 1 - |B \cup -B| = n - 1 - 2k + 4\ell$ pairs $\{a, -a\}$ to choose from to extend B to a set A satisfying the desired conditions and we need to choose ℓ such pairs.

Conversely, given $A \subseteq \{1, \ldots, n-1\}$ with $|A| = k$ and $|A \cap -A| = 2\ell$, the set $B = A \setminus -A$ is uniquely determined and satisfies $|B| = k - |A \cap -A|, |B \cap -B| = 0$. $\square$

With this claim in hand, and since

$$\sum_{A \subseteq \{1,\ldots,n-1\}} \zeta^{\sum_{a\in A} a} = \sum_{\substack{0 \leq k \leq n-1 \\ \max\{0, k - \frac{n-1}{2}\} \leq \ell \leq \frac{k}{2}}} \sum_{\substack{A \subseteq \{1,\ldots,n-1\} \\ |A|=k,\, |A\cap -A|=2\ell}} \zeta^{\sum_{a\in A} a},$$

by Equations (4) and (5) we obtain

$$c_k = (-1)^k - \sum_{1 \leq \ell \leq \frac{k}{2}} \binom{\frac{n-1}{2} - k + 2\ell}{\ell} c_{k-2\ell}. \tag{6}$$

In order to solve the recurrence (6) we shall make use of the following identity:

$$\sum_{0 \leq j \leq k} (-1)^j \binom{x-j}{j} \binom{x-2j}{k-j} = 1, \tag{7}$$

for every $k \geq 0$.

Proof of Identity (7): For a polynomial $p(x)$ let $D(p(x)) = p(x) - p(x-1)$. Then for integer $k \geq 0$ we have

$$D^k(p(x)) = \sum_{0 \leq j \leq k} (-1)^j \binom{k}{j} p(x-j).$$

If $p(x)$ has degree k then $D^k(p(x))$ is a constant. In particular, setting $p(x) = \binom{x}{k}$, we have

$$\sum_{0 \leq j \leq k} (-1)^j \binom{k}{j} \binom{x-j}{k} = 1,$$

since the left-hand side is a constant, equal to 1 upon setting $x = k$. All that remains to do now is make the easy verification that

$$\binom{k}{j} \binom{x-j}{k} = \binom{x-j}{j} \binom{x-2j}{k-j},$$

and the identity (7) follows. $\square$

If we rewrite the recurrence in Equation (6) as

$$\sum_{0 \leq \ell \leq \frac{k}{2}} (-1)^k \binom{\frac{n-1}{2} - k + 2\ell}{\ell} c_{k-2\ell} = 1,$$

we see that, with boundary conditions $c_0 = 1$, $c_1 = -1$, its solution is given by

$$c_{2j} = (-1)^j \binom{\frac{n-1}{2} - j}{j},$$

upon setting $x = \frac{n-1}{2}$ in Equation (7), and

$$c_{2j+1} = (-1)^{j+1} \binom{\frac{n-1}{2} - j - 1}{j}.$$

upon setting $x = \frac{n-1}{2} - 1$ in Equation (7).

Thus, for odd n,

$$q(t) = \sum_{0 \le j < \frac{n-1}{4}} (-1)^j t^{\frac{n-1}{2} - 2j - 1} \left[\binom{\frac{n-1}{2} - j - 1}{j} + t \binom{\frac{n-1}{2} - j}{j} \right]. \tag{8}$$

A similar derivation shows that for even n, if we write $q(t) = \sum_{0 \le k \le \frac{n}{2}} (-1)^k c_k t^{\frac{n}{2} - k}$, then, for $0 \le j < \frac{n}{4}$,

$$c_{2j} = (-1)^j \binom{\frac{n}{2} - j - 1}{j},$$

$$c_{2j+1} = (-1)^{j+1} 2 \binom{\frac{n}{2} - j - 1}{j}.$$

Thus, for even n,

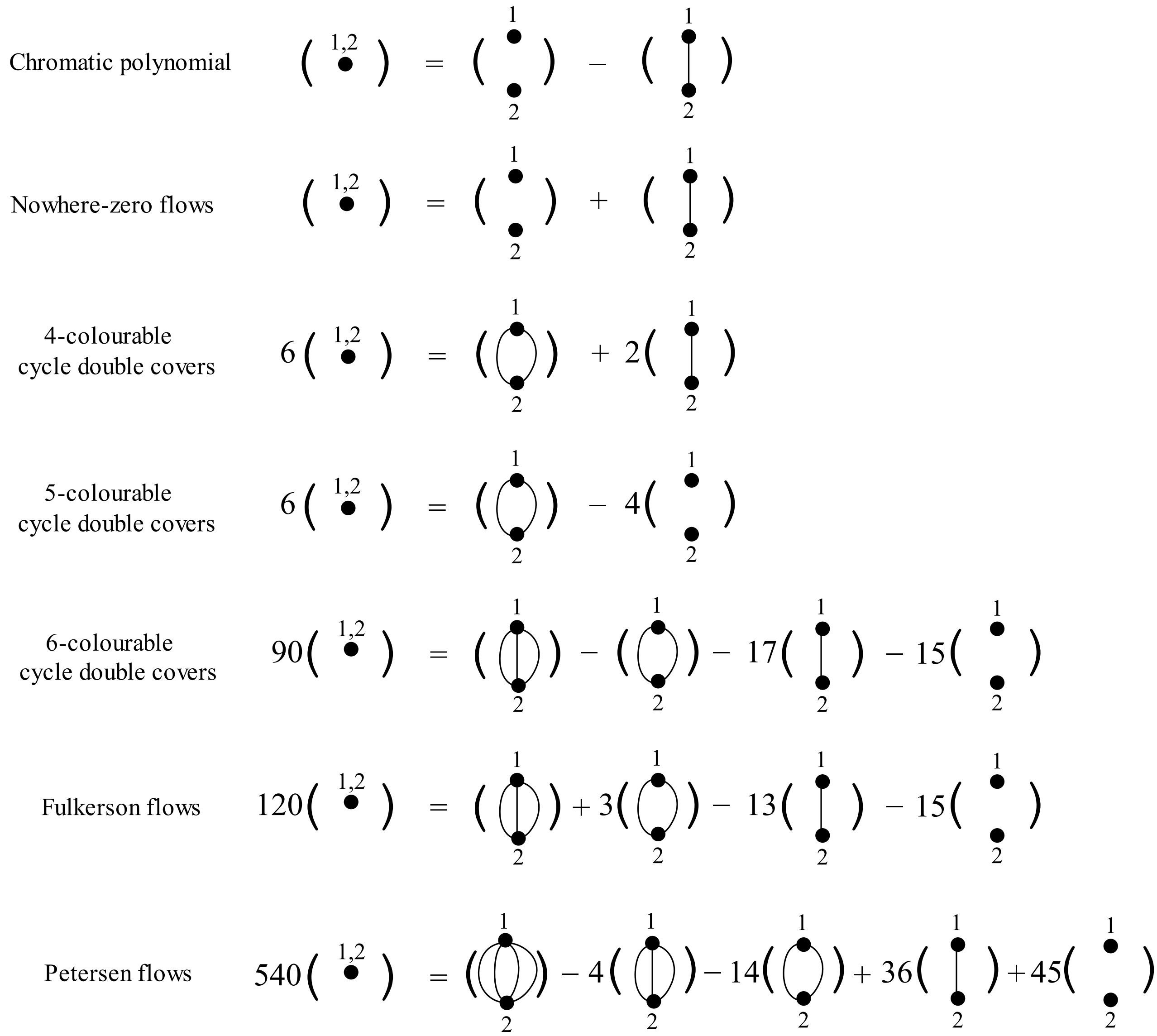
$$\begin{aligned} q(t) &= \sum_{0 \le j < \frac{n}{4}} (-1)^j t^{\frac{n}{2} - 2j - 1} \left[2\binom{\frac{n-1}{2} - j - 1}{j} + t \binom{\frac{n-1}{2} - j - 1}{j} \right] \qquad (9) \\ &= (t+2) \sum_{0 \le j < \frac{n}{4}} (-1)^j \binom{\frac{n}{2} - j - 1}{j} t^{\frac{n}{2} - 2j - 1}. \end{aligned}$$

By Theorem 2.7, for odd n the coefficient $[t^{\frac{n-1}{2} - k}]q(t)$, given by Equation (8), is equal to the coefficient of P_1^k in the quantum graph contractor for nowhere-zero $n,1$-flows. Likewise, for even n we can use Equation (9) to find $\frac{1}{2}[t^{\frac{n}{2} - k}]q(t)$, corresponding by Theorem 2.7 to the coefficient of P_1^k in the contractor for nowhere-zero $n,1$-flows. In other words, Equations (8) and (9) provide a continuation of Table 1 for general n.

4. Conclusion

In Figure 1 we display some of the contractors constructed in this paper. The contractors for the chromatic polynomial and flow polynomial allow inductive proofs of some of their properties. Likewise, when H is a strongly regular graph, the contractor in Example 2.5 for the graph parameter $\hom(\cdot, H)$ allows the recursive computation of $\hom(G, H)$ when G is a series-parallel graph, as shown in different terminology in [4]. Can the contractors in Figure 1 be used to make similar inductive arguments for the number of Petersen flows or 5-colourable cycle double covers, for instance? (For 5-colourable cycle double covers see also Equation (3).)

Figure 1: *Some examples of contractors.* The identities are as seen modulo the graph parameter in question. The first example is to be understood as saying that $\overline{K}_2 - K_2$ is a contractor for the chromatic polynomial, i.e., $P(XK_1;q) = P(X\overline{K}_2;q) - P(XK_2;q)$ for every graph X with two independent vertices labelled by numbers $1,2$, where K_1 has labels 1 and 2 on its single vertex, K_2 has one vertex label 1 and the other label 2, and similarly for $\overline{K}_2$. The graph XK_1 is the result of identifying the vertices labelled 1 and 2 in X, XK_2 adds an edge between 1 and 2, etc.



For the last of our contractor constructions, in Section 3.5, we began with the observation that the number of Eulerian orientations is expressible as the limit as n tends to infinity of the graph parameters counting the number of nowhere-zero $n,1$-flows. Equivalently, Eulerian orientations are counted by the number of homomorphisms to the Fourier dual of the infinite graph Cayley$(\mathbb{Z},\{-1,+1\})$, which is the graph on vertex set the real interval $[0,1]$ with edges ij weighted $2\cos 2\pi(i-j)$, equal to the Fourier transform of the edge weights on Cayley$(\mathbb{Z},\{-1,+1\})$. (The dual group of $\mathbb{Z}$ is the circle group, which in the usual way can be identifed with the additive group of reals in $[0,1]$ modulo 1.) This is the limiting graph density for the graph parameter counting Eulerian orientations that is described in [8, 9]. (In the notation of these two papers, in terms of "graphons" the number of Eulerian orientations of G can be expressed as $t(G, W)$, where $W(x, y) = 2\cos(2\pi(x - y))$.) Is there any sense in which we can say these contractors for nowhere-zero $n, 1$-flows converge to a "limit contractor" for Eulerian orientations?

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