The fractional chromatic number of triangle-free subcubic graphs*

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Abstract

Heckman and Thomas conjectured that the fractional chromatic number of any triangle-free subcubic graph is at most $14/5$. Improving on estimates of Hatami and Zhu and of Lu and Peng, we prove that the fractional chromatic number of any triangle-free subcubic graph is at most $32/11 \approx 2.909$.

1 Introduction

When considering the chromatic number of certain graphs, one may notice colourings which are best possible (in that they use as few colours as possible) but which are in some sense wasteful. For instance, an odd cycle cannot be properly coloured with two colours but can be coloured using three colours in such a way that the third colour is used only once.

Indeed if C_7 has vertices $v_1, v_2, v_3, \ldots, v_7$, then we can colour v_1, v_3, v_5 red, v_2, v_4, v_6 blue and v_7 green. If, however, our aim is instead to assign multiple colours to each vertex such that adjacent vertices receive disjoint lists of colours, then we could double-colour C_9 using five (rather than six) colours

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and triple-colour it using seven (rather than nine) colours in such a way that each colour is used exactly three times — colour v_i with colours $3i, 3i+1, 3i+2$ (mod 7). Asking for the minimum of the ratio of colours required to the number of colours assigned to each vertex gives us a generalisation of the chromatic number.

Alternatively, for a graph $G = (V, E)$ we can consider a function w assigning to each independent set of vertices I a real number $w(I) \in [0, 1]$. We call such a function a *weighting*. The weight $w[v]$ of a vertex $v \in V$ with respect to w is then defined to be the sum of $w(I)$ over all independent sets containing v. A weighting w is a *fractional colouring* of G if for each $v \in V$ $w[v] \geq 1$. The size $|w|$ of a fractional colouring is the sum of $w(I)$ over all independent sets I. The fractional chromatic number $\chi_f(G)$ is then defined to be the infimum of $|w|$ over all possible fractional colourings. We refer the reader to [8] for more information on fractional colourings and the related theory.

By a folklore result, the above two definitions of the fractional chromatic number are equivalent to each other and to a third, probabilistic, definition. It is this third definition which we will make most use of:

Lemma 1. *Let G be a graph and k a positive rational number. The following are equivalent:*

(i) $\chi_f(G) \leq k$,

(ii) there exists an integer N and a multi-set $\mathcal{W}$ of kN independent sets in G such that each vertex is contained in exactly N sets of $\mathcal{W}$,

(iii) there exists a probability distribution π on the independent sets of G such that for each vertex v, the probability that v is contained in a random independent set (with respect to π) is at least $1/k$.

In this paper, we consider the problem of bounding the fractional chromatic number of a graph that has maximum degree at most three (we call such graphs *subcubic*) and contains no triangle. Brooks' theorem (see, e.g., [1, Theorem 5.2.4]) asserts that such graphs have chromatic number at most three, and, thus, also have fractional chromatic number at most three. On the other hand, Fajtlowicz [2] observed that the independence number of the generalised Petersen Graph $P(7, 2)$ (Figure 1) equals 5, which implies that $\chi_f(P(7, 2)) = 14/5 = 2.8$.

In 2001, Heckman and Thomas [4] made the following conjecture:

Conjecture 2. *The fractional chromatic number of any triangle-free subcubic graph G is at most* 2.8.

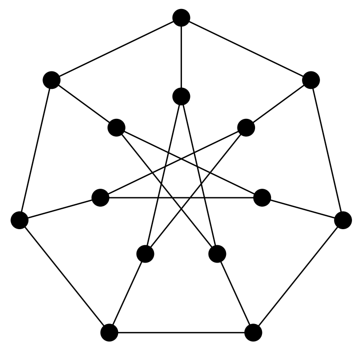


Figure 1: The generalised Petersen graph.

Conjecture 2 is based on the result of Staton [9] (see also [5, 4]) that any triangle-free subcubic graph contains an independent set of size at least $5n/14$, where n is the number of vertices of G. As shown by the graph $P(7, 2)$, this result is optimal.

Hatami and Zhu [3] proved that under the same assumptions, $\chi_f(G) \le 3 - 3/64 \approx 2.953$. More recently, Lu and Peng [7] were able to improve this bound to $\chi_f(G) \le 3 - 3/43 \approx 2.930$. We offer a new probabilistic proof which improves this bound as follows:

Theorem 3. *The fractional chromatic number of any triangle-free subcubic graph is at most* $32/11 \approx 2.909$.

In the rest of this section, we review the necessary terminology. The *length* of a path P, denoted by $|P|$, is the number of its edges. We use the following notation for paths. If P is a path and $x, y \in V(P)$, then xPy is the subpath of P between x and y. The same notation is used when P is a cycle with a specified orientation, in which case xPy is the subpath of P between x and y which follows x with respect to the orientation. In both cases, we write $d_P(x, y)$ for $|xPy|$.

We distinguish between edges in undirected graphs and arcs in directed graphs. If xy is an arc, then x is its *tail* and y its *head.*

If G is a graph and $X \subseteq V(G)$, then $\partial(X)$ denotes the set of edges of G with precisely one endvertex in X. For a subgraph $H \subseteq G$, we let $\partial(H)$ denote $\partial(V(H))$. The *neighbourhood* of a vertex u of G is the set $N(u)$ of its neighbours. We define $N[u] = N(u) \cup \{u\}$ and call this set the *closed neighbourhood* of u.

2 An algorithm

Let G be a simple cubic bridgeless graph. By a well-known theorem of Petersen (see, e.g., [1, Corollary 2.2.2]), G has a 2-factor. It will be helpful in our proof to pick a 2-factor with special properties, namely one satisfying the condition in the following result of Kaiser and Škrekovski [6, Corollary 4.5]:

Theorem 4 ([6])**.** *Every cubic bridgeless graph contains a 2-factor whose edge set intersects each inclusionwise minimal edge-cut in G of size 3 or 4.*

The condition in Theorem 4 will simplify the case analysis found in Section 6 by ruling out the cases where there are only 3 or 4 edges from a certain cycle of the 2-factor to the rest of the graph.

Among all 2-factors of G satisfying Theorem 4, choose a 2-factor F with as many components as possible. Let M be the complementary perfect matching. If $u \in V(G)$, then u' denotes the opposite endvertex of the edge of M containing u. We call u' the *mate* of u. We fix a reference orientation of each cycle of F, and let u_{+k} (where k is a positive integer) denote the vertex reached from u by following k consecutive edges of F in accordance with the fixed orientation. The symbol u_{-k} is defined symmetrically. We write u_+ and u_- for u_{+1} and u_{-1}. These vertices are referred to as the *F-neighbours* of u. To simplify the notation for mates we write, for example, u'_{+2} instead of $(u_{+2})'$.

Observation 5. *Let $w, z \in V(G)$. If both wz and w_+z_- are chords of the cycle of F containing w, then either $d_C(z, w)$ or $d_C(w_+, z_-)$ equals 3.*

Proof. We can use the chords to split the cycle of F into two shorter cycles and obtain a 2-factor with more components. If the length of both of these shorter cycles were at least 5, we would have a contradiction with the choice of F. Thus, one of $d_C(z, w)$ and $d_C(w_+, z_-)$ is less than or equal to 3. On the other hand, it cannot be less than 3 since G is assumed to be simple and triangle-free. □

We now describe **Algorithm 1**, an algorithm to construct a random independent set I in G. We will make use of a random operation, which we define next. An independent set is said to be *maximum* if no other independent set has larger cardinality. Given a set $X \subseteq V(G)$, we define $\Phi(X) \subseteq X$ as follows:

(a) if $F[X]$ is a path, then $\Phi(X)$ is either a maximum independent set of $F[X]$ or its complement in X, each with probability 1/2,

(b) if $F[X]$ is a cycle, then $\Phi(X)$ is a maximum independent set in $F[X]$, chosen uniformly at random,

(c) if $F[X]$ is disconnected, then $\Phi(X)$ is the union of the sets $\Phi(X \cap V(K))$, where K ranges over all components of $F[X]$.

In **Phase 1** of the algorithm, we choose an orientation $\vec{\sigma}$ of M by directing each edge of M independently at random, choosing each direction with probability 1/2. A vertex u is *active* (with respect to $\vec{\sigma}$) if u is a head of $\vec{\sigma}$, otherwise it is *inactive*.

An *active run* of $\vec{\sigma}$ is a maximal set R of vertices such that the induced subgraph $F[R]$ is connected and each vertex in R is active. Thus, $F[R]$ is either a path or a cycle. We let

$$\sigma^1 = \bigcup_R \Phi(R),$$

where R ranges over all active runs of $\vec{\sigma}$. The independent set I (which will be modified by subsequent phases of the algorithm and eventually become its output) is defined as σ^1. The vertices of σ^1 are referred to as those *added in Phase 1*. This terminology will be used for the later phases as well.

In **Phase 2**, we add to I all the active vertices u such that each neighbour of u is inactive. Observe that if an active run consists of a single vertex u, then u will be added to I either in Phase 1 or in Phase 2.

In **Phase 3**, we consider the set of all vertices of G which are not contained in I and have no neighbour in I. We call such vertices *feasible*. Note that each feasible vertex must be inactive. A *feasible run* R is defined analogously to an active run, except that each vertex in R is required to be feasible.

We define $\sigma^3 = \bigcup_R \Phi(R)$, where R now ranges over all feasible runs. All of the vertices of σ^3 are added to I.

In **Phase 4**, we add to I all the feasible vertices with no feasible neighbours. As with Phase 2, a vertex which forms a feasible run by itself is certain to be added to I either in Phase 3 or in Phase 4.

When referring to the random independent set I in Sections 3–6, we mean the set output from Phase 4 of Algorithm 1. It will, however, turn out that this set needs to be further adjusted in certain special situations. This augmentation step will be performed in Phase 5, whose discussion we defer to Section 7.

We represent the random choices made during an execution of Algorithm 1 by the triple $\sigma = (\vec{\sigma}, \sigma^1, \sigma^3)$ which we call a *situation*. Thus, the set Ω of all situations is the sample space in our probabilistic scenario. As usual for finite probabilistic spaces, an *event* is any subset of Ω.

Note that if we know the situation σ associated with a particular run of Algorithm 1, we can determine the resulting independent set $I = I(\sigma)$. We will say that an event $\Gamma \subseteq \Omega$ *forces* a vertex $u \in V(G)$ if u is included in $I(\sigma)$ for any situation $\sigma \in \Gamma$.

3 Templates and diagrams

Throughout this and the subsequent sections, let u be a fixed vertex of G, and let $v = u'$. Furthermore, let Z be the cycle of F containing u. All cycles of F are taken to have a preferred orientation, which enables us to use notation such as uZv for subpaths of these cycles.

We will analyze the probability of the event $u \in I(\sigma)$, where σ is a random situation produced by Algorithm 1. To this end, we classify situations based on what they look like in the vicinity of u.

A *template* in G is a 5-tuple $\Delta = (\vec{\Delta}, \Delta^1, \Delta^{\bar{1}}, \Delta^3, \Delta^{\bar{3}})$, where:

- $\vec{\Delta}$ is an orientation of a subgraph of M,
- Δ^1 and $\Delta^{\bar{1}}$ are disjoint sets of heads of $\vec{\Delta}$,
- Δ^3 and $\Delta^{\bar{3}}$ are disjoint sets of tails of $\vec{\Delta}$.

We set $\Delta^* = \Delta^1 \cup \Delta^{\bar{1}} \cup \Delta^3 \cup \Delta^{\bar{3}}$. The *weight* of Δ, denoted by $w(\Delta)$, is defined as

$$w(\Delta) = \left| E(\vec{\Delta}) \right| + |\Delta^*| .$$

A situation $\sigma = (\vec{\sigma}, \sigma^1, \sigma^3)$ *weakly conforms to* Δ if the following hold:

- $\vec{\Delta} \subseteq \vec{\sigma}$,
- $\Delta^1 \subseteq \sigma^1$ and
- $\Delta^{\bar{1}} \cap \sigma^1 = \emptyset$.

If, in addition,

- $\Delta^3 \subseteq \sigma^3$ and $\Delta^{\bar{3}} \cap \sigma^3 = \emptyset$,

then we say that σ *conforms to* Δ. The *event defined by* Δ, denoted by $\Gamma(\Delta)$, consists of all situations conforming to Δ.

By the above definition, we can think of Δ^1 and $\Delta^{\bar{1}}$ as specifying which vertices must or must not be added to I in Phase 1. However, note that a vertex u in an active run of length 1 will be added to I in Phase 2 even if $u \in \Delta^{\bar{1}}$. Similarly, Δ^3 and $\Delta^{\bar{3}}$ specify which vertices will (not) be added to I in Phase 3, with an analogous provision for feasible runs of length one.

To facilitate the discussion, we represent templates by pictorial *diagrams*. These usually show only the neighbourhood of the distinguished vertex u, and the following conventions apply for a diagram representing a template Δ:

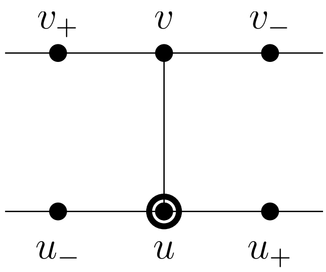


Figure 2: The location of neighbours of u and v.

- the vertex u is circled, solid and dotted lines represent edges and non-edges of G, respectively, dashed lines represent subpaths of F (see Figure 14),
- cycles and subpaths of F are shown as circles and horizontal paths, respectively, and the edge uv is vertical,
- u_- is shown to the left of u, while v_- is shown to the right of v (see Figure 2),
- the arcs of $\vec{\Delta}$ are shown with arrows,
- the vertices in Δ^1 ($\Delta^{\bar{1}}$, Δ^3, $\Delta^{\bar{3}}$, respectively) are shown with a star (crossed star, triangle, crossed triangle, respectively),
- only one endvertex of an arc may be shown (so an edge of G may actually be represented by one or two arcs of the diagram), but the other endvertex may still be assigned one of the above symbols.

An arc with only one endvertex in a diagram is called an *outgoing* or an *incoming* arc, depending on its direction. A diagram is *valid* in a graph G if all of its edges are present in G, and each edge of G is given at most one orientation in the diagram. Thus, a diagram is valid in G if and only if it determines a template in G. An event defined by a diagram is *valid* in G if the diagram is valid in G.

A sample diagram is shown in Figure 3. The corresponding event (more precisely, the event given by the corresponding template) consists of all situations $(\vec{\sigma}, \sigma^1, \sigma^3)$ such that v, v_+, u_{-2} and u'_+ are heads of $\vec{\sigma}$, σ^1 includes v_+ and u_{-2} but does not include u'_+, and σ^3 includes u.

Let us call a template Δ *admissible* if $\Delta^3 \cup \Delta^{\bar{3}}$ is either empty or contains only u, and in the latter case, u is feasible in any situation weakly conforming to Δ. All the templates we consider in this paper will be admissible. Therefore, we state the subsequent definitions and results in a form restricted to this case.

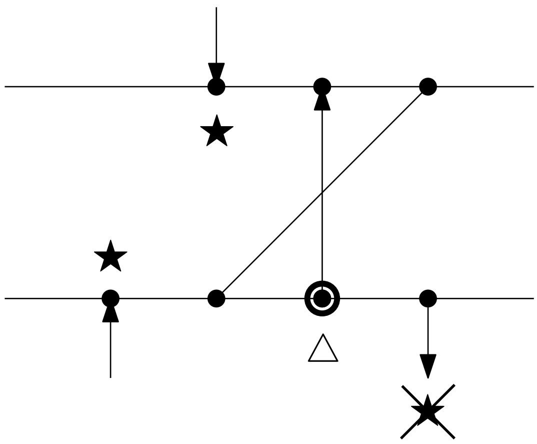


Figure 3: A diagram.

We will need to estimate the probability of an event defined by a given template. If it were not for the sets Δ^1, $\Delta^{\bar{3}}$ etc., this would be simple as the orientations of distinct edges represent independent events. However, the events, say, $u_1 \in \Delta^1$ and $u_2 \in \Delta^1$ (where u_1 and u_2 are vertices) are in general not independent, and the amount of their dependence is influenced by the orientations of certain edges of F. To keep the dependence under control, we introduce the following concept.

A *sensitive pair* of a template Δ is an ordered pair (x, y) of vertices in $\Delta^1 \cup \Delta^{\bar{1}}$, such that x and y are contained in the same cycle W of F, the path xWy has no internal vertex in Δ^* and one of the following conditions holds:

(a) $x, y \in \Delta^1$ or $x, y \in \Delta^{\bar{1}}$, $x \neq y$, the path xWy has odd length and contains no tail of $\vec{\Delta}$,

(b) $x \in \Delta^1$ and $y \in \Delta^{\bar{1}}$ or vice versa, the path xWy has even length and contains no tail of $\vec{\Delta}$,

(c) $x = y \in \Delta^1$, W is odd and contains no tail of $\vec{\Delta}$.

Sensitive pairs of the form (x, x) are referred to as *circular*, the other ones are *linear*.

A sensitive pair (x, y) is *k-free* (where k is a positive integer) if xWy contains at least k vertices which are not heads of $\vec{\Delta}$. Furthermore, any pair of vertices which is not sensitive is considered k-free for any integer k.

We define a number $q(\Delta)$ in the following way: If $u \in \Delta^3$ and Z is an odd cycle, then $q(\Delta)$ is the probability that all vertices of Z are feasible with respect to a random situation from $\Gamma(\Delta)$; otherwise, $q(\Delta)$ is defined as 0.

Observation 6. *Let Δ be a template in G. Then:*

(i) $q(\Delta) = 0$ if $u \notin \Delta^3$ or Z contains a head of $\vec{\Delta}$ or Z is even,

(ii) $q(\Delta) \le 1/2^t$ *if* Z *contains at least* t *vertices which are not tails of* $\vec{\Delta}$.

The following lemma is a basic tool for estimating the probability of an event given by a template.

Lemma 7. *Let* G *be a graph and* Δ *an admissible template in* G *such that:*

(i) Δ *has* ℓ *linear sensitive pairs, the* i*-th of which is* x_i*-free* ($i = 1, \dots, \ell$)*, and*

(ii) Δ *has* c *circular sensitive pairs, the* i*-th of which is* y_i*-free* ($i = 1, \dots, c$)*.*

Then

$$\mathbf{P}(\Gamma(\Delta)) \ge \Big(1 - \sum_{i=1}^{\ell} \frac{1}{2^{x_i}} - \sum_{i=1}^{c} \frac{1}{5 \cdot 2^{y_i}} - \frac{q(\Delta)}{5}\Big) \cdot \frac{1}{2^{w(\Delta)}}.$$

Proof. Consider a random situation σ. We need to estimate the probability that σ conforms to Δ. We begin by investigating the probability P_1 that σ weakly conforms to Δ.

In Phase 1, the orientation $\vec{\sigma}$ is chosen by directing each edge of M independently at random, each direction being chosen with probability 1/2. Therefore, the probability that the orientation of each edge in the subgraph specified by $\vec{\Delta}$ agrees with the orientation chosen at random is $(1/2)^{|E(\vec{\Delta})|}$.

As noted above, the sets $\Delta^1, \Delta^{\bar{1}}, \Delta^3$ and $\Delta^{\bar{3}}$ prescribe vertices to be added or not added in Phases 1 and 3 of the algorithm.

Suppose, for now, that every active run R has $\big|R \cap (\Delta^1 \cup \Delta^{\bar{1}})\big| = 1$ and is either a path or an even cycle. Then a given vertex in Δ^1 is added in Phase 1 with probability 1/2. Likewise, a given vertex in $\Delta^{\bar{1}}$ is not added in Phase 1 with probability 1/2. Indeed, R has either one or two maximum independent sets and $\Phi(R)$ either chooses either between the maximum independent set and its complement or between the two maximum independent sets.

There are $|\Delta^1|$ vertices required to be added in Phase 1 and $\big|\Delta^{\bar{1}}\big|$ vertices required to not be added in Phase 1. These events are independent each with probability 1/2, giving the resultant probability

$$\mathbf{P}(\Delta^1 \subseteq \sigma_1, \Delta^{\bar{1}} \cap \sigma_1 = \emptyset) = \Big(\frac{1}{2}\Big)^{|\Delta^1| + |\Delta^{\bar{1}}|}. \tag{1}$$

The probability P_1 is obtained by multiplying (1) by $(1/2)^{|E(\vec{\Delta})|}$.

We now assess the probability that σ conforms to Δ under the assumption that it conforms weakly. By this assumption and the fact that Δ is admissible, u is feasible with respect to σ. Let R be the feasible run containing u.

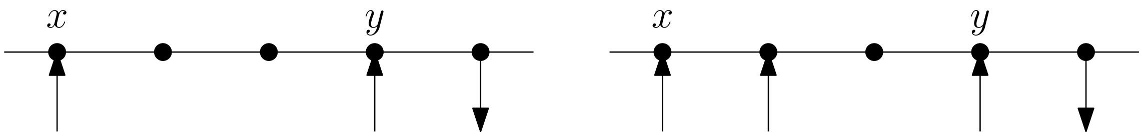


Figure 4: The probability that x and y are in distinct active runs in a conforming random situation is: (a) 3/4, (b) 1/2.

Suppose that R is a path or an even cycle. Then if $u \in \Delta^3$, it is added in Phase 3 with probability 1/2, and if $u \in \Delta^{\bar{3}}$, it is not added in Phase 3 with probability 1/2. Since u is the only vertex allowed in $\Delta^3 \cup \Delta^{\bar{3}}$, we obtain

$$\mathbf{P}(\Delta^3 \subseteq \sigma^3, \Delta^{\bar{3}} \cap \sigma^3 = \emptyset) = \begin{cases} \frac{1}{2} & \text{if } u \in \Delta^3 \cup \Delta^{\bar{3}}, \\ 1 & \text{otherwise.} \end{cases}$$

The assumption that u is feasible whenever σ weakly conforms to Δ implies that the addition of u to σ^3 is independent of the preceding random choices.

Observe that we can relax the assumptions above to allow, for instance, $|R \cap \Delta^1| \geq 1$, provided that the vertices of Δ^1 are appropriately spaced. Suppose that x, y are in the same component W of F and all vertices of xWy are active after the choice of orientations in Phase 1. Let R be the active run R containing xWy.

Observe that if $d_W(x, y)$ is even, Φ will choose both x and y with probability 1/2 for addition to I in Phase 1, an increase compared to the probability 1/4 if they are in different active runs. On the other hand, if $d_W(x, y)$ is odd, then the probability of adding both x and y is zero as x and y cannot both be contained in $\Phi(R)$. Thus, if $x, y \in \Delta^1$ and $d_W(x, y)$ is odd, then x and y must be in distinct active runs with respect to any situation conforming to Δ. As a result, we will in general get a lower value for the probability in (1); the estimate will depend on the sensitive pairs involved in Δ.

Let (x, y) be a k-free sensitive pair contained in a cycle W of F, and let the internal vertices of xWy which are not heads of Δ be denoted by $x_1, \dots, x_k$. Suppose that (x, y) is of type (a); say, $x, y \in \Delta^1$. The active runs of x and y with respect to σ will be separated if we require that at least one of $x_1, x_2, ..., x_k$ is the tail of an arc of $\vec{\sigma}$, which happens with probability $1 - (1/2)^k$. The same computation applies to a sensitive pair of type (b).

Now suppose that (x, x) is sensitive of type (c), i.e., x is the only member of Δ^1 belonging to an odd cycle W of length ℓ. If some vertex of W is the tail of an arc of $\vec{\sigma}$, then x will be added in Phase 1 with probability 1/2 as usual. It can happen, however (with probability $1/2^{\ell-1}$), that all the vertices of W are heads in $\vec{\sigma}$, in which case $\Phi(V(W))$ is one of ℓ maximum independent sets in W. If this happens, x will be added to I with probability $1/2 \cdot (\ell-1)/\ell \geq 2/5$ rather than $1/2$; this results in a reduction in $\mathbf{P}(\Gamma(\Delta))$ of at most $1/5 \cdot 1/2^{\ell-1} \cdot 1/2^{w(\Delta)}$.

Finally, let us consider the situation where $u \in \Delta^3$ and the feasible run containing u is cyclic, that is, the case where every vertex in C_u is feasible. If Z is even, then this has no effect as u is still added in Phase 3 with probability $1/2$. If Z is odd, then u is added in Phase 3 with probability at least $2/5$ instead. Thus, if the probability of all vertices in C_u being feasible is $q(\Delta)$, then the resultant loss of probability from $\mathbf{P}(\Gamma(\Delta))$ is at most $q(\Delta)/(5 \cdot 2^{w(\Delta)})$.

Putting all this together gives:

$$\mathbf{P}(\Gamma(\Delta)) = \left(1 - \sum_{i=1}^{\ell} \frac{1}{2^{x_i}} - \sum_{i=1}^{c} \frac{1}{5 \cdot 2^{y_i}} - \frac{q(\Delta)}{5}\right) \cdot \frac{1}{2^{w(\Delta)}}$$

as required. □

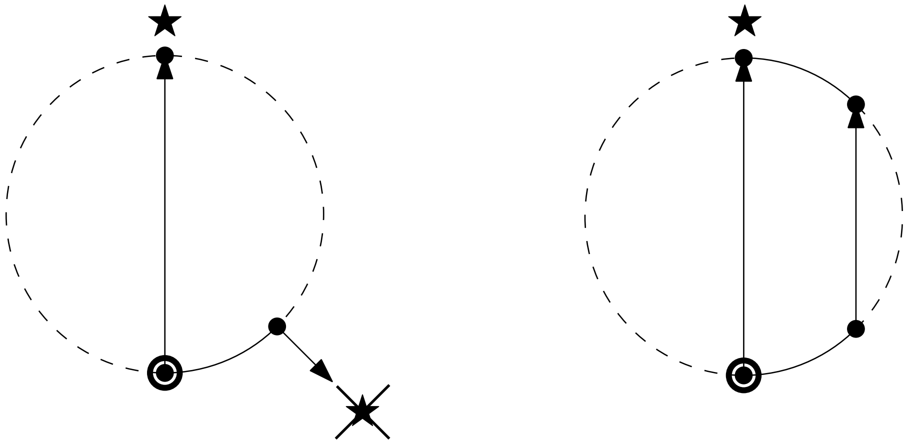
We remark that by a careful analysis of the template in question, it is sometimes possible to obtain a bound better than that given by Lemma 7; however, the latter bound will usually be sufficient for our purposes.

A template without any sensitive pairs is called *weakly regular*. If a weakly regular template Δ has $q(\Delta) = 0$, then it is *regular*. By Lemma 7, if Δ is a regular template, then $\mathbf{P}(\Gamma(\Delta)) \geq 1/2^{w(\Delta)}$. When using Lemma 7 in this way, we will usually just state that the template in question is regular and give its weight, and leave the straightforward verification to the reader.

The analysis is often more involved if sensitive pairs are present. To allow for a brief description of a template Δ, we say that Δ is *covered (in G) by* ordered pairs of vertices (x_i, y_i), where $i = 1, \ldots, k$, if every sensitive pair of Δ is of the form (x_i, y_i) for some i. In most cases, our information on the edge set of G will only be partial; although we will not be able to tell for sure whether any given pair of vertices is sensitive, we will be able to restrict the set of possibly sensitive pairs.

For brevity, we also use $(x, y)^\ell$ to denote an ℓ-free pair of vertices (x, y). Thus, we may write, for instance, that a template Δ is covered by pairs $(x, y)^2$ and $(z, z)^4$. By Lemma 7, we then have $\mathbf{P}(\Gamma(\Delta)) \geq 1/2^{w(\Delta)} \cdot (1 - 1/4 - 1/80)$.

In some cases, the structure of G may make some of the symbols in a diagram redundant. For instance, consider the diagram in Figure 5(a) and let R_1 be the event corresponding to the associated template Δ_1. Since the weight of Δ_1 is 4, Lemma 7 implies a lower bound for $\mathbf{P}(R_1)$ which is slightly below $1/16$. However, if we happen to know that the mate of u_+ is v_-, then we can remove the symbol at u'_+; the resulting diagram encodes the same event and comes with a better bound of $1/8$. We will describe this situation



(a) Δ_1. (b) Δ_2.

Figure 5: The symbol at u'_+ in the diagram defining the template Δ_1 becomes removable if we add the assumption that $u'_+ = v_-$.

by saying that the symbol at u'_+ in the diagram for Δ_1 is *removable* (under the assumption that $u'_+ = v_-$).

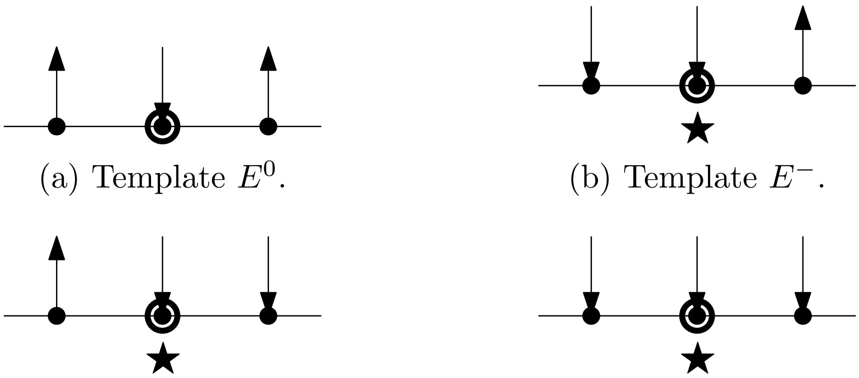
We extend the terminology used for templates to events defined by templates. Suppose that Δ is a template in G. The properties of $\Gamma(\Delta)$ simply reflect those of Δ. Thus, we say that the event $\Gamma(\Delta)$ is *regular* (*weakly regular*) if Δ is regular (weakly regular), and we set $q(\Gamma(\Delta)) = q(\Delta)$. A pair of vertices is said to be *k-free* for $\Gamma(\Delta)$ if it is k-free for Δ; $\Gamma(\Delta)$ is *covered by* a set of pairs of vertices if Δ is.

4 Events forcing a vertex

In this section, we build up a repertoire of events forcing the distinguished vertex u. (Recall that u is forced by an event Γ if u is contained in $I(\sigma)$ for every situation $\sigma \in \Gamma$.) In our analysis, we will distinguish various cases based on the local structure of G and show that in each case, the total probability of these events (and thus the probability that $u \in I$) is large enough.

Suppose first that σ is a situation for which u is active. By the description of Algorithm 1, we will have $u \in I$ if either both u_+ and u_- are inactive, or $u \in \sigma^1$. Thus, each of the templates $E^0, E^-, E^+, E^\pm$ represented by the diagrams in Figure 6 defines an event which forces u. These events (which will be denoted by the same symbols as the templates, e.g., E^0) are pairwise disjoint. Observe that by the assumption that G is simple and triangle-free, each of the diagrams is valid in G.

It is not difficult to estimate the probabilities of these events. The event E^0 is regular of weight 3, so $\mathbf{P}(E^0) \geq 1/8 = 32/256$ by Lemma 7. Similarly,



(a) Template E^0. (b) Template E^-. (c) Template E^+. (d) Template $E^{\pm}$.

Figure 6: Some templates defining events which force u.

E^+ and E^- are regular of weight 4 and have probability at least 16/256 each. The weakly regular event $E^{\pm}$ has weight 4 and the only potentially sensitive pair is (u, u). If the pair is sensitive, the length of Z must be odd and hence at least 5; thus, the pair is 2-free. By Lemma 7,

$$\mathbf{P}(E^{\pm}) \geq \frac{1}{16} \cdot \frac{19}{20} = 15.2/256.$$

Note that if Z has a chord (for instance, uv), then $E^{\pm}$ is actually regular, which improves the above estimate to 16/256.

By the above,

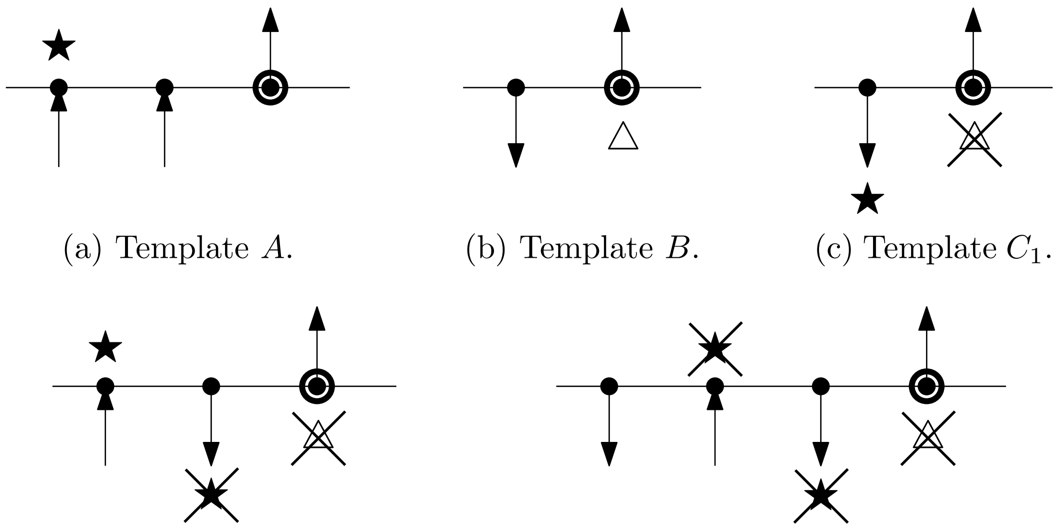
$$\mathbf{P}(E^0 \cup E^+ \cup E^- \cup E^{\pm}) \geq \frac{32 + 16 + 16 + 15.2}{256} = \frac{79.2}{256}.$$

These events cover most of the situations where $u \in I$. To prove Theorem 3, we will need to find other situations which also force u and their total probability is at least about one tenth of the above. Although this number is much smaller, finding the required events turns out to be a more difficult task.

Since Figure 6 exhausts all the possibilities where u is active, we now turn to the situations where u is inactive.

Assume an event forces u although u is inactive. We find that if u_- is active, then u_{-2} must be added in Phase 1. If u_- is inactive, then there are several configurations which allow u to be forced, for instance if u is added in Phase 3. However, the result also depends on the configurations around u_+ and v. We will express the events forcing u as combinations of certain 'primitive' events.

Let us begin by defining templates A, B, C_1, C_2, C_3 (so called *left templates*). We remind the reader that the vertex v is the mate of u. Diagrams corresponding to the templates are given in Figure 7:



(a) Template A. (b) Template B. (c) Template C_1.

(d) Template C_2. (e) Template C_3.

Figure 7: Left templates.

template	heads of $\vec{\sigma}$	other conditions
A	v, u_-, u_{-2}	$u_{-2} \in \sigma^1$
B	v, u'_-	$u \in \sigma^3$
C_1	v, u'_-	$u \notin \sigma^3, u'_- \in \sigma^1$
C_2	v, u'_-, u_{-2}	$u \notin \sigma^3, u'_- \notin \sigma^1, u_{-2} \in \sigma^1$
C_3	v, u'_-, u_{-2}, u'_{-3}	$u \notin \sigma^3, u'_- \notin \sigma^1, u_{-2} \notin \sigma^1$

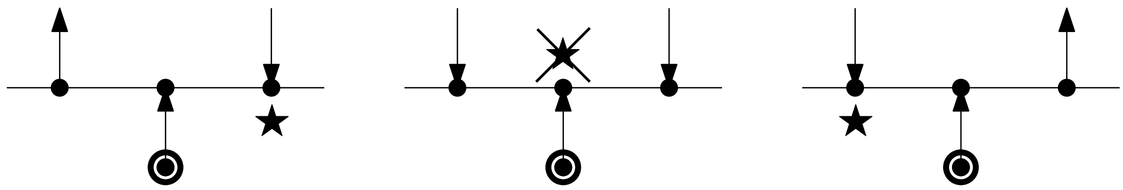
In addition, for $P \in \{A, B, C_1, C_2, C_3\}$, the template P^* is obtained by exchanging all '$-$' signs for '$+$' in this description. These are called *right templates*. In our diagrams, templates such as A or C_1 restrict the situation to the left of u, while templates such as A^* or C_1^* restrict the situation to the right.

We also need primitive templates related to v and its neighbourhood (*upper templates*), for the configuration here is also relevant. These are simpler (see Figure 8):

template	heads of $\vec{\sigma}$	other conditions
D^-	v, v_-, v'_+	$v_- \in \sigma^1$
D^0	v, v_-, v_+	$v \notin \sigma^1$
D^+	v, v'_-, v_+	$v_+ \in \sigma^1$

We can finally define the templates obtained from the left, right and upper events as their combinations. More precisely, for $P, Q \in \{A, B, C_1, C_2, C_3\}$ and $R \in \{D^-, D^0, D^+\}$, we define PQR to be the template Δ such that

$$\vec{\Delta} = \vec{P} \cup \vec{Q}^* \cup \vec{R},$$
$$\Delta^1 = P^1 \cup (Q^*)^1 \cup R^1,$$



(a) Template D^-. (b) Template D^0. (c) Template D^+.

Figure 8: Upper templates.

and so on for the other constituents of the template. The same symbol PQR will be used for the event defined by the template. If the result is not a legitimate template (for instance, because an edge is assigned both directions, or because u is required to be both in Δ^3 and $\Delta^{\bar{3}}$), then the event is an empty one and is said to be *invalid*, just as if it were defined by an invalid diagram.

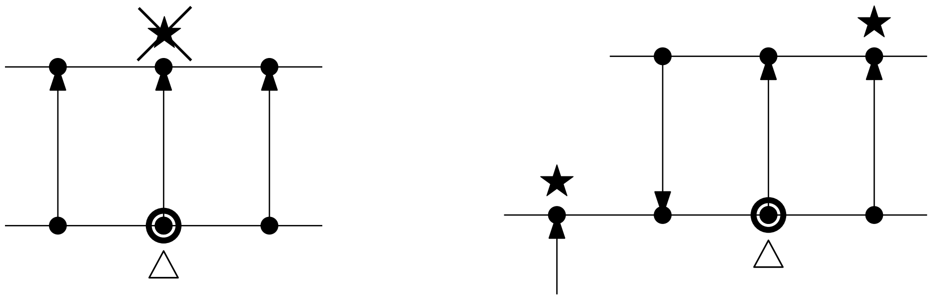
Let Σ be the set of all valid events PQR given by the above templates. Thus, Σ includes, e.g., the events AAD^0 or BC_1D^+. However, some of them (such as BC_1D^+) may be invalid, and the probability of others will in general depend on the structure of G. We will examine this dependence in detail in the following section. It is not hard to check (using the description of Algorithm 1) that each of the valid events in Σ forces u and also that each of them is given by an admissible template, as defined in Section 3.

5 Analysis: uv is not a chord

We are going to use the setup of the preceding sections to prove Theorem 3 for a cubic bridgeless graph G. Recall that v denotes the vertex u' and Z denotes the cycle of F containing u. If we can show that $\mathbf{P}(u \in I) \geq 11/32$, then by Lemma 1, $\chi_f(G) \leq 32/11$ as required. Thus, our task will be accomplished if we can present disjoint events forcing the fixed vertex u whose probabilities sum up to at least $11/32 = 88/256$. It will turn out that this is not always possible, which will make it necessary to use a compensation step discussed in Section 7.

In this section, we begin with the case where v is contained in a cycle $C_v \neq Z$ of F (that is, uv is not a chord of Z). We define a number $\varepsilon(u)$ as follows:

$$\varepsilon(u) = \begin{cases} 1 & \text{if } uv \text{ is contained in a 4-cycle,} \\ 0 & \text{if } u \text{ has no } F\text{-neighbour contained in a 4-cycle intersecting } C_v, \\ -1 & \text{otherwise.} \end{cases}$$



(a) BBD^0. (b) ABD^-.

Figure 9: The events used in Case 1 of the proof of Lemma 8.

The vertices with $\varepsilon(u) = -1$ will be called *deficient of type 0*.

The end of each case in the proof of the following lemma is marked by ▲.

Lemma 8. *If uv is not a chord of Z, then*

$$\mathbf{P}(u \in I) \geq \frac{88 + \varepsilon(u)}{256}.$$

Proof. As observed in Section 4, the probability of the event $E^0 \cup E^- \cup E^+$ is at least $64/256$. For the event $E^\pm$, we only get the estimate $\mathbf{P}(E^\pm) \geq 15.2/256$, which yields a total of $79.2/256$.

Case 1. *The edge uv is contained in two 4-cycles.*

Consider the event BBD^0 of weight 5 (see the diagram in Figure 9(a)). We claim that $\mathbf{P}(BBD^0) \geq 8/256$. Note that for any situation $\sigma \in BBD^0$, at least one of the vertices v_-, v_+ is added to I in Phase 1. It follows that for any such situation, u_- or u_+ is infeasible. Thus, $q(BBD^0) = 0$, and by Lemma 7,

$$\mathbf{P}(BBD^0) \geq \frac{1}{2^5} = \frac{8}{256}.$$

Next, we use the event ABD^- of weight 7 (see Figure 9(b)). Since $u_{-2} \in \sigma^1$ for any situation $\sigma \in ABD^-$, it is infeasible and hence $q(ABD^-) = 0$. Furthermore, ABD^- contains no sensitive pair and thus it is regular. Lemma 7 implies that $\mathbf{P}(ABD^-) \geq 2/256$. This shows that

$$\mathbf{P}(u \in I) \geq 89.2/256.$$

We remark that a further contribution of $2/256$ could be obtained from the event AC_1D^-, but it will not be necessary. ▲

Case 2. *uv is contained in one 4-cycle.*

We may assume that u_+ is adjacent to v_-. From Figure 10(a), we see that the event BBD^0 is weakly regular; we will estimate $q(BBD^0)$. Let σ be a random situation from BBD^0. If C_v is even, then $v_- \in \sigma^1$, which makes u_+ infeasible, so $q(BBD^0) = 0$. Assume then that C_v is odd; since G is triangle-free, the length of C_v is at least 5. Thus it contains at least two vertices other than v, v_-, v_+; consequently, the probability that all the vertices of C_v are active is at most $1/4$. If all the vertices of C_v are active, then $v_- \in \sigma^1$ (and hence u_+ is infeasible) with probability at least $2/5$. It follows that

$$q(BBD^0) \leq \mathbf{P}(v_- \notin \sigma^1 \mid \sigma \in BBD^0) \leq \frac{1}{10}.$$

By Lemma 7, $\mathbf{P}(BBD^0) \geq 98/100 \cdot 1/64 = 3.92/256$.

Consider the weakly regular event BBD^- (Figure 10(b)). Observe first that the event is valid in G as u_- and v_+ are not neighbours. Since u_+ is infeasible with respect to any situation from BBD^-, we have $q(BBD^-) = 0$ and so BBD^- is regular. Lemma 7 implies that $\mathbf{P}(BBD^-) \geq 4/256$.

Finally, consider the events ABD^- and AC_1D^- (Figure 10(c) and (d)); note that the only difference between them is that for $\sigma \in ABD^-$, $u \in \sigma^3$, whereas for $\sigma \in AC_1D^-$ it is the opposite. Both events, however, force u. Observe that their validity does not depend on whether u_{-2} and v_+ are neighbours: even if they are, the diagram prescribes consistent orientations at both ends of the edge $u_{-2}v_+$. The events are regular of weight 8, and thus $\mathbf{P}(ABD^- \cup AC_1D^-) \geq 2/256$. This proves that $\mathbf{P}(u \in I) > 89.1/256$. ▲

Having dealt with the above cases, we may now assume that the set $\{u_-, u_+, v_-, v_+\}$ is independent.

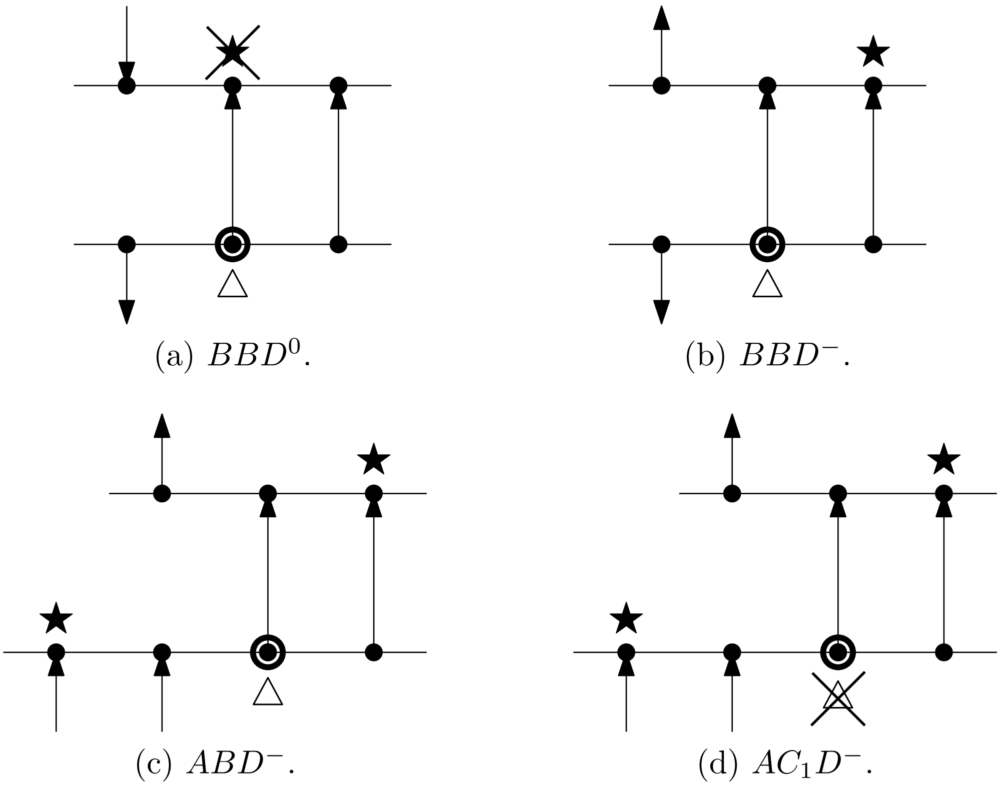
Case 3. *M includes the edges $u_{-2}v_+$ and $u_{+2}v_-$.*

The event BBD^+ (Figure 11(a)) is regular of weight 7; thus, $\mathbf{P}(BBD^+) \geq 2/256$. Similarly, $\mathbf{P}(BBD^-) \geq 2/256$. We also have $\mathbf{P}(BBD^0) = 2/256$ since v_+ and v_- have mates on Z, ensuring that one of the vertices of Z is infeasible and thus $q(BBD^0) = 0$. Furthermore, $\mathbf{P}(ABD^- \cup BAD^+) \geq 2/256$ by Lemma 7.

We may assume that $u'_- \neq v_{+2}$ and $u'_+ \neq v_{-2}$, for otherwise u has a neighbour contained in a 4-cycle and $\varepsilon(u) = -1$. In that case, the bound $\mathbf{P}(u \in I) \geq 87.2/256$, proved so far, would be sufficient.

If u_+ or u_- have a mate on Z, then $E^\pm$ is regular and hence $\mathbf{P}(E^\pm) = 16/256$. This adds further $0.8/256$ to $\mathbf{P}(u \in I)$, making it reach $88/256$, which is sufficient. Thus, we may assume that u'_- and u'_+ are not contained in Z.

Consider the event C_1AD^+ given by the diagram in Figure 11(b). Since this is the first time that the analysis of its probability involves a sensitive pair, we explain it in full detail. Assume that there exists a sensitive pair for this event. The only vertices which can be included in the pair are u'_-,



(a) BBD^0. (b) BBD^-. (c) ABD^-. (d) AC_1D^-.

Figure 10: The events used in Case 2 of the proof of Lemma 8.

u_{+2} and v_+. None of (u_{+2}, u_{+2}) and (v_+, v_+) is a circular sensitive pair, since both Z and C_v contain a tail in C_1AD^+ (u_- and v_-, respectively). Hence, the only possible circular sensitive pair is (u'_-, u'_-). As for linear sensitive pairs, the only possibility is (v_+, u'_-): the vertex u_{+2} is ruled out since none of u'_- and v_+ is contained in Z, and the pair (u'_-, v_+) cannot be sensitive as v_- is a tail in C_1AD^+. (Note that the sensitivity of a pair depends on the order of the vertices in the pair.) Summarizing, the sensitive pair is (u'_-, u'_-) or (v_+, u'_-), and it is clear that not both pairs can be sensitive at the same time.

If (u'_-, u'_-) is sensitive, then the cycle of F containing u'_- contains at least four vertices which are not heads in C_1AD^+. Consequently, the pair (u'_-, u'_-) is 4-free, and Lemma 7 implies $\mathbf{P}(C_1AD^+) \geq 79/80 \cdot 0.5/256 > 0.49/256$.

On the other hand, if (v_+, u'_-) is sensitive, we know that $d_{C_v}(v_+, u'_-)$ is odd, and our assumption that $u'_- \neq v_{+2}$ implies that the pair (v_+, u'_-) is 2-free. By Lemma 7, $\mathbf{P}(C_1AD^+) \geq 3/4 \cdot 0.5/256 = 0.375/256$. As this estimate is weaker than the preceding one, C_1AD^+ is guaranteed to have probability at least 0.375/256. Symmetrically, $\mathbf{P}(AC_1D^-) \geq 0.375/256$.

So far, we have accumulated a probability of 87.95/256. The missing bit can be supplied by the event $C_1C_2D^+$ of weight 10 (Figure 11(c)). Since u_- and u_+ do not have mates on Z, any sensitive pair will involve only the vertices u'_-, u'_+ and v_+, and it is not hard to check that there will be at most two such pairs. Since $u'_- \neq v_{+2}$, each of these pairs is 1-free. If one of them is 2-free, then $\mathbf{P}(C_1C_2D^+) \geq 1/4 \cdot 0.25/256 > 0.06/256$ by Lemma 7, which is more than the amount missing to 88/256.

We may thus assume that none of these pairs is 2-free. This implies that (v_+, u'_-) is not a sensitive pair, as d_{C_v} would have to be odd and strictly between 1 and 3. Thus, there are only two possibilities: (a) $C_1C_2D^+$ is covered by (u'_-, u'_+) and (u'_+, u'_-), or (b) it is covered by (v_+, u'_+) and (u'_+, u'_-). The former case corresponds to u'_+ and u'_- being contained in a cycle W of F of length 4, which is impossible by the choice of F. In the latter case, u'_+ and u'_- are contained in C_v; in fact, $u'_+ = v_{+3}$ and $u'_- = v_{+5}$. Although Lemma 7 does not give us a nonzero bound for $\mathbf{P}(C_1C_2D^+)$, we can get one by exploiting the fact that G is triangle-free. Since $v_{+2}v_{+4} \notin E(M)$, the probability that both v_{+2} and v_{+4} are tails with respect to the random situation σ is 1/4, and these events are independent of orientations of the other edges of G. Thus, the probability that σ weakly conforms to the template for $C_1C_2D^+$ and v_{+2}, v_{+4} are tails is $1/2^7 = 2/256$. Under this condition, σ will conform to the template with probability $1/2^5$ (a factor 1/2 for each symbol in the diagram). Consequently, $\mathbf{P}(C_1C_2D^+) > 0.06/256$, again a sufficient amount. ▲

Case 4. *M includes the edge $u_{-2}v_+$ but not $u_{+2}v_-$.*

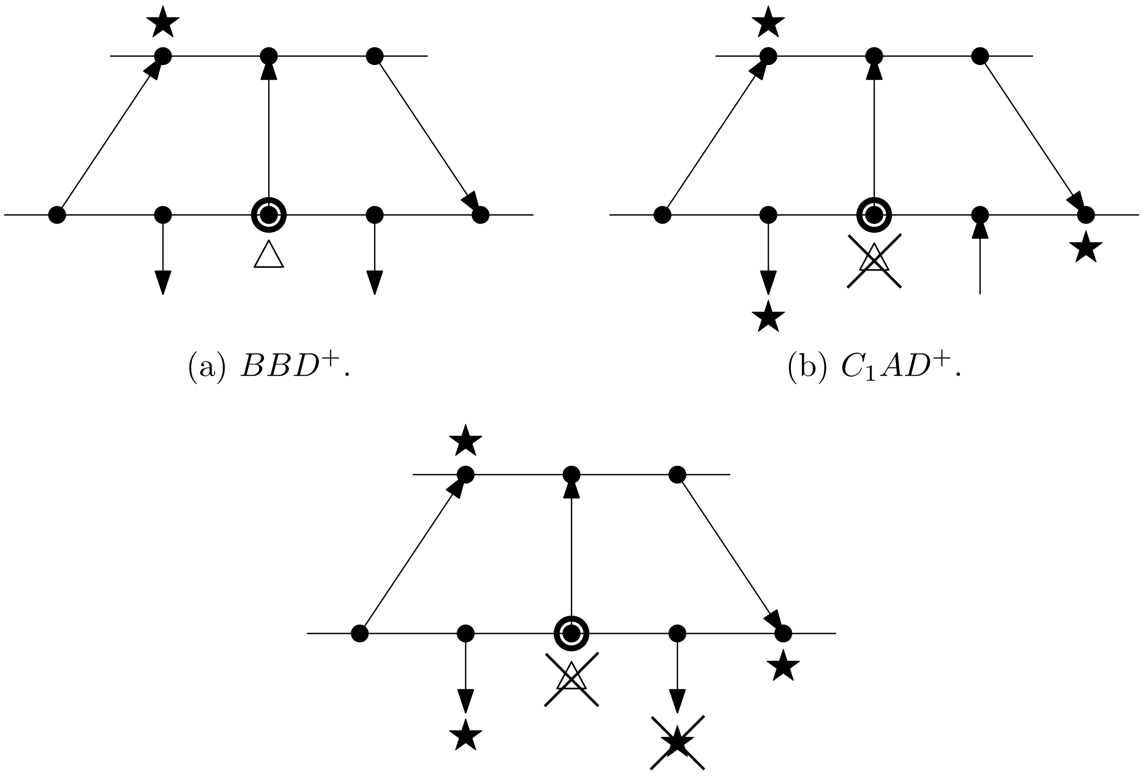
As in the previous case, $\mathbf{P}(BBD^-) \geq 2/256$. Consider the weakly regular event BBD^+ (Figure 12(a)). Since $u'_{-2} = v_+ \in \sigma^1$ for any $\sigma \in BBD^+$, we have $q(BBD^+) = 0$. By Lemma 7, $\mathbf{P}(BBD^+) \geq 2/256$.

The event BBD^0 is also weakly regular, and it is not hard to see that $q(BBD^0) \leq 1/10$ (using the fact that the length of C_v is at least 5). Lemma 7 implies that $\mathbf{P}(BBD^0) \geq 98/100 \cdot 2/256 = 1.96/256$.

Each of the events BAD^0 (Figure 12(b)), BAD^+ and BAD^- is regular and has weight 9. By Lemma 7, it has probability at least 0.5/256. Furthermore, $\mathbf{P}(ABD^-) \geq 1/256$, also by regularity. So far, we have shown that $\mathbf{P}(u \in I) \geq 87.68/256$. As in the previous case, this enables us to assume that u'_- and u'_+ are not vertices of Z. Furthermore, it may be assumed that $u'_- \neq v_{+2}$, for otherwise $\varepsilon(u) = -1$ and the current estimate on $\mathbf{P}(u \in I)$ is sufficient.

If M includes the edge u_-v_{-2}, then AC_1D^- is regular and $\mathbf{P}(AC_1D^-) \geq 0.5/256$, which would make the total probability exceed 88/256. Let us therefore assume the contrary.

The event C_1AD^- is covered by $(u'_-, v_-)^2$ and $q(C_1AD^-) = 0$, so the probability of C_1AD^- is at least $3/4 \cdot 0.25/256$. Similarly, C_1AD^+ is covered



(a) BBD^+. (b) C_1AD^+. (c) $C_1C_2D^+$.

Figure 11: Some of the events used in Case 3 of the proof of Lemma 8.

by (v_+, u'_-). Suppose for a moment that this pair is 2-free; we then get $\mathbf{P}(C_1AD^+) \geq 3/4 \cdot 0.25/256$. The event C_1AD^0 is covered by (v, u'_-) and (u'_-, v). Our assumptions imply for each of the pairs that it is 2-free. By Lemma 7, $\mathbf{P}(C_1AD^0) \geq 1/2 \cdot 0.25/256$. The contribution we have obtained from $C_1AD^+ \cup C_1AD^- \cup C_1AD^0$ is at least 0.5/256, which is sufficient to complete the proof in this subcase.

It remains to consider the possibility that (v_+, u'_-) is not 2-free in the diagram for C_1AD^+. It must be that the path $v_+C_vu'_-$ includes v'_- and has length 3. The probability bound for C_1AD^+ is now reduced to $1/2 \cdot 0.25/256$. However, now, C_1AD^0 is covered by (u'_-, v), and we find that $\mathbf{P}(C_1AD^0) \geq 3/4 \cdot 0.25/256$. In other words,

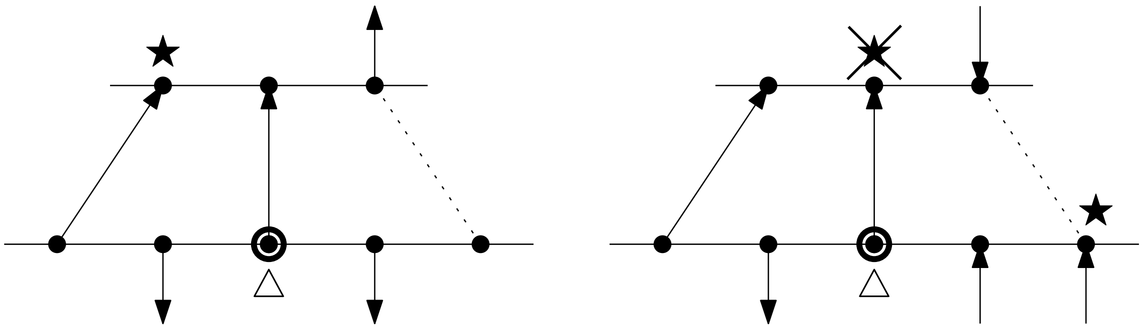
$$\mathbf{P}(C_1AD^+ \cup C_1AD^- \cup C_1AD^0) \geq 0.5/256$$

as before. ▲

By symmetry, it remains to consider the following case. Note that our assumption that the set $\{u_-, u_+, v_-, v_+\}$ is independent remains in effect.

Case 5. *G contains no edge from the set $\{u_{-2}, u_{+2}\}$ to $\{v_-, v_+\}$.*

Consider the weakly regular event BBD^+ (Figure 13). As before, the fact that $|V(Z)| \geq 5$ if Z is odd, together with Observation 6(ii), implies that



(a) BBD^+. (b) BAD^0.

Figure 12: Some of the events used in Case 4 of the proof of Lemma 8.

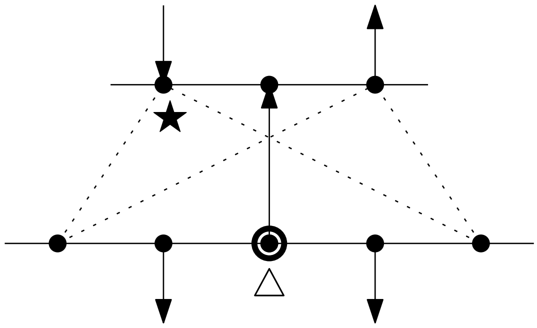


Figure 13: The event BBD^+ used in the final part of the proof of Lemma 8.

$q(BBD^+) \le 1/4$. Since the event has weight 7, $\mathbf{P}(BBD^+) \ge 1.9/256$ by Lemma 7. We get the same estimate for BBD^- and BBD^0.

Since u_- is not adjacent to either of v_- and v_+, the event ABD^+ is valid. It is regular, so $\mathbf{P}(ABD^+) \ge 0.5/256$. The same applies to the events ABD^-, ABD^0, BAD^+, BAD^- and BAD^0. Thus, the probability of the union of these six events is at least 3/256. Together with the other events described so far, the probability is at least 87.9/256. As in the previous cases, this means that we may assume that the mate of u_+ is not contained in Z, for otherwise we would obtain a further 0.8/256 from the event $E^\pm$ and reach the required amount.

Since the length of C_v is at least 5, u'_+ is not adjacent to both v_- and v_+. Suppose that it is not adjacent to v_+ (the other case is symmetric). Then AC_1D^+ is covered by the pair $(v_+, u'_+)^2$. Hence, $\mathbf{P}(AC_1D^+) \ge 3/4 \cdot 0.5/256 = 0.375/256$. The total probability of $u \in I$ is therefore larger than 88/256, which concludes the proof. ▲

□

type of u	condition	$\varepsilon(u)$
I	the path v_-vv_+ is contained in a 4-cycle in G, neither the path u_-uu_+ nor the edge uv are contained in a 4-cycle, and u is not of types Ia, Ib, Ia* or Ib* (see text)	-0.5
Ia	$\lvert uZv\rvert = 4$ and M includes the edges $u_{+2}v_+$, $u_{-2}v_-$, while $u_+v_{+2} \notin E(M)$	-2
Ib	$\lvert uZv\rvert = 4$ and M includes the edges $u_{+2}v_+$, $u_{-2}v_-, u_+v_{+2}$,	-1.5
II	$\lvert uZv\rvert = 4$, $\lvert vZu\rvert \geq 7$ and M includes all of the edges u_-v_{+2}, $u_{-2}v_+$, $u_{-3}u_+$, while $v_{+3}v_- \notin E(M)$	-0.125
IIa	$\lvert uZv\rvert = 4$, $\lvert vZu\rvert = 6$, and M includes all of the edges $u_{-2}v_+$, $u_{-3}u_+$ and u_-u_{-4},	-0.5
III	$\lvert uZv\rvert = 4$, $\lvert vZu\rvert = 8$ and M includes all of the edges $u_{-2}v_+$, $u_{-3}u_+$, $v_{+3}v_-$ and u_-u_{-4}	-0.125

Table 1: The type of a deficient vertex u provided that uv is a chord of Z, and the associated value $\varepsilon(u)$.

6 Analysis: uv is a chord

In the present section, we continue the analysis of Section 5, this time confining our attention to the case where uv is a chord of Z. Although this case is more complicated, one useful simplification is that by Observation 6(i), we now have $q(\Delta) = 0$ for any template Δ. In particular, $\mathbf{P}(E^\pm) \geq 16/256$, which implies

$$\mathbf{P}(E^0 \cup E^- \cup E^+ \cup E^\pm) \geq \frac{80}{256}.$$

Roughly speaking, since the probability needed to prove Theorem 3 is 88/256, we need to find events in Σ whose total probability is at least 8/256. However, like in Section 5, we may actually require a higher probability or be satisfied with a lower one, depending on the type of the vertex. The surplus probability will be used to compensate for the deficits in Section 7.

Recall that at the beginning of Section 5, we defined deficient vertices of type 0, and we associated a number $\varepsilon(u)$ with the vertex u provided that uv is not a chord of a cycle of F. We are now going to provide similar definitions for the opposite case, introducing a number of new types of deficient vertices.

Suppose that uv is a chord of Z which is not contained in any 4-cycle of G. The vertex u is *deficient* if it satisfies one of the conditions in Table 1. (See the illustrations in Figure 14.) Since the conditions are mutually exclusive, this also determines the *type* of the deficient vertex u.

We now extend the definition to cover the symmetric situations. Suppose that u satisfies the condition of type II when the implicit orientation of Z is replaced by its reverse — which also affects notation such as u_+, uZv etc. In this case, we say that u is deficient of type II*. (As seen in Figure 15, the picture representing the type is obtained by a flip about the vertical axis.) The same notation is used for all the other types except types 0 and I. A type such as II* is called the *mirror* type of type II.

Note that even with this extension, the types of a deficient vertex remain mutually exclusive. Furthermore, we have the following observation which will be used repeatedly without explicit mention:

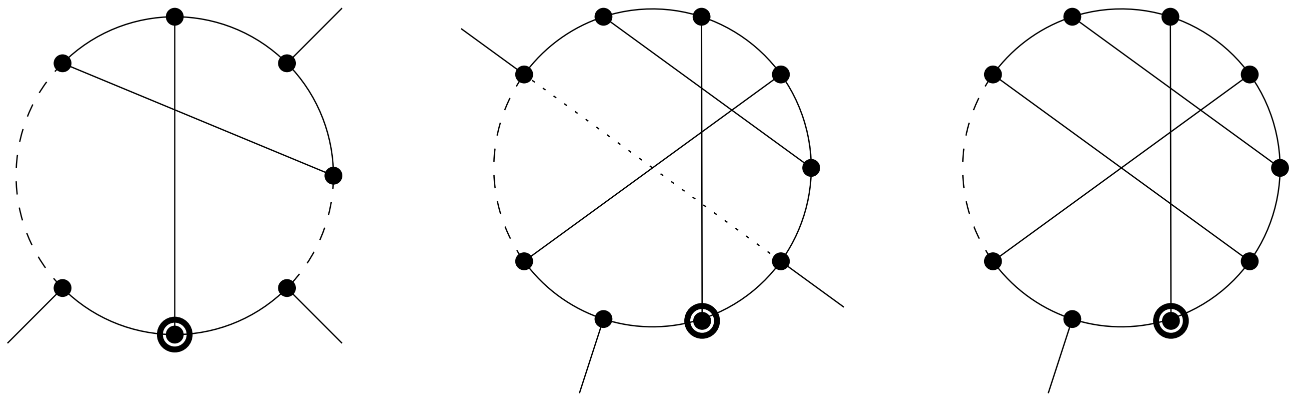
Observation 9. *If u is deficient (of type different from 0), then its mate v is not deficient.*

Proof. Let u be as stated. A careful inspection of Table 1 and Figure 14 shows that the path u_-uu_+ is not contained in any 4-cycle. It follows that v is not deficient of type I, Ia, Ib or their mirror variants. Suppose that v is deficient. By symmetry, u also does not belong to the said types, and hence the types of both u and v are II, IIa, III or the mirror variants. As seen from Figure 14, when u is of any of these types, the path u_-uu_+ belongs to a 5-cycle in G. By symmetry again, the same holds for v_-vv_+. The only option is that u belongs to type III and v to III*, or vice versa. But this is clearly impossible: if u is of type III or III*, then one of its neighbours on Z is contained in a 4-cycle, and this is not the case for any neighbour of v on Z. Hence, v cannot be of type III or III*. This contradiction shows that v is not deficient. □

We will often need to apply the concept of a type to the vertex v rather than u. This may at first be somewhat tricky; for instance, to obtain the definition of 'v is of type IIa*', one needs to interchange u and v in the definition of type IIa in Table 1 and then perform the reversal of the orientation of Z. In this case, the resulting condition will be that $|uZv| = 4$, $|vZu| = 6$ (here the two changes cancel each other) and M includes the edges $v_{+2}u_-$, $v_{+3}v_-$ and v_+v_{+4}. To spare the reader from having to turn Figure 14 around repeatedly, we picture the various cases where v is deficient in Figure 16.

Table 1 also associates the value $\varepsilon(u)$ with each type. By definition, a type with an asterisk (such as II*) has the same value assigned as the corresponding type without an asterisk.

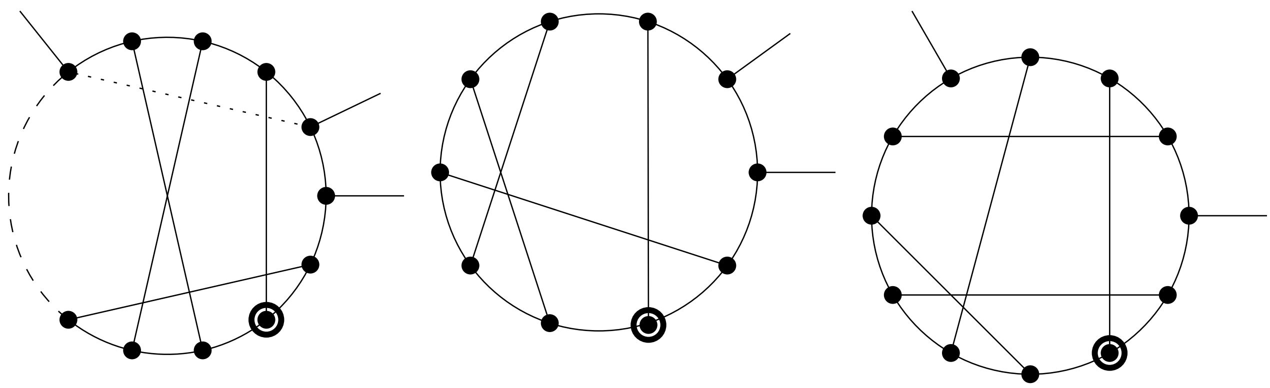
We now extend the function ε to all vertices of G. It has been defined for all deficient vertices, as well as for all vertices whose mate is contained in a different cycle of F. Suppose that w is a non-deficient vertex whose mate w'



(a) Type I (only one of the two possibilities shown).

(b) Type Ia.

(c) Type Ib.



(d) Type II.

(e) Type IIa.

(f) Type III.

Figure 14: Deficient vertices.

is contained in the same cycle of F. We set

$$\varepsilon(w) = \begin{cases} -\varepsilon(w') & \text{if } w' \text{ is deficient,} \\ 0 & \text{otherwise.} \end{cases}$$

Our goal in this section is to prove the following proposition, which is the main technical result of this paper. As in the proof of Lemma 8, we mark the end of each case by ▲; furthermore, the end of each subcase is marked by △.

Proposition 10. *If uv is a chord of Z, then for the total probability of the events in Σ we have*

$$\mathbf{P}(\bigcup \Sigma) \geq \frac{8 + \varepsilon(u)}{256}.$$

Proof. We distinguish a number of cases based on the structure of the neighbourhood of u in G.

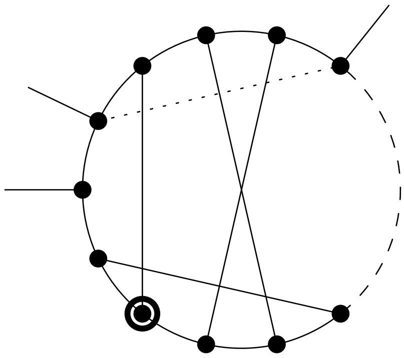
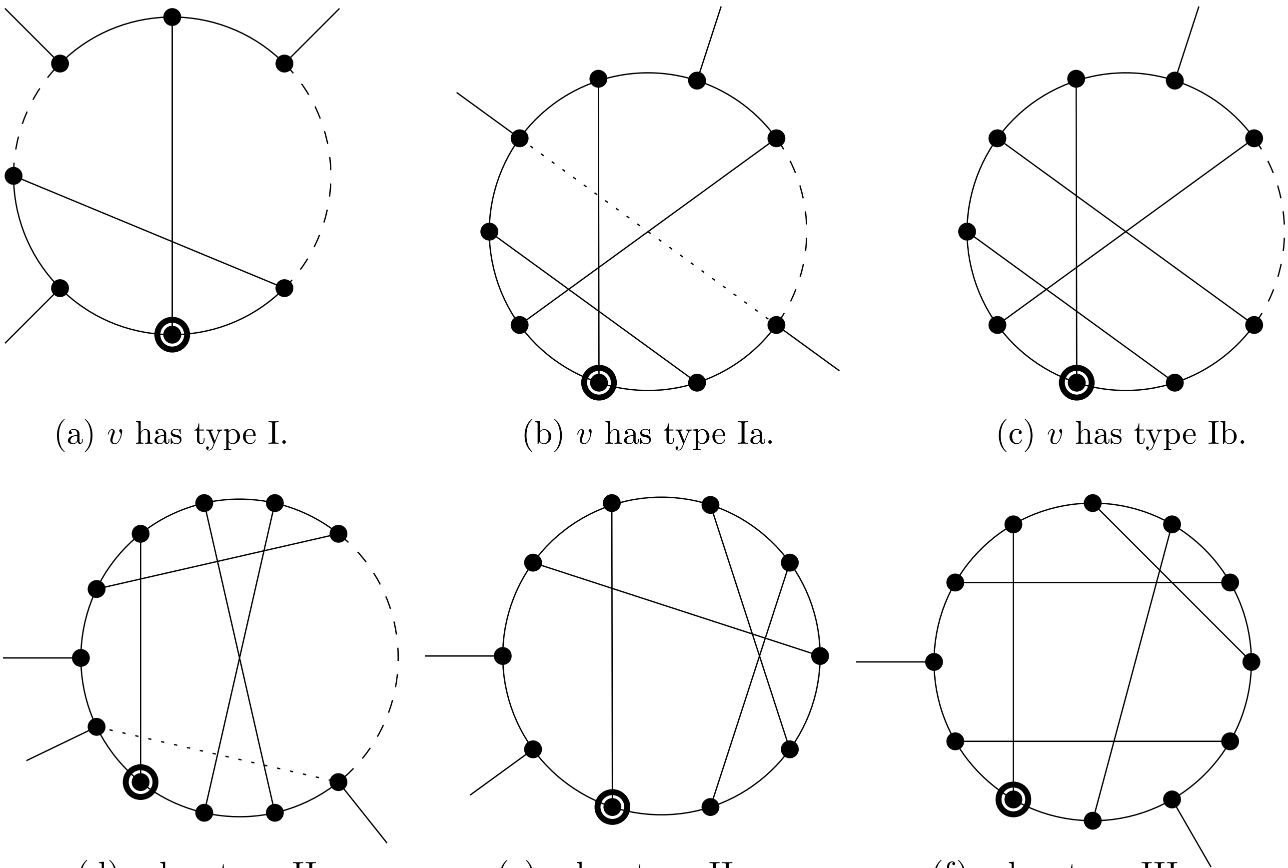


Figure 15: A deficient vertex u of type II*.



(a) v has type I.

(b) v has type Ia.

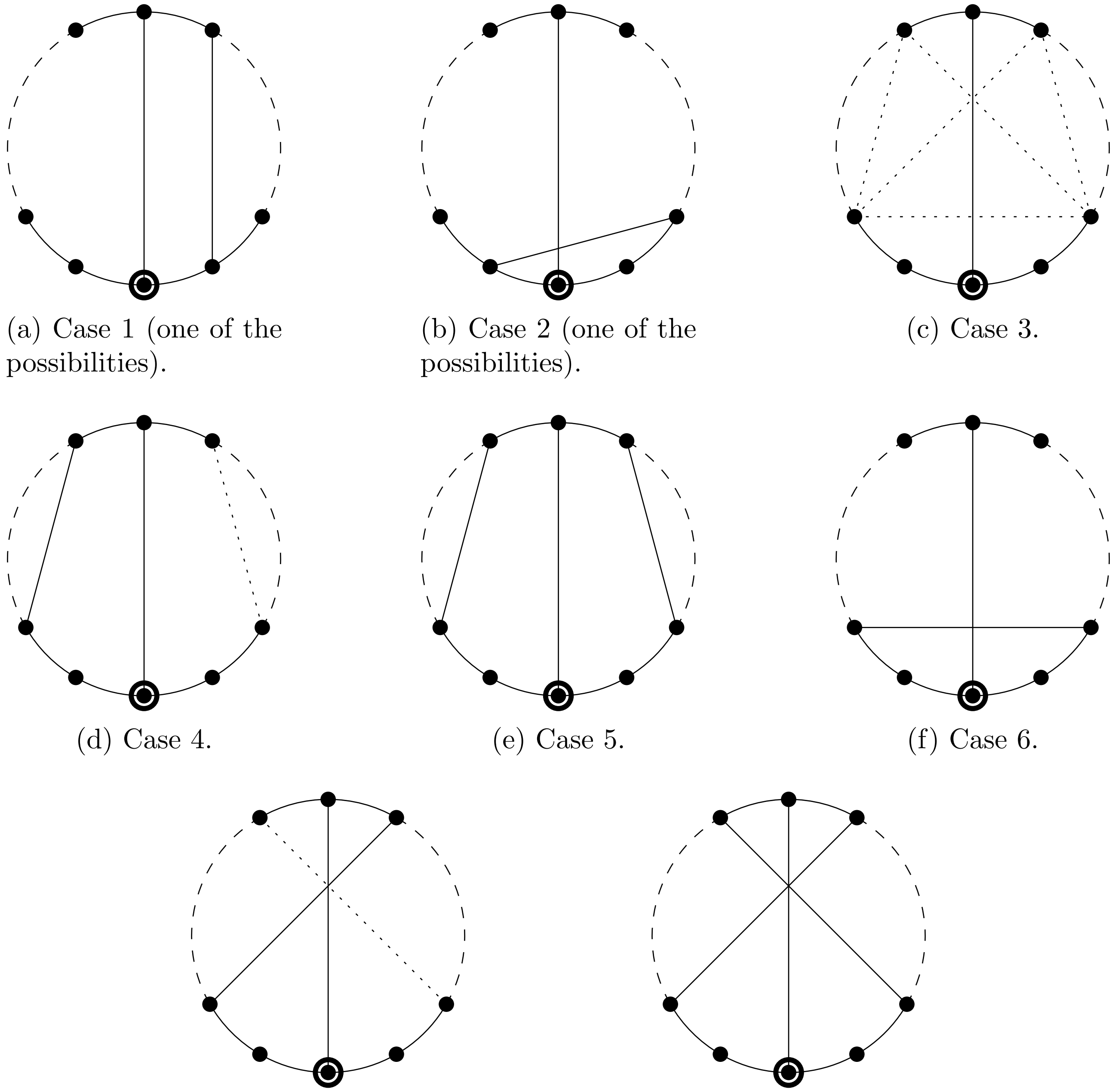
(c) v has type Ib.

(d) v has type II.

(e) v has type IIa.

(f) v has type III.

Figure 16: The situation when the vertex v is deficient. As usual, the vertex u is circled.



(a) Case 1 (one of the possibilities).

(b) Case 2 (one of the possibilities).

(c) Case 3.

(d) Case 4.

(e) Case 5.

(f) Case 6.

(g) Case 7.

(h) Case 8.

Figure 17: The main cases in the proof of Proposition 10. Relevant non-edges are represented by dotted lines, paths are shown as dashed lines.

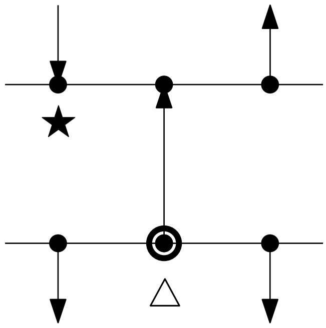


Figure 18: The event BBD^+.

Case 1. *The edge uv is contained in a 4-cycle.*

Observe that in this case, neither u nor v is deficient.

Suppose that uvv_-u_+ is a 4-cycle (the argument in the other cases is the same). Consider first the possibility that v_-u_+ is an edge of M. The event BBD^0 is (valid and) regular. By Lemma 7, $\mathbf{P}(BBD^0) \geq 4/256$. Since this lower bound increases to 8/256 if u_-v_+ is an edge of M (and since v is not deficient), we may actually assume that this is not the case. Consequently, $\mathbf{P}(BBD^-) \geq 4/256$ as BBD^- is regular. The total contribution is 8/256 as desired.

We may thus assume that v_-u_+ is an edge of F and no edge of M has both endvertices in $\{u_-, u_+, v_-, v_+\}$. Since the events BBD^0 and BBD^- are regular, we have $\mathbf{P}(BBD^0 \cup BBD^-) \geq 4/256$.

A further probability of 4/256 is provided by the regular events BAD^0 and BAD^-. Indeed, although the template BAD^0 has weight 8, which would only yield $\mathbf{P}(BAD^0) \geq 1/256$ by Lemma 7, the estimate is improved to 2/256 by the fact that the associated diagram has a removable symbol at v. The same applies to the event BAD^-. We conclude

$$\mathbf{P}(BBD^0 \cup BBD^- \cup BAD^0 \cup BAD^-) \geq 8/256$$

as required. ▲

We will henceforth assume that uv is not contained in a 4-cycle. Note that this means that the set $\{u_-, u_+, v_-, v_+\}$ is independent. Consider the regular event BBD^+ (Figure 18). By Lemma 7, we have

$$\mathbf{P}(BBD^+) \geq \frac{2}{256}.$$

The same applies to the events BBD^0 and BBD^-. Thus, in the subsequent cases, it suffices to find additional events of total probability at least $(2 + \varepsilon(u))/256$.

Case 2. *The path u_-uu_+ is contained in a 4-cycle.*

Suppose that $u_-uu_+u_{+2}$ is such a 4-cycle. (The other case is symmetric.) Consider the events C_1AD^+ and BAD^+. Since the condition of Case 1 does not hold, and by the assumption that G is triangle-free, the set $\{u_+, v_-, v_+\}$ is independent in G. Furthermore, each of the events is regular and by Lemma 7, each of them has probability at least 1/256. Thus, it remains to find an additional contribution of $\varepsilon(u)$.

We distinguish several subcases based on the deficiency and type of the vertex v. Since u_-uu_+ is contained in a 4-cycle, v is either not deficient, or is deficient of type I, Ia, Ib, Ia* or Ib*.

Subcase 2.1. *v is not deficient.*

In this subcase, $\varepsilon(u) \leq 0$, so there is nothing to prove. △

Subcase 2.2. *v is deficient of type I.*

By the definition of type I, both of the following conditions hold:

- $u_+v_{+2} \notin E(M)$ or $|uZv| \geq 5$,
- $u_-v_{-2} \notin E(M)$ or $|vZu| \geq 5$.

Moreover, we have $\varepsilon(u) = 0.5$.

We may assume that M includes the edge $u_{-2}v_-$, for otherwise the event ABD^- is regular (see Figure 19(a)) and has probability at least 0.5/256 as required.

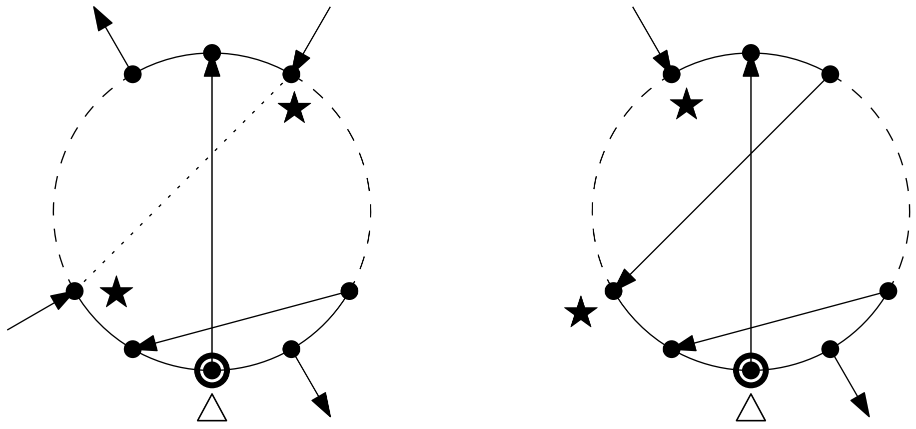
The event ABD^+ (Figure 19(b)) is covered by the pair (v_+, u_{-2}). Consequently, we may assume that $|vZu| = 4$: otherwise the pair is 1-free, and since the event has weight 8, we have $\mathbf{P}(ABD^+) \geq 0.5/256$ by Lemma 7.

By a similar argument applied to the event C_1AD^-, we infer that $|uZv| = 4$. Thus, the length of Z is 8 and the structure of $G[V(Z)]$ is as shown in Figure 20(a). The regular event $C_1C_2D^+$ (Figure 20(b)) has probability at least 0.5/256, which is sufficient. This concludes the present subcase.

△

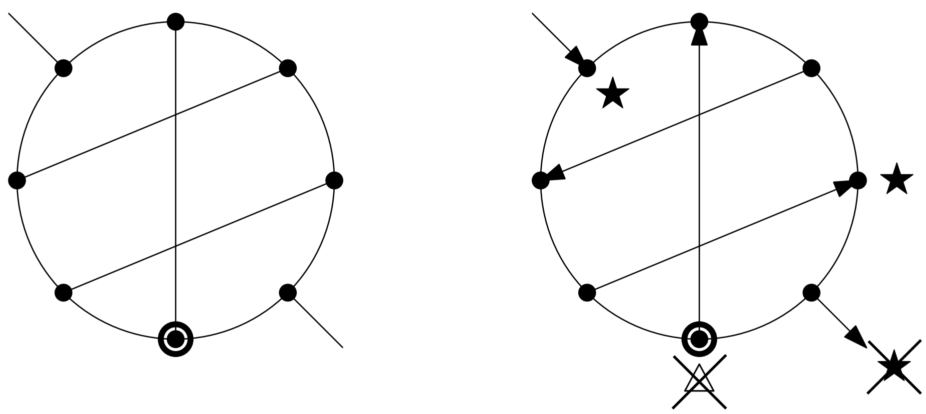
Subcase 2.3. *v is deficient of type Ia, Ib, Ia* or Ib*.*

By symmetry, we may assume that v is either of type Ia* (if $u_{-2}v_-$ is not an edge of M) or Ib* (otherwise). Accordingly, we have either $\varepsilon(u) = 2$ or $\varepsilon(u) = 1.5$.



(a) (b)

Figure 19: Subcase 2.2 of the proof of Proposition 10: (a) The event ABD^- if $u_{-2}v_- \notin E(M)$. (b) The event ABD^+.



(a) (b)

Figure 20: (a) A configuration in Subcase 2.2 of the proof of Proposition 10. (b) The event $C_1C_2D^+$.

The regular event $C_1C_2D^+$ provides a contribution of $1/256$. If $u_{-2}v_- \notin E(M)$ (thus, v is of type Ia* and $\varepsilon(u) = 2$), then the event ABD^- is also regular (including when $|vZu| = 4$) and $\mathbf{P}(ABD^-) \geq 1/256$, a sufficient amount.

It remains to consider the case that $u_{-2}v_- \in E(M)$. The required additional probability of $0.5/256$ is supplied by the event ABD^+, which is covered by the 1-free pair (v_+, u_{-2}). △

The discussion of Case 2 is complete. ▲

From here on, we assume that none of the conditions of Cases 1 and 2 holds. In particular, v is not deficient of type I, Ia, Ib or their mirror types. We distinguish further cases based on the set of edges induced by M on the set

$$U = \{u_{-2}, u_{+2}, v_-, v_+\}.$$

Note that the length of the paths uZv and vZu is now assumed to be at least 4. We call a path *short* if its length equals 4.

Case 3. $E(M[U]) = \emptyset$.

We claim that if v is deficient, then its type is III or III*. Indeed, for types I, Ia, Ib and their mirror types, u_-uu_+ would be contained in a 4-cycle and this configuration has been covered by Case 2. For types II, IIa and their mirror variants, U would not be an independent set. Since type 0 is ruled out for trivial reasons, types III and III* are the only ones that remain. The only subcase compatible with these types is Subcase 3.2; in the other subcases, v is not deficient and we have $\varepsilon(u) \leq 0$. This will simplify the discussion in the present case.

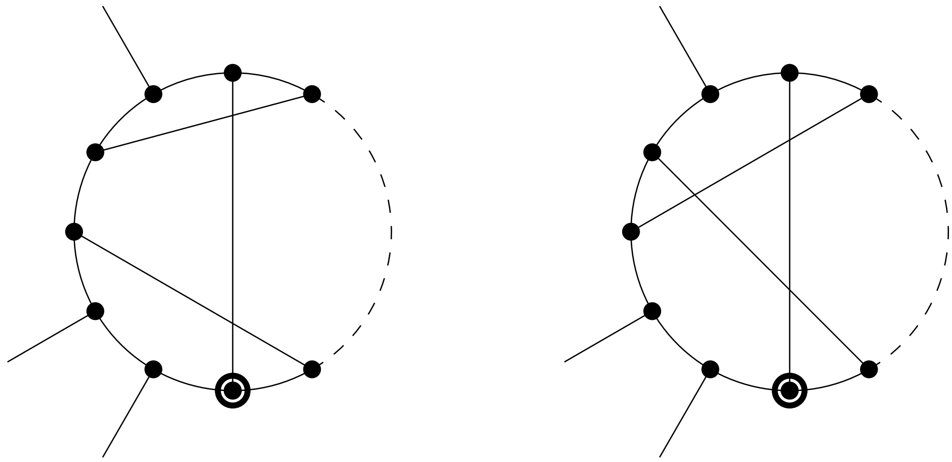
We begin by considering the event ABD^-. By the assumptions, it is valid. Since neither (u_{-2}, v_+) nor its reverse is a sensitive pair, the event is regular. Thus, $\mathbf{P}(ABD^-) \geq 0.5/256$. By symmetry, we have $\mathbf{P}(BAD^+) \geq 0.5/256$.

We distinguish several subcases, in each of which we try to accumulate further $(1 + \varepsilon(u))/256$ worth of probability.

Subcase 3.1. *None of uZv and vZu is short.*

Consider the event ABD^0. By the assumptions, it is valid and covered by (v_+, u_{-2}). Since vZu is not short and the diagram of ABD^0 contains only one outgoing arc (namely $u_+u'_+$), the pair is 1-free. By Lemma 7, $\mathbf{P}(ABD^0) \geq 1/2 \cdot 0.5/256 = 0.25/256$. By symmetry, $\mathbf{P}(BAD^0) \geq 0.25/256$.

The argument for ABD^0 also applies to the event ABD^+ (whose diagram has two outgoing arcs), unless the vertex set of the path vZu is



(a) (b)

Figure 21: Two cases where the event ABD^+ cannot be used in Subcase 3.1 of the proof of Proposition 10.

$\{v, v_+, u'_+, v'_-, u_-, u\}$ (in which case we get the two possibilities in Figure 21). If this does not happen, then we obtain a contribution of at least 0.25/256 again.

Let us examine the exceptional case in Figure 21(a) (i.e., $u'_+ = u_{-3}$ and $v'_- = v_{+2}$). The event $C_1C_1D^+$ is covered by $(u'_-, u'_-)^4$. By Lemma 7, $\mathbf{P}(C_1C_1D^+) \geq 79/80 \cdot 1/256 > 0.98/256$.

Consider now the situation of Figure 21(b). The event AAD^+ is valid, since $\{u_{-2}, u_{+2}, v_+\}$ is an independent set by assumption, and it is regular. We infer that $\mathbf{P}(AAD^+) \geq 0.25/256$.

To summarize the above three paragraphs, we proved

$$\mathbf{P}(ABD^+ \cup C_1C_1D^+ \cup AAD^+) \geq 0.25/256.$$

By symmetry, we have

$$\mathbf{P}(BAD^- \cup C_1C_1D^- \cup AAD^-) \geq 0.25/256.$$

Together with the events ABD^0 and BAD^0 considered earlier, this makes for a total contribution of at least 1/256. As noted at the beginning of Case 3, $\varepsilon(u) \leq 0$, so this is sufficient. △

Subcase 3.2. *The path vZu is short, but uZv is not.*

In this subcase, v may be deficient of type III*, in which case $\varepsilon(u) = 0.125$; otherwise, $\varepsilon(u) \leq 0$.

The event BAD^- is covered by the pair (u_{+2}, v_-) which is 1-free unless v'_+ and u'_- are the only internal vertices of the path $u_{+2}Zv_-$. However, this situation would be inconsistent with our choice of F, since $\partial(Z)$ would have size 4. (Recall that $\partial(Z)$ is the set of edges of G with one end in $V(Z)$.) Consequently, $\mathbf{P}(BAD^-) \geq 1/2 \cdot 0.5/256 = 0.25/256$. Moreover, if u'_+ (which is a tail in BAD^-) is contained in $u_{+2}Zv_-$, then $\mathbf{P}(BAD^-) \geq 0.5/256$.

The same discussion applies to the event BAD^0. In particular, if $u'_+ \in V(u_{+2}Zv_-)$, then the probability of the union of these two types is at least 1/256. This is a sufficient amount, unless v is deficient of type III*, in which case a further 0.25/256 is obtained from the regular event AAD^-.

We may thus assume that $u'_+ \notin V(u_{+2}Zv_-)$ (so v is not deficient). The event AC_1D^- is then covered by $(u'_+, u'_+)^3$ (we are taking into account the arc incident with v_+) and hence $\mathbf{P}(AC_1D^-) \geq 39/40 \cdot 0.25/256 > 0.24/256$ by Lemma 7.

The event AC_2D^- is covered by the pair $(u_{+2}, v_-)^1$ and has probability at least $1/2 \cdot 0.0625/256 > 0.03/256$. We claim that $\mathbf{P}(BAD^- \cup BAD^0 \cup AAD^-) \geq 0.75/256$. Since the total amount will exceed 1/256, this will complete the present subcase.

Suppose first that $v'_+ \in V(uZv)$. Then the event BAD^0 is regular and $\mathbf{P}(BAD^0) \geq 0.5/256$. In addition, BAD^- has only one sensitive pair (u_{+2}, v_-). This pair is 1-free, for otherwise v'_+ and u'_- would be the only internal vertices of the path $u_{+2}Zv_-$, and Z would be incident with exactly four non-chord edges of M, a contradiction with the choice of F. Thus, $\mathbf{P}(BAD^-) \geq 0.25/256$ and the claim is proved.

Let us therefore assume that $v'_+ \notin V(uZv)$. We again distinguish two possibilities according to whether u'_- is contained in uZv or not. If $u'_- \in V(uZv)$, then $\mathbf{P}(BAD^-) \geq 1/2 \cdot 0.5/256 = 0.25/256$ as BAD^- is covered by $(u_{+2}, v_-)^1$. Similarly, $\mathbf{P}(BAD^0) \geq 0.25/256$. The event AAD^- is regular of weight 10, whence $\mathbf{P}(AAD^-) \geq 0.25/256$. The total probability of these three events is at least 0.75/256 as claimed.

To complete the proof of the claim, we may assume that $u'_- \notin V(uZv)$. The only possibly sensitive pair of BAD^- and BAD^0 is now 2-free, implying a probability bound of $3/4 \cdot 0.5/256$ for each event. Thus, $\mathbf{P}(BAD^- \cup BAD^0) \geq 0.75/256$, finishing the proof of the claim and the whole subcase. △

Subcase 3.3. *Both vZu and uZv are short.*

In this subcase, Z is an 8-cycle; by our assumptions, it has only one chord

uv. Recall also that in this subcase, $\varepsilon(u) \leq 0$.

Consider the event AC_1D^-. Since it is covered by $(u'_+, u'_+)^3$, we have $\mathbf{P}(AC_1D^-) \geq 39/40 \cdot 0.25/256 > 0.24/256$ by Lemma 7. By symmetry, $\mathbf{P}(C_1AD^+) \geq 0.24/256$, so the total probability so far is 0.48/256.

Suppose now that the vertices u'_+ and u'_- are located on different cycles of F. By Lemma 7, $\mathbf{P}(C_1C_1D^0) \geq 39/40 \cdot 0.5/256 > 0.48/256$. Similarly, $\mathbf{P}(C_1C_1D^+) \geq 77/80 \cdot 0.5/256 > 0.48/256$, which makes for a sufficient contribution.

We may thus assume that u'_+ and u'_- are on the same cycle, say Z', of F. Suppose that they are non-adjacent, in which case $C_1C_1D^0$ is covered by $(u'_+, u'_-)^2$ and $(u'_-, u'_+)^2$, and its probability is at least $1/2 \cdot 0.5/256 = 0.25/256$. If neither v'_- nor v'_+ are on Z', then the same computation applies to $C_1C_1D^+$ and $C_1C_1D^-$, so the total probability accumulated so far is $(0.48 + 0.25 + 0.25 + 0.25)/256 > 1/256$ by Lemma 7. We may thus assume, without loss of generality, that $v'_+ \in V(u'_+Z'u'_-)$. Under this assumption, $C_1C_1D^-$ is covered by $(u'_-, u'_+)^1$ and thus $\mathbf{P}(C_1C_1D^-) \geq 1/2 \cdot 0.5/256 = 0.25/256$. At the same time, $\mathbf{P}(C_1C_1D^+)$ is similarly seen to be at least 0.125/256, which makes the total probability at least $(0.48 + 0.25 + 0.25 + 0.125)/256 > 1/256$.

It remains to consider the possibility that u'_+ and u'_- are adjacent. In this case, $\mathbf{P}(AC_1D^- \cup C_1AD^+) \geq 0.5/256$, so we need to find additional 0.5/256. The event $C_1C_2D^+$ has a template covered by $(u'_-, u'_-)^2$, and hence its probability is at least $19/20 \cdot 0.25/256 > 0.23/256$. Similarly, $\mathbf{P}(C_2C_1D^-) \geq 0.23/256$. The same argument applies to the events $C_1C_3D^+$ and $C_3C_1D^-$, resulting in a total probability of $(0.5+4 \cdot 0.23)/256 > 1/256$. This finishes Case 3. △

▲

Case 4. $E(M[U]) = \{u_{-2}v_+\}$.

In this case, two significant contributions are from the regular events ABD^- and BAD^+:

$$\mathbf{P}(ABD^-) \geq \frac{1}{256},$$
$$\mathbf{P}(BAD^+) \geq \frac{0.5}{256}.$$

We distinguish several subcases; in each of them, we try to accumulate a contribution of $(0.5 + \varepsilon(u))/256$ from other events. In particular, if u is deficient of type I, IIa or IIa* (and $\varepsilon(u) = -0.5$), we are done.

Let us consider the vertex v. We claim that if v is deficient, then it must be of type II* or IIa*. Indeed, the assumption that u_-uu_+ is not contained

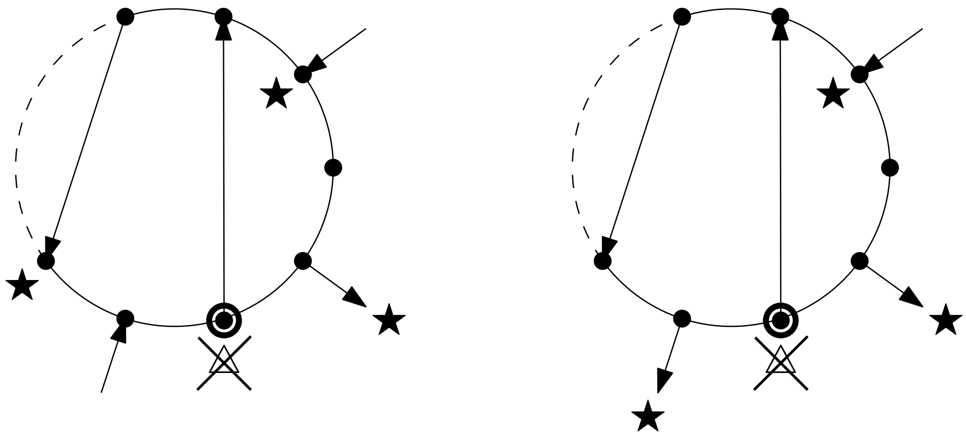


Figure 22: Two events used in Subcase 4.2 of the proof of Proposition 10: (a) AC_1D^-, (b) $C_1C_1D^-$.

in a 4-cycle excludes types I, Ia, Ib and their mirror variants. Type 0 is excluded for trivial reasons. An inspection of the type definitions shows that if v is of type II or IIa, then M includes the edge $u_{+2}v_-$, which we assume not to be the case. Finally, if v is of type III or III*, then $u_{-2}v_+$ is not an edge of M, another contradiction with our assumption.

The only types that remain for v are II* and IIa*. Observe that if v is of one of these types, then uZv is short.

Subcase 4.1. *The path uZv is not short.*

By the above, v is not deficient of either type, whence $\varepsilon(u) \leq 0$. The event BAD^- is covered by $(u_{+2}, v_-)^1$ (consider the outgoing arc incident with u_-). It follows that $\mathbf{P}(BAD^-) \geq 0.25/256$. The same argument applies to BAD^0, and thus

$$\mathbf{P}(ABD^- \cup BAD^+ \cup BAD^- \cup BAD^0) \geq \frac{1 + 0.5 + 0.25 + 0.25}{256} = \frac{2}{256}.$$

△

We have observed that if v is deficient, then it must be of type II* or IIa*. Since this requires that the F-neighbours of u'_- are v_+ and v'_-, it can only happen in the following subcase.

Subcase 4.2. *The vertices u'_+ and u'_- are non-adjacent.*

Consider the events AC_1D^- and $C_1C_1D^-$ (Figure 22). If the event AC_1D^- has a sensitive pair, it is either (u'_+, u'_+) or (u'_+, u_{-2}).

Suppose first that u'_+ is distinct from u_{-3}. In this case, Lemma 7 implies that $\mathbf{P}(AC_1D^-) \geq 3/4 \cdot 0.5/256$ no matter whether $u'_+ \in V(Z)$ or not. Secondly, $\mathbf{P}(C_1C_1D^-) \geq 1/2 \cdot 0.5/256$ (by Lemma 7 again), so the total contribution is at least $0.625/256$, which is sufficient if v is not deficient, or is deficient of type II*. It remains to consider the possibility that v is deficient of type IIa*. In this case, AC_1D^- is covered by $(u'_+, u'_+)^4$; by Lemma 7, $\mathbf{P}(AC_1D^-) \geq 79/80 \cdot 0.5/256 > 0.49/256$. Similarly, we obtain $\mathbf{P}(C_1C_1D^-) > 0.49/256$ and $\mathbf{P}(C_2C_1D^-) \geq 0.24/256$. The total contribution is $1.22/256 > (0.5 + \varepsilon(u))/256$.

We may thus suppose that $u'_+ = u_{-3}$; since this is incompatible with v being of type II* as well as IIa*, we find that v is not deficient and $\varepsilon(u) \leq 0$. We have $\mathbf{P}(C_1C_1D^-) \geq 3/4 \cdot 0.5/256$ (whether u'_- is contained in vZu or outside Z) since the event $C_1C_1D^-$ is covered by a single 2-free pair (either (u'_-, u'_-) or (u'_-, u_{-3})) and the weight of the event is 9. It remains to find a further contribution of $0.125 + \varepsilon(u)$ to reach the target amount. In particular, we may assume that u is not deficient of type II.

If $u'_- \neq v_{+2}$, the event $C_1C_1D^0$ is covered by (v_+, u'_-) and (u'_-, u_{-3}). Using Lemma 7, we find that $\mathbf{P}(C_1C_1D^0) \geq 1/2 \cdot 0.5/256$, which is sufficient.

Thus, the present subcase boils down to the situation where u'_- is adjacent to v_+ (i.e., $u'_- = v_{+2}$) and $u'_+ = u_{-3}$. Since u is not deficient of type II, it must be that v_-v_{+3} is an edge of M. In this case, the only events of nonzero probability in Σ are the events ABD^-, BAD^+ and $C_1C_1D^-$ considered above. Fortunately, the condition that $v_-v_{+3} \in E(M)$ increases the probability bound for $C_1C_1D^+$ from $3/4 \cdot 0.5/256$ to $0.5/256$ as required. △

As all the subcases where v is deficient have been covered in Subcase 4.2, we may henceforth assume that $\varepsilon(u) \leq 0$. It is thus sufficient to find a further contribution of $0.5/256$.

Subcase 4.3. *The vertices u'_+ and u'_- are adjacent, uZv is short and $u'_+ \neq u_{-3}$.*

Suppose first that u'_- (and u'_+) is contained in Z. The event $C_2C_1D^-$ is then covered by the 1-free pair (u'_-, u_{-2}) or (u'_+, u_{-2}). Since its weight is 9, we have $\mathbf{P}(C_2C_1D^-) \geq 1/2 \cdot 0.5/256 = 0.25/256$. Note that the event $C_3C_1D^-$ is valid; it is also regular, so $\mathbf{P}(C_3C_1D^-) \geq 0.25/256$. Together, this yields $0.5/256$, which is sufficient.

We may therefore assume that u'_- (and u'_+) are not contained in Z. The event $C_2C_1D^-$ is covered by (u'_+, u'_-) or its reverse, each of which is 3-free. By Lemma 7, $\mathbf{P}(C_2C_1D^-) \geq 39/40 \cdot 0.5/256 > 0.48/256$. The event $C_3C_1D^-$, if irregular, has the same sensitive pair and it is now 2-free. Since the weight of its diagram is 10, $\mathbf{P}(C_3C_1D^-) \geq 19/20 \cdot 0.25/256 >$

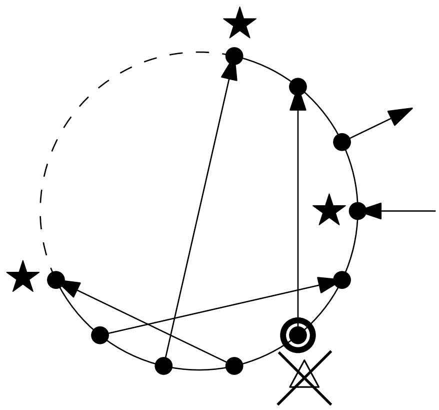


Figure 23: The event C_1AD^+ used in Subcase 4.4 of the proof of Proposition 10.

0.23/256. The total contribution exceeds the desired 0.5/256. △

Subcase 4.4. *The vertices u'_+ and u'_- are adjacent, uZv is short and $u'_+ = u_{-3}$.*

Suppose first that the path v_+Zu_{-4} contains at least two vertices distinct from v'_-. Then the event C_1AD^+ (see Figure 23) is covered by $(v_+, u_{-4})^2$. Since the weight of C_1AD^+ is 10, we have $\mathbf{P}(C_1AD^+) \geq 3/4 \cdot 0.25/256$. The events $C_1C_2D^+$ and $C_1C_3D^+$ have weight 11, but the diagram of each of them has a removable symbol at u_{-3}, so we get the same bound of $3/4 \cdot 0.25/256$ for each of $C_1C_2D^+$ and $C_1C_3D^+$, since each of the diagrams is covered by one 2-free pair. The total contribution is at least 0.56/256.

If v'_- is the only internal vertex of v_+Zu_{-4}, then the above events are in fact regular and we obtain an even higher contribution. Thus, we may assume that either v_+ and u_{-4} are neighbours on Z, or v_+Zu_{-4} contains two internal vertices and one of them is v'_-.

The former case is ruled out since we are assuming (from the beginning of Case 4) that u is not deficient of type IIa. It remains to consider the latter possibility. Here, v'_- is either v_{+2} or v_{+3}. In fact, it must be v_{+3}, since otherwise u would be deficient of type I, which has also been excluded at the beginning of Case 4. But then u is deficient of type III, so $\varepsilon(u) = -0.125$. At the same time, the unique sensitive pair for each of the events C_1AD^+, $C_1C_2D^+$ and $C_1C_3D^+$, considered above, is now 1-free; the probability of the union of these events is thus at least $3 \cdot 1/2 \cdot 0.25/256 = 0.375/256 = (0.5 + \varepsilon(u))/256$ as necessary. △

▲

Case 5. $E(M[U]) = \{u_{-2}v_+, u_{+2}v_-\}$.

As in Case 4, the probability of the event ABD^- is at least 1/256; by symmetry, $\mathbf{P}(BAD^+) \geq 1/256$. We claim that the resulting contribution of 2/256 is sufficient because $\varepsilon(u) \leq 0$. Clearly, v is not of type 0. Applying the definitions of the remaining types to v, we find that none of them is compatible with the presence of the edges $u_{-2}v_+$ and $u_{+2}v_-$ in M. This shows that $\varepsilon(u) \leq 0$. ▲

Case 6. $E(M[U]) = \{u_{-2}u_{+2}\}$.

Recall our assumption that the set $J = \{u_-, u_+, v_-, v_+\}$ is independent. If we suppose that, moreover, both the paths uZv and vZu were short, then the mate of each vertex in J must be outside Z. This means that $|\partial(Z)| = 4$, a contradiction with F satisfying the condition in Theorem 4. Thus, we may assume by symmetry that the path vZu is not short.

The event ABD^- is regular of weight 9, so $\mathbf{P}(ABD^-) \geq 0.5/256$. Similarly, $\mathbf{P}(BAD^+) \geq 0.5/256$. We need to find additional $(1 + \varepsilon(u))/256$ to add to the probabilities of ABD^- and BAD^+ above. Note also that if v is deficient, then it must be of type III* and this only happens in Subcase 6.3.

Subcase 6.1. *uZv is not short.*

Assume that u'_+ is not contained in vZu, and consider the events ABD^+ and ABD^0. If v'_- is not contained in vZu, then ABD^+ is covered by the pair $(v_+, u_{-2})^2$, and it follows that $\mathbf{P}(ABD^+) \geq 3/4 \cdot 0.5/256 = 0.375/256$. Similarly, $\mathbf{P}(ABD^0) \geq 0.375/256$. On the other hand, if v'_- is contained in vZu, then the pair (v_+, u_{-2}) may only be 1-free for ABD^+, whence $\mathbf{P}(ABD^+) \geq 1/2 \cdot 0.5/256 = 0.25/256$, but this decrease is compensated for by the fact that $\mathbf{P}(ABD^0) \geq 0.5/256$ as ABD^0 is now regular. Summarizing, if u'_+ is not contained in vZu, then the probability of $ABD^+ \cup ABD^0$ is at least 0.75/256.

The event BAD^0 of weight 9 is covered by the pair (u_{+2}, v), which is 1-free since uZv is not short. Hence, $\mathbf{P}(BAD^0) \geq 1/2 \cdot 0.5/256 = 0.25/256$. Putting this together, for $u'_+ \notin V(vZu)$ we have:

$$\mathbf{P}(ABD^- \cup BAD^+ \cup ABD^+ \cup ABD^0 \cup BAD^0) \geq \frac{0.5 + 0.5 + 0.375 + 0.375 + 0.25}{256} = \frac{2}{256}.$$

Since this is the required amount, we may assume by symmetry that $u'_+ \in V(vZu)$ and $u'_- \in V(uZv)$ (Figure 24).

If v'_+ is not contained in uZv, then in addition to $\mathbf{P}(BAD^0) \geq 0.25/256$ as noted above, we have $\mathbf{P}(BAD^-) \geq 0.25/256$ for the same reasons. On

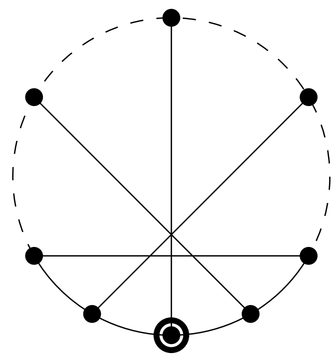


Figure 24: A configuration in Subcase 6.1 of the proof of Proposition 10.

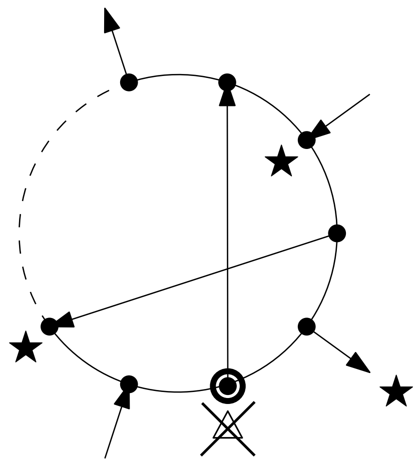


Figure 25: The event AC_1D^- used in Subcase 6.2 of the proof of Proposition 10.

the other hand, $v'_+ \in V(uZv)$ increases the probability bound for BAD^0 to $\mathbf{P}(BAD^0) \geq 0.5/256$ as the event is regular in this case. All in all, the contribution of $BAD^- \cup BAD^0$ is at least $0.5/256$.

By symmetry, $ABD^+ \cup ABD^0$ also contributes at least $0.5/256$. Together with the events ABD^- and BAD^+, which have each a probability of at least $0.5/256$ as discussed above, we have found the required $2/256$. $\triangle$

Thus, the path uZv may be assumed to be short.

Subcase 6.2. $u'_+ \notin V(Z)$.

As in the previous subcase, $\mathbf{P}(ABD^+ \cup ABD^0) \geq 0.75/256$.

The event AC_1D^- has weight 10 (see Figure 25). If the cycle of F containing u'_+ is odd, it contains at least 3 vertices different from u'_+ and v'_+. Thus, AC_1D^- is covered by $(u'_+, u'_+)^3$. By Lemma 7, $\mathbf{P}(AC_1D^-) \geq 39/40 \cdot 0.25/256 > 0.24/256$.

Similarly, AC_1D^0 has a diagram of weight 10 and is covered by $(u'_+, u'_+)^4$ and $(v_-, u_{-2})^2$. By Lemma 7, $\mathbf{P}(AC_1D^0) \geq 59/80 \cdot 0.25/256 > 0.18/256$.

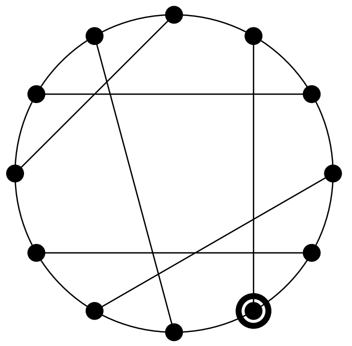


Figure 26: The situation where v is deficient of type III* in Subcase 6.3 of the proof of Proposition 10.

The probability of $AC_1D^- \cup AC_1D^0$ is thus at least $(0.24 + 0.18)/256 = 0.42/256$, more than the missing $0.25/256$. $\triangle$

Subcase 6.3. $u'_+ \in V(Z)$ *and the length of* vZu *is at least 7.*

We will show that the assumption about vZu increases the contribution of $ABD^+ \cup ABD^0$. Suppose that $v'_- \in V(Z)$. Then ABD^+ is covered by $(v_+, u_{-2})^2$ and ABD^0 is regular, so $\mathbf{P}(ABD^+ \cup ABD^0) \geq (3/4 + 1) \cdot 0.5/256 = 0.875/256$. On the other hand, if $v'_- \notin V(Z)$, then the pair (v_+, u_{-2}) is 3-free for both ABD^+ and ABD^0, and we get the same result:

$$\mathbf{P}(ABD^+ \cup ABD^0) \geq 2 \cdot \frac{7}{8} \cdot \frac{0.5}{256} = 0.875/256.$$

We need to find the additional $(0.125 + \varepsilon(u))/256$.

Suppose first that v is deficient, necessarily of type III*, so $\varepsilon(u) = 0.125$. The induced subgraph of G on $V(Z)$ is then as shown in Figure 26; in this case, the event $C_1C_1D^-$ is regular and $\mathbf{P}(C_1C_1D^-) \geq 0.5/256$, a sufficient amount.

We may thus assume that $\varepsilon(u) \leq 0$. Suppose that u'_+ is not adjacent to either u_{-2} or v_+. Then the event AC_1D^0 is covered by $(v_+, u'_+)^2$ and $(u'_+, u_{-2})^2$. By Lemma 7, $\mathbf{P}(AC_1D^0) \geq 1/2 \cdot 0.25/256 = 0.125/256$ as required.

The vertex u'_+ can therefore be assumed to be adjacent to u_{-2} or v_+. The event $C_1C_1D^-$ has only one sensitive pair, namely (u'_-, u'_+) or its reverse (if $u'_- \in V(Z)$) or (u'_-, u'_-) (if u'_- is outside Z). If this is a 1-free pair, then by Lemma 7, $\mathbf{P}(C_1C_1D^-) \geq 1/2 \cdot 0.5/256 > 0.125/256$ as required. In the opposite case, it must be that u'_- is a neighbour of u'_+. Then, however, we observe that $\mathbf{P}(ABD^+)$ and $\mathbf{P}(ABD^0)$ are both at least $0.5/256$ (as the events are regular), and this increase provides the missing $0.125/256$. $\triangle$

To complete the discussion of Case 6, it remains to consider the following subcase.

Subcase 6.4. *uZv is short, $u'_+ \in V(Z)$, and the length of vZu is 6.*

The vertex u'_+ equals either v_{+2} or v_{+3}. Suppose first that $u'_+ = v_{+2}$. Then each edge in $\partial(Z)$ is incident with a vertex in $\{v_{+3}, u_-, v_-, v_+\}$. By the choice of F, M must contain an edge with both ends in the latter set. For trivial reasons, the only candidate is $v_{+3}v_-$, but this is also not an edge of M for we could replace Z with two 5-cycles $u_{-2}Zu_{+2}$ and v_-Zv_{+3}, a contradiction.

Thus, $u'_+ = v_{+3}$. Here, each edge of $\partial(Z)$ is incident with a vertex in $\{u_-, v_-, v_+, v_{+2}\}$, and it is easy to see that one of these edges must be incident with v_+. There are two possibilities for an edge with both ends in $\{u_-, v_-, v_+, v_{+2}\}$, namely v_-v_{+2} or u_-v_{+2}. In either case, the event ABD^0 is easily seen to be regular and thus $\mathbf{P}(ABD^0) \geq 0.5/256$. In fact, this concludes the discussion if $v_-v_{+2} \in E(M)$, since then u is deficient of type I and $\varepsilon(u) = -0.5$, and the contribution of 0.5/256 is sufficient.

In the remaining case that $u_-v_{+2} \in E(M)$, we need a further 0.5/256, and it is provided by the regular event ABD^+. △

▲

Case 7. $E(M[U]) = \{u_{-2}v_-\}$.

If both the paths uZv and vZu are short, then each edge of $\partial(Z)$ is incident with a vertex in $\{u_-, u_+, u_{+2}, v_+\}$. Our assumptions imply that no edge of M joins two of these vertices, so $|\partial(Z)| = 4$ — a contradiction with the choice of F. We may therefore assume that at least one of vZu and uZv is not short.

In all the subcases, we can use the regular event BAD^+, for which we have $\mathbf{P}(BAD^+) \geq 0.5/256$. Hence, we need to find an additional probability of $(1.5 + \varepsilon(u))/256$.

Subcase 7.1. *vZu is short.*

In this subcase, the path v_-vv_+ is contained in a 4-cycle and it is not hard to see that u must be deficient of type I (neither uv nor u_-uu_+ is contained in a 4-cycle, and the missing edge $u_{+2}v_+$ rules out cases Ia* and Ib*). Thus, $\varepsilon(u) = -0.5$ and we need to find further 1/256 worth of probability.

Observe first that by our assumptions, the set $\{u_-, u_+, u_{+2}, v_+\}$ is independent. We will distinguish several cases based on whether u'_-, u'_+ and v'_+ are contained in Z (and hence in $u_{+3}Zv_{-2}$) or not.

If $u'_+ \in V(Z)$, then the events BAD^0 and BAD^- are regular, and each of them has probability $0.5/256$, which provides the necessary $1/256$.

Suppose thus that $u'_+ \notin V(Z)$ and consider first the case that $u'_- \notin V(Z)$. The event C_1AD^+ is covered by the pair $(u'_-, u'_-)^4$, so by Lemma 7 its probability is $\mathbf{P}(C_1AD^+) \geq 79/80 \cdot 0.25/256 > 0.24/256$. The event C_1AD^0 has up to two sensitive pairs: it is covered by $(u'_-, u'_-)^4$ and $(u_{+2}, v_-)^2$, where the latter pair is 2-free because uZv is not short. We obtain $\mathbf{P}(C_1AD^0) \geq 59/80 \cdot 0.25/256 > 0.18/256$.

To find the remaining $0.58/256$ (still for $u'_- \notin V(Z)$), we use the events BAD^0 and BAD^-. We claim that their probabilities add up to at least $0.75/256$. Indeed, if $v'_+ \notin V(Z)$, then both BAD^0 and BAD^- are covered by the pair $(u_{+2}, v_-)^2$ (which is 2-free because uZv is not short and $u'_- \notin V(Z)$). By Lemma 7, they have probability at least $0.375/256$ each. On the other hand, if $v'_+ \in V(Z)$, then BAD^0 is regular and BAD^- is covered by $(u_{+2}, v_-)^1$, so $\mathbf{P}(BAD^0) \geq 0.5/256$ and $\mathbf{P}(BAD^-) \geq 0.25/256$. For both of the possibilities, $\mathbf{P}(BAD^0 \cup BAD^-) \geq 0.75/256$ as claimed.

We can therefore assume that $u'_- \in V(Z)$ (and $u'_+ \notin V(Z)$, of course). A large part of the required $1/256$ is provided by the event $C_1C_1D^+$, which is covered by the pair $(u'_+, u'_+)^4$, so $\mathbf{P}(C_1C_1D^+) \geq 79/80 \cdot 0.5/256 > 0.49/256$.

A final case distinction will be based on the location of v'_+. Suppose first that $v'_+ \notin V(Z)$. We claim that the length of uZv is at least 7. If not, then since uZv is not short, the length of Z is 9 or 10. At the same time, Z has at least 3 chords (incident with u, u_- and u_{-2}) and therefore $|\partial(Z)| \leq 4$. By the choice of F, $\partial(Z)$ and the assumption that the mates of u_+ and v_+ are outside Z, Z has length 10 and $\partial(Z)$ is of size 2. In addition, u_{+2} is incident with a chord of Z whose other endvertex w is contained in $u_{+3}Zv_{-2}$. However, $|uZv| = 6$ implies that $w \in \{u_{+3}, u_{+4}\}$, contradicting the assumption that G is simple and triangle-free. We conclude that $|uZv| \geq 7$ as claimed.

This observation implies that for the event BAD^0, the only possibly sensitive pair, namely (u_{+2}, v_-), is 2-free. Hence, $\mathbf{P}(BAD^0) \geq 3/4 \cdot 0.5/256 = 0.375/256$. Hence, $\mathbf{P}(BAD^-) \geq 0.375/256$ and this amount is sufficient.

It remains to consider the case that $v'_+ \in V(Z)$. Being regular, the event BAD^0 has probability at least $0.5/256$. Thus, it is sufficient to find further events forcing u of total probability at least $0.01/256$. It is easiest to consider the mutual position of u'_- and v'_+ on $u_{+3}Zv_{-2}$. If $u'_- \in V(v'_+Zv_{-2})$, then the event C_1AD^+ is regular and has probability at least $0.25/256$. In the opposite case, $C_1C_1D^0$ is covered by the pair $(u'_+, u'_+)^4$, which means that $\mathbf{P}(C_1C_1D^0) \geq 79/80 \cdot 0.5/256 > 0.49/256$.

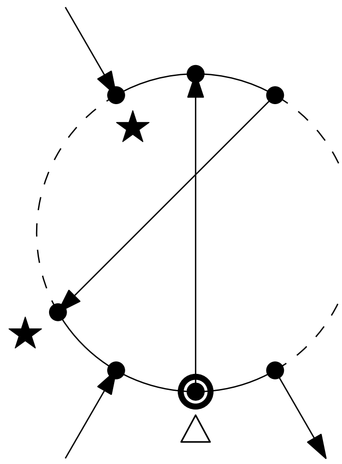


Figure 27: The event ABD^+ used in Subcase 7.2 of the proof of Proposition 10.

In both cases, the probability is sufficiently high. △

Having dealt with Subcase 7.1, we can use the event AAD^+, which is covered by $(v_+, u_{-2})^2$. By Lemma 7, $\mathbf{P}(AAD^+) \geq 3/4 \cdot 0.5/256 = 0.375/256$ and hence $\mathbf{P}(BAD^+ \cup AAD^+) \geq 0.875/256$. Since v is not deficient, we seek a further contribution of at least 1.125/256.

Subcase 7.2. *Neither vZu nor uZv is short.*

Consider the event ABD^+ of weight 8 (Figure 27) which is covered by the pair (v_+, u_{-2}). Since vZu is not short, the vertices in the pair are not neighbours. Furthermore, if the pair is sensitive, then the path v_+Zu_{-2} contains at least two internal vertices, one of which is different from u'_+. Thus, the pair is 1-free and by Lemma 7, $\mathbf{P}(ABD^+) \geq 1/2 \cdot 1/256$. If the pair (v_+, u_{-2}) is actually 2-free in ABD^+, then the estimate increases to $3/4 \cdot 1/256$.

The event BAD^0 is covered by the pair (u_{+2}, v_-), which is 1-free as uZv is not short; moreover, if $u'_- \notin V(uZv)$, then the pair is 2-free. Thus, $\mathbf{P}(BAD^0) \geq 1/2 \cdot 0.5/256 = 0.25/256$ or $3/4 \cdot 0.5/256 = 0.375/256$ in the respective cases.

If the higher estimates hold for both the events ABD^+ and BAD^0 considered above, then the contributions of these events total

$$\frac{0.75 + 0.375}{256} = \frac{1.125}{256},$$

which is sufficient.

Suppose first that we get the higher estimate for $\mathbf{P}(ABD^+)$, that is, that (v_+, u_{-2}) is 2-free in ABD^+. By the above, it may be assumed that $u'_- \in V(uZv)$ and the pair (u_{+2}, v_-) is not 2-free. We need to find an additional 0.125/256. To this end, we use the event BAD^- of weight 9. The probability of BAD^- is at least $1/2 \cdot 0.5/256$ (which is sufficient) if (u_{+2}, v_-) is 1-free in BAD^-. This could be false only if $\{u'_-, v'_+\} = \{u_{+3}, v_{-2}\}$, but the corresponding two possibilities are easily ruled out: if $u'_- = v_{-2}$, then we may replace Z in F by the cycles u_-Zv_{-2}, v_-Zu_{-2} and obtain a contradiction with the choice of F; in the other case, u is deficient of type I, so $\varepsilon(u) = -0.5$ and the contribution from BAD^- is actually unnecessary.

It remains to discuss the possibility that (v_+, u_{-2}) is not 2-free in ABD^+ — thus, the length of vZu is 6 and $u'_+ \in \{v_{+2}, v_{+3}\}$. Since the lower bound to $\mathbf{P}(AAD^+)$ increases to 0.5/256 in this case, the total probability of BAD^+, AAD^+ and ABD^+ is at least 1.5/256. In addition, we have a contribution of $1/2 \cdot 0.5/256$ from BAD^0.

Consider the diagram $C_1C_1D^-$. We claim that $\mathbf{P}(C_1C_1D^-) \geq 1/2 \cdot 0.5/256$. This is certainly true if $u'_- \notin V(Z)$ since $C_1C_1D^-$ has weight 9 and it is covered by $(u'_-, u'_-)^4$. Suppose thus that $u'_- \in V(Z)$. There is at most one sensitive pair for $C_1C_1D^-$ ((u'_-, u'_+) or (u'_-, v_-) or none). If the event is regular or the sensitive pair is 1-free, then $\mathbf{P}(C_1C_1D^-) \geq 1/2 \cdot 0.5/256$ as required. Otherwise, since there is only one outgoing arc in the diagram for $C_1C_1D^-$, u'_- must be adjacent either to u'_+ or to v_-. In the former case, we can reach a contradiction with our assumptions by replacing Z with the cycles $u'_+u_+Zv_-u_{-2}$ and $uvZv_{+2}u_-$. In the latter case, the cycles u_-Zv_{-2}, v_-Zu_{-2} do the job. This finishes Subcase 7.2.

△

Subcase 7.3. *uZv is short and either the length of vZu is at least 7, or $u'_+ \notin V(vZu)$.*

The event ABD^+ is covered by $(v_+, u_{-2})^2$ by the assumption. Thus, $\mathbf{P}(ABD^+) \geq 3/4 \cdot 1/256$. In view of the events BAD^+ (probability at least 0.5/256) and AAD^+ (probability at least $3/4 \cdot 0.5/256$), we need to collect further 0.375/256.

Suppose first that $u'_+ \notin V(vZu)$. The event AC_1D^+ of weight 9 is covered by $(u'_+, u'_+)^4$ and $(v_+, u_{-2})^2$. By Lemma 7, $\mathbf{P}(AC_1D^+) \geq 59/80 \cdot 0.5/256 > 0.36/256$. The event AC_2D^+ of weight 11 is covered by $(v_+, u_{-2})^2$; thus, $\mathbf{P}(AC_2D^+) \geq 3/4 \cdot 0.125/256$, which together with $\mathbf{P}(AC_1D^+)$ yields more than the required 0.375/256.

We may therefore assume that $u'_+ \in V(vZu)$, which increases $\mathbf{P}(AAD^+)$ to at least 0.5/256 (so the missing probability is now 0.25/256).

Suppose that u'_- and u'_+ are non-adjacent. If $u'_- \notin V(Z)$, then $C_1C_1D^-$ is covered by $(u'_-, u'_-)^3$. Otherwise, it is covered by $(u'_-, u'_+)^1$

(we have to consider v'_+ here). In either case, $\mathbf{P}(C_1C_1D^-) \geq 1/2 \cdot 0.5/256$ as required.

We may thus assume that u'_- and u'_+ are adjacent. The event AC_1D^+ has weight 9 and at most one possibly sensitive pair; this pair is (u'_+, u_{-2}) if $(u'_-)_+ = u'_+$, or (u'_+, v_+) otherwise. If the sensitive pair is 2-free, we are done since $\mathbf{P}(AC_1D^+) \geq 3/4 \cdot 0.5/256$. In the opposite case, we get two possibilities.

The first possibility is that u'_+ is adjacent to u_{-2}, so $u'_+ = u_{-3}$. The cycle Z may then be replaced by the cycles $u_{-3}u_+Zv_-u_{-2}$ and $uvZu_{-4}u_-$, a contradiction.

The second possibility is that u'_+ is adjacent to v_+, i.e., $u'_+ = v_{+2}$. Here, the event AC_2D^+ is regular, and $\mathbf{P}(AC_2D^+) \geq 0.25/256$ as desired.

$\triangle$

Subcase 7.4. *uZv is short, the length of vZu is 6, and $u'_+ \in V(vZu)$.*

The vertex u'_+ equals either v_{+2} or v_{+3}. Each of the events ABD^+, BAD^+, AAD^+ (considered earlier) now have probability at least 0.5/256. We need to find an additional 0.5/256.

If $u'_+ = v_{+2}$, then each edge of $\partial(Z)$ is incident with a vertex from the set $\{u_-, u_{+2}, v_+, v_{+3}\}$. By the choice of F, some edge of M must join two of these vertices; our assumptions imply that the only candidate is the edge $u_{-3}u_{+2}$. The events AC_2D^+, C_2AD^+ and $C_2C_2D^+$ are regular, with AC_2D^+ having a removable symbol, and their probabilities are easily computed to be at least 0.25/256, 0.125/256 and 0.125/256, respectively. This adds up to the required 0.5/256.

On the other hand, if $u'_+ = v_{+3}$, then each edge of $\partial(Z)$ is incident with $\{u_-, u_{+2}, v_+, v_{+2}\}$. In each case, we obtain a contradiction with the choice of F by replacing Z with two cycles of length 5: if $u_{+2}v_{+2} \in E(M)$, then the cycles are $u_{-3}Zu_+$ and $u_{+2}Zv_{+2}$, while if $u_-v_{+2} \in E(M)$, then Z may be replaced by u_-uvZv_{+2} and $u_+2Zv_-u_{-2}u_{-3}$. All the other cases are ruled out by the assumptions (notably, the assumption that $u_{+2}v_+ \notin E(M)$). $\triangle$

The only possibility in Case 7 not covered by the above subcases is that uZv is short, vZu has length 5 and $u'_+ \in V(vZu)$. This is, however, excluded by our choice of Z: the cycle Z of length 9 would have at least three chords, implying $|\partial(Z)| \in \{1, 3\}$, which is impossible. $\blacktriangle$

Case 8. $E(M[U]) = \{u_{-2}v_-, u_{+2}v_+\}$.

We will call a chord f of Z *bad* if $f = u_-u_{+3}$ or $f = u_+u_{-3}$.

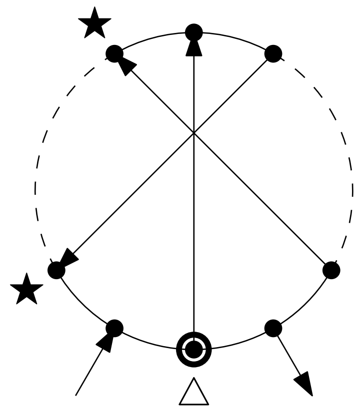


Figure 28: The event ABD^+ used in Subcase 8.1 of the proof of Proposition 10.

Subcase 8.1. *Neither uZv nor vZu is short and Z has no bad chord.*

The event ABD^+ has one sensitive pair, namely (v_+, u_{-2}) (see Figure 28). We claim that this pair is 2-free. Suppose not; then it must be that u'_+ is an internal vertex of v_+Zu_{-2} and there is exactly one other internal vertex in the path. Since $u'_+ \neq u_{-3}$ (that would correspond to a bad chord), we have $u'_+ = u_{-4} = v_{+2}$. However, we may then replace Z with the cycles $v_{+2}Zu_+$ and $u_{+2}Zv_+$ and obtain a contradiction with the choice of F. Hence (v_+, u_{-2}) is 2-free in ABD^+ and $\mathbf{P}(ABD^+) \geq 3/4 \cdot 1/256$ as the weight of ABD^+ is 8.

For a similar reason, $\mathbf{P}(BAD^-) \geq 3/4 \cdot 1/256$. Since v is not deficient in this subcase, it suffices to find a further $0.5/256$ to reach the desired bound.

Suppose first that u'_- and u'_+ are not neighbours.

If u'_- and u'_+ are contained in two distinct cycles of F, both different from Z, then by Lemma 7, we have $\mathbf{P}(C_1C_1D^+) \geq 39/40 \cdot 0.5/256$ and the same estimate holds for $C_1C_1D^0$ and $C_1C_1D^-$. Thus

$$\mathbf{P}(C_1C_1D^+ \cup C_1C_1D^0 \cup C_1C_1D^-) \geq \frac{1.46}{256},$$

much more than the required amount.

If u'_+ and u'_- are contained in the same cycle $Z' \neq Z$ of F, then the event $C_1C_1D^+$ is covered by $(u'_+, u'_-)^2$ and $(u'_-, u'_+)^2$. By Lemma 7, $\mathbf{P}(C_1C_1D^+) \geq 1/2 \cdot 0.5/256$. Since the same holds for $C_1C_1D^0$ and $C_1C_1D^-$, we find a sufficient contribution of $0.75/256$.

If, say, u'_+ is contained in Z and u'_- is not, then $C_1C_1D^+$ is covered by the pairs $(v_+, u'_+)^2$ and $(u'_-, u'_-)^4$ (note that the first pair is 2-free since if v_+ is a neighbour of u'_+, then we can replace Z by two shorter cycles in F and obtain a contradiction). Using Lemma 7, we find that $\mathbf{P}(C_1C_1D^+) \geq$

$59/80 \cdot 0.5/256 > 0.36/256$. Similarly, C_1AD^- is covered by $(u_{+2}, v_-)^2$ and $(u'_-, u'_-)^4$, so by Lemma 7, $\mathbf{P}(C_1AD^-) \geq 59/80 \cdot 0.5/256 > 0.36/256$. Thus,

$$\mathbf{P}(C_1C_1D^+ \cup C_1AD^-) \geq \frac{0.36 + 0.36}{256} = \frac{0.72}{256}$$

and we are done.

Thus, still in the case that u'_+ and u'_- are not neighbours, we may assume that they are both contained in Z. Consider the event AC_1D^+. If $u'_- \in V(vZu)$, then the event is covered by a single 2-free pair, namely (v_+, u'_+) or (u'_+, u_{-2}), so $\mathbf{P}(AC_1D^+) \geq 3/4 \cdot 0.5/256$. On the other hand, if $u'_- \in V(uZv)$, then AC_1D^+ is regular if $u'_+ \in V(uZv)$ or covered by $(v_+, u'_+)^2$ and $(u'_+, u_{-2})^2$ otherwise. Summing up, $\mathbf{P}(AC_1D^+) \geq 1/2 \cdot 0.5/256$. Symmetrically, $\mathbf{P}(C_1AD^-) \geq 1/2 \cdot 0.5/256$ and we have found the necessary $0.5/256$.

We may thus assume that u'_- and u'_+ are neighbours.

If they are in contained in a cycle of F different from Z, then the event C_1AD^- is covered by the 2-free pair (u_{+2}, v_-), so $\mathbf{P}(C_1AD^-) \geq 3/4 \cdot 0.5/256$. By symmetry, $\mathbf{P}(AC_1D^+) \geq 3/4 \cdot 0.5/256$, making for a sufficient contribution of $1.5/256$.

We may thus suppose that u'_- and u'_+ are both contained in vZu. By the choice of F and the absence of bad chords, u'_+ is not a neighbour of v_+ nor u_{-2}. Thus, the event AC_1D^+ is covered by a single 2-free pair, namely (v_+, u'_+) or (u'_+, u_{-2}), and $\mathbf{P}(AC_1D^+) \geq 3/4 \cdot 0.5/256$. Moreover, $\mathbf{P}(C_1AD^-) \geq 3/4 \cdot 0.5/256$ since the event is covered by $(u_{+2}, v_-)^1$, so

$$\mathbf{P}(AC_1D^+ \cup C_1AD^-) \geq \frac{0.375 + 0.375}{256} = \frac{0.75}{256}$$

as required. This finishes Subcase 8.1. △

Subcase 8.2. *Neither uZv nor vZu is short, but Z has a bad chord.*

By symmetry, we may assume that u_+u_{-3} is a bad chord. Note that by the choice of F, the length of vZu is at least 6, since otherwise Z could be replaced in F by the cycles $u_{-3}Zu_+$ and $u_{+2}Zv_+$. Furthermore, $u'_- \neq u_{-4}$ in view of the possible replacement of Z by the cycles $u_+Zv_-u_{-2}u_{-3}$ and u_-uvZu_{-4}.

We distinguish three cases based on the position of u'_-. Assume first that u'_- is contained in vZu. The regular event ABD^+ has probability at least $1/256$. The event BAD^- is covered by the pair (u_{+2}, v_-) which is 2-free since uZv is not short. Thus $\mathbf{P}(BAD^-) \geq 3/4 \cdot 1/256$. Finally, the event $C_1C_1D^-$ is covered by the pair (u'_-, u_{-3}) which is 2-free since $u'_- \neq u_{-4}$ as noted above. Consequently,

$$\mathbf{P}(ABD^+ \cup BAD^- \cup C_1C_1D^-) \geq \frac{1}{256} + \frac{0.75}{256} + \frac{0.75 \cdot 0.5}{256} = \frac{2.125}{256},$$

more than the required 2/256.

Suppose next that u'_- is contained in uZv. Note that $u'_- \neq v_{-2}$ by the choice of F. Since the event ABD^+ is covered by $(v_+, u_{-2})^1$, $\mathbf{P}(ABD^+) \geq 1/2 \cdot 1/256$. Similarly, BAD^- is covered by $(u_{+2}, v_-)^1$ and so $\mathbf{P}(BAD^-) \geq 1/2 \cdot 1/256$. The event $C_1C_1D^-$ is covered by the 2-free pair (v_+, u_{-3}) and thus $\mathbf{P}(C_1C_1D^-) \geq 3/4 \cdot 0.5/256 = 0.375/256$. The same bound is valid for $C_1C_1D^+$. Finally, $\mathbf{P}(C_1C_1D^0) \geq 1/2 \cdot 0.5/256$ as the event is covered by $(u'_-, v_-)^2$ and $(v_+, u_{-3})^2$. Altogether, we have

$$\mathbf{P}(ABD^+ \cup BAD^- \cup C_1C_1D^- \cup C_1C_1D^+ \cup C_1C_1D^0) \geq \frac{0.5 + 0.5 + 0.375 + 0.375 + 0.25}{256} = \frac{2}{256}.$$

The last remaining possibility is that u'_- is not contained in Z. We have $\mathbf{P}(ABD^+) \geq 0.5/256$ and $\mathbf{P}(BAD^-) \geq 0.75/256$ by standard arguments. The event $C_1C_1D^-$ is covered by the pair $(u'_-, u'_-)^4$, so $\mathbf{P}(C_1C_1D^-) \geq 79/80 \cdot 0.5/256 > 0.49/256$ by Lemma 7. Similarly, $\mathbf{P}(C_1C_1D^+) \geq 59/80 \cdot 0.5/256 > 0.36/256$ since the event is covered by $(u'_-, u'_-)^4$ and $(v_+, u_{-3})^2$. The total contribution is at least 2.1/256. This concludes Subcase 8.2. △

We may now assume that the path uZv is short; note that this means that u is deficient of type Ia or Ib. In the former case, there is nothing to prove as $2 + \varepsilon(u) = 0$. Therefore, suppose that u is of type Ib (i.e., $u'_+ = v_{+2}$). Since $\varepsilon(u) = -1.5$, it remains to find events forcing u with total probability at least 0.5/256. It is sufficient to consider the event ABD^+ of weight 8, which is covered by the 1-free pair (v_+, u_{-2}), and therefore $\mathbf{P}(ABD^+) \geq 0.5/256$ by Lemma 7. This finishes the proof of Case 8 and the whole proposition. ▲

□

7 Augmentation

In this section, we show that it is possible to apply the augmentation step mentioned in the preceding sections.

Suppose that u is a deficient vertex of G and $v = u'$. Let us continue to use Z to denote the cycle of the 2-factor F containing u. The *sponsor* $s(u)$ of u is one of its neighbours, defined as follows:

- if u is deficient of type 0 (recall that this type was defined at the beginning of Section 5), then $s(u)$ is the F-neighbour u with $\varepsilon(s(u)) = 1$; if there are two such F-neighbours, we choose $s(u) = u_-$,
- if u is deficient of any other type (in particular, $v \in V(Z)$), then $s(u) = v$.

Observation 11. *Every vertex is the sponsor of at most one other vertex.*

Proof. Clearly, a given vertex can only sponsor its own neighbours, that is, its mate and F-neighbours. Suppose that u is the sponsor of its mate v; thus, $u \in C_v$. Suppose also that u is the sponsor of one of its F-neighbours, say u_+. Then uv belongs to a 4-cycle intersecting $C_{u'_+}$, but this is not possible since $C_{u'_+} \neq C_v$.

The only remaining possibility is that u is the sponsor of both of its F-neighbours. In that case, both u_+ and u_- are deficient of type 0 and $\varepsilon(u) = 1$. Thus, uv is contained in a 4-cycle, but neither u_+ or u_- is, giving rise to a contradiction. □

We let $N[u]$ denote the closed neighbourhood of u, i.e., $N[u] = N(u) \cup \{u\}$. An independent set J in G is *favourable* for u if $N[u] \cap J = \{s(u)\}$. The *receptivity* of u, denoted $\rho(u)$, is the probability that a random independent set (with respect to the distribution given by Algorithm 1) is favourable for u. We say that u is *k-receptive* ($k \geq 0$) if the receptivity of u is at least $k/256$.

For an independent set J, we let $p(J)$ denote the probability that the random independent set produced by Algorithm 1 is equal to J. We fix an ordering $J_1, \dots, J_s$ of all independent sets J in G such that $p(J) > 0$. Furthermore, an ordering $u_1, \dots, u_r$ of all deficient vertices is chosen in such a way that $|\varepsilon(u_i)| \leq |\varepsilon(u_j)|$ if $1 \leq i < j \leq s$ (to which we refer as the *monotonicity* of the ordering).

Let u_i be a deficient vertex. We let $\tilde{N}(u_i)$ be the set of all deficient neighbours u_j of u_i such that $j < i$; furthermore, we put $\tilde{N}[u_i] = \tilde{N}(u_i) \cup \{u_i\}$. We define $\eta(u_i)$ as

$$\eta(u_i) = \sum_{u_j \in \tilde{N}[u_i]} |\varepsilon(u_j)| \, .$$

We aim to replace $s(u_i)$ with u_i in some of the independent sets that are favourable for u_i, thereby boosting the probability of the inclusion of u_i in the random independent set I. Clearly, this requires that the receptivity of u_i is at least $|\varepsilon(u_i)|/256$, for otherwise the probability of $u_i \in I$ cannot be increased to the required $88/256$ in this way. We also need to take into account the fact that an independent set may be favourable for u_i and its neighbour at the same time, but the replacement can only take place once. To dispatch the replacements in a consistent way, the following lemma will be useful. We remark that the number $p(u_i, J_j)$ which appears in the statement will turn out to be the probability that u_i is added to the random independent set during Phase 5 of the execution of the algorithm.

Lemma 12. *If the receptivity of each deficient vertex u_i is at least $\eta(u_i)$, then we can choose a nonnegative real number $p(u_i, J_j)$ for each deficient vertex u_i and each independent set J_j in such a way that the following holds:*

(i) $p(u_i, J_j) = 0$ *whenever J_j is not favourable for u_i,*

(ii) for each deficient vertex u_i, $\sum_j p(u_i, J_j) \cdot p(J_j) = |\varepsilon(u_i)|/256$*,*

(iii) for each independent set J_j and deficient vertex u_i, $\sum_{u_t \in \tilde{N}[u_i]} p(u_t, J_j) \leq 1$*.*

Proof. We may view the numbers $p(u_i, J_j)$ as arranged in a matrix (with rows corresponding to vertices) and choose them in a simple greedy manner as follows. For each $i = 1, \dots, r$ in this order, we determine $p(u_i, J_1)$, $p(u_i, J_2)$ and so on. Let $\vec{r_i}$ be the i-th row of the matrix, with zeros for the entries that are yet to be determined. Furthermore, let $\vec{p} = (p(J_1), \dots, p(J_s))$.

For each i, j such that J_j is favourable for u_i, $p(u_i, J_j)$ is chosen as the maximal number such that $\vec{r_i} \cdot \vec{p}^T \leq |\varepsilon(u_i)|/256$, and its sum with any number in the j-th column corresponding to a vertex in $\tilde{N}(u_i)$ is at most one. In other words, we set

$$p(u_i, J_j) = \min\Big(\frac{|\varepsilon(u_i)|/256 - \sum_{\ell=1}^{j-1} p(u_i, J_\ell) \cdot p(J_\ell)}{p(J_j)}, 1 - \sum_{u_\ell \in \tilde{N}(u_i)} p(u_\ell, J_j)\Big) \tag{2}$$

if J_j is favourable for u_i, and $p(u_i, J_j) = 0$ otherwise. Note that the denominator in the fraction is nonzero since every independent set J_j with $1 \leq j \leq s$ has $p(J_j) > 0$. By the construction, properties (i) and (iii) in the lemma are satisfied, and so is the inequality $\vec{r_i} \cdot \vec{p}^T \leq |\varepsilon(u_i)|/256$ in property (ii). We need to prove the converse inequality.

Suppose that for some i, $\vec{r_i} \cdot \vec{p}^T$ is strictly smaller than $|\varepsilon(u_i)|/256$. This means that in (2), for each j such that J_j is favourable for u_i, $p(u_i, J_j)$ equals the second term in the outermost pair of brackets. In other words, for each such j, we have

$$\sum_{u_\ell \in \tilde{N}[u_i]} p(u_\ell, J_j) = 1.$$

Thus, we can write

$$\sum_{\substack{j \\ J_j \text{ favourable for } u_i}} \Big(\sum_{u_\ell \in \tilde{N}[u_i]} p(u_\ell, J_j) \Big) \cdot p(J_j) = \sum_{\substack{j \\ J_j \text{ favourable for } u_i}} p(J_j) \tag{3}$$

$$= \rho(u_i) \geq \eta(u_i) = \sum_{u_\ell \in \tilde{N}[u_i]} \frac{|\varepsilon(u_\ell)|}{256},$$

where the inequality on the second line follows from our assumption on the receptivity of u_i.

On the other hand, the expression on the first line of (3) is dominated by the sum of the scalar products of $\vec{p}$ with the rows corresponding to vertices in $\tilde{N}[u_i]$. For each such vertex u_ℓ, we know from the first part of the proof that $\vec{r_\ell} \cdot \vec{p}^T \leq |\varepsilon(u_\ell)|/256$. Comparing with (3), we find that we must actually have equality both here and in (3); in particular,

$$\vec{r_i} \cdot \vec{p}^T = \frac{|\varepsilon(u_i)|}{256},$$

a contradiction. □

For brevity, we will say that an event $X \subseteq \Omega$ is *favourable for* u if the independent set $I(\sigma)$ is favourable for u for every situation $\sigma \in X$. We lower-bound the receptivity of deficient vertices as follows:

Proposition 13. *Let u be a deficient vertex. The following holds:*

(i) u is 1.9-receptive,

(ii) if u is of type 0, then it is 3-receptive,

(iii) if u is of type Ia or Ib (or their mirror types), then it is 8-receptive.

Proof. All the event(s) discussed in this proof will be favourable for u, as it is easy to check. To avoid repetition, we shall not state this property in each of the cases.

(i) First, let u be a deficient vertex of type I. We distinguish three cases, in each case presenting an event which is favourable for u and has sufficient probability. If $u_{-2}u_{+2}$ is not an edge of M, then the event Q_1 given by the diagram in Figure 29(a) is valid. Since it is a regular diagram of weight 7, $\mathbf{P}(Q_1) \geq 2/256$ by Lemma 7. Thus, $\rho(u) \geq 2/256$ as Q_1 is favourable for u.

We may thus assume that $u_{-2}u_{+2} \in E(M)$. Suppose that neither u'_- nor u'_+ is contained in vZu. Consider the event Q_2, given by the diagram in Figure 29(b). Since the edge uv is not contained in a 4-cycle (u being deficient), neither v_- nor v_+ is the mate of u_+, so the diagram is valid. The event is covered by the pair (u'_+, u'_+). If the pair is sensitive, then the cycle of F containing u'_+ has length at least 5, and hence it contains at least two vertices different from u'_+, v'_- and v'_+. Thus, the pair is 2-free, and we have $\mathbf{P}(Q_2) \geq 19/20 \cdot 2/256 = 1.9/256$ by Lemma 7.

By symmetry, we may assume that each of uZv and vZu contain one of u'_- and u'_+. Hence, the event Q_3, defined by Figure 29(c), is regular and $\mathbf{P}(Q_3) \geq 2/256$. (The event is valid for the same reason as Q_2.)

To finish part (i), it remains to discuss deficient vertices of types other than I. In view of parts (ii) and (iii), it suffices to look at types II, IIa, III and their mirror variants. Each of these types is consistent with the diagram in Figure 29(d) or its symmetric version. The diagram of weight 6 defines a regular event Q_4, whose probability is at least 4/256 by Lemma 7. This proves part (i).

We prove (ii). Let u be deficient of type 0. We may assume that u_- is contained in a 4-cycle intersecting the cycle C_v; in particular, the mates of u_- and u_{-2} are contained in C_v. By the definition of type 0, we also know that neither u_{-2} nor u_{+2} has a neighbour in $\{v_-, v_+\}$.

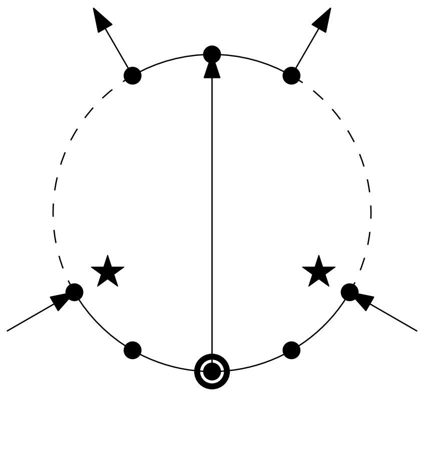
Suppose that the set $\{u_{-2}, u_{+2}, v_-, v_+\}$ is independent. Since $u'_- \in V(C_v)$, the event R defined by the diagram in Figure 30(a) is regular and it is easy to see that it is favourable for u and its probability is at least 1/256. Since the same holds for the events R^+ and R^- obtained by reversing the arrow at v_- or v_+, respectively, we have shown that u is 3-receptive in this case.

If M includes the edge $u_{-2}v_+$, then both R and R^+ remain valid events, and the probability of each of them increases to at least 2/256, showing that u is 4-receptive. An analogous argument applies if M includes $u_{-2}v_-$.

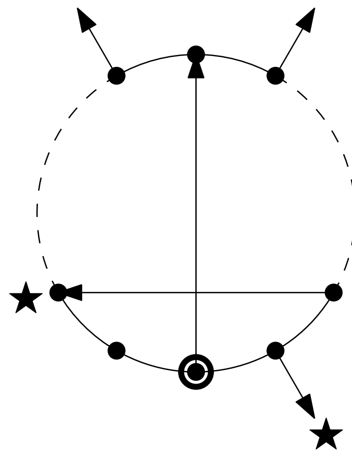
It remains to consider the possibility that $u_{+2}v_-$ or $u_{+2}v_+$ is in M. Suppose that $u_{+2}v_- \in E(M)$. The event R^- remains valid and regular; its probability increases to at least 2/256. Let S^+ and T^+ be the events given by diagrams in Figure 30(b) and (c), respectively. It is easy to check that R^+, S^+ and T^+ are pairwise disjoint and favourable for u. The event S^+ is covered by the pair (u_{+2}, u_-)[1] and Lemma 7 implies that $\mathbf{P}(S^+) \geq 0.5/256$. The event T^+ is regular and $\mathbf{P}(T^+) \geq 0.5/256$. Since $\mathbf{P}(R^+ \cup S^+ \cup T^+) \geq 3/256$, u is 3-receptive.

In the last remaining case, namely $u_{+2}v_+ \in E(M)$, we argue similarly. Let S^- and T^- be the events obtained by reversing both arcs incident with v_+ and v_- in the diagram for S^+ or T^+, respectively. It is routine to check that $\mathbf{P}(R^- \cup S^- \cup T^-) \geq 3/256$ and the events are favourable for u. Hence, u is 3-receptive. The proof is finished.

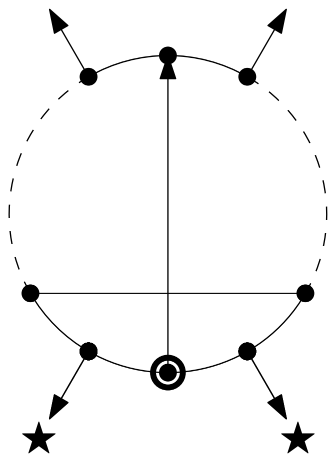
Part (iii) follows by considering the event defined by the diagram in Figure 31. Note that the event is regular and its probability is at least



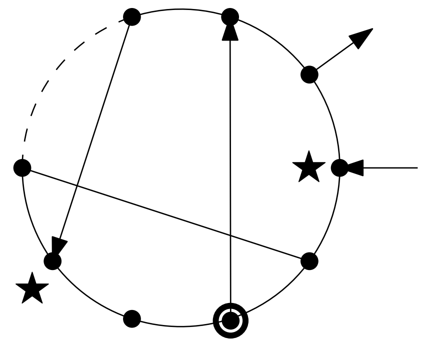
(a) Q_1.



(b) Q_2.

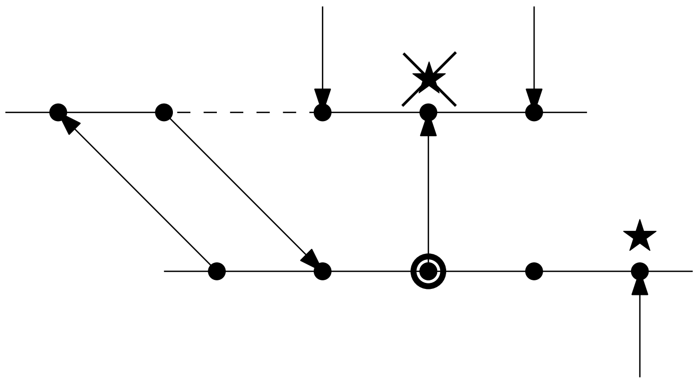


(c) Q_3 (note that each of uZv, vZu contains one of u'_+, u'_-).

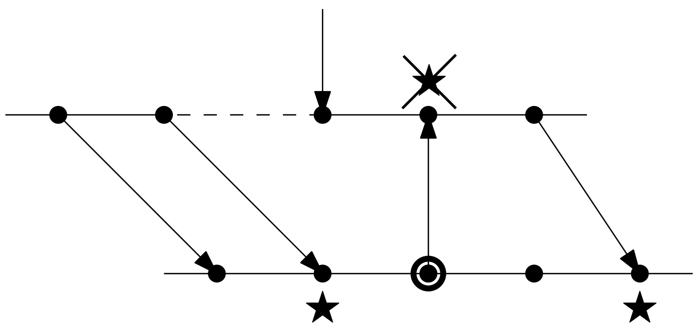


(d) Q_4.

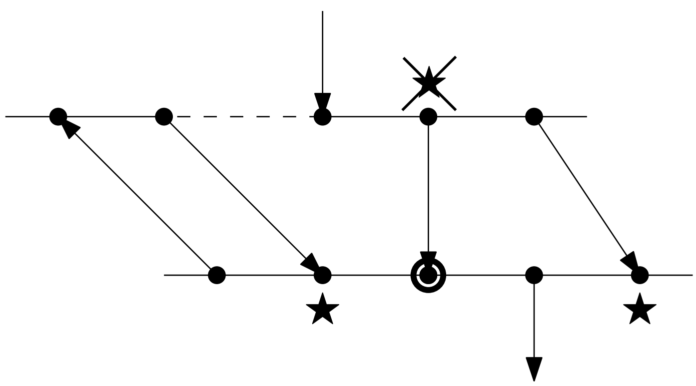
Figure 29: Events used in the proof of Proposition 13(i).



(a) R.



(b) S^+.



(c) T^+.

Figure 30: Events used in the proof of Proposition 13(ii) for vertices of type 0. Only the possibility that $u'_{-2} = (u'_-)_-$ is shown, but the events remain valid if $u'_{-2} = (u'_-)_+$ (i.e., if the chords of Z incident with u_- and u_{-2} cross).

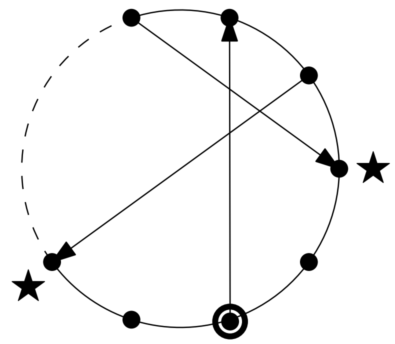


Figure 31: The event used in the proof of Proposition 13(iii) for vertices of type Ia and Ib and their mirror types.

$1/2^5 = 8/256$. Furthermore, the event is favourable for the vertex u. Thus, u is 8-receptive. $\square$

We now argue that Proposition 13 implies the assumption of Lemma 12 that the receptivity of a deficient vertex u_i is at least $\eta(u_i)$. By the monotonicity of the ordering $u_1, \dots, u_r$ and the fact that $\left|\tilde{N}[u_i]\right| \leq 4$ and each deficient vertex has at least one non-deficient neighbour (namely its sponsor), we have $\eta(u_i) \leq 3\,|\varepsilon(u_i)|$. From Proposition 13 and the definition of $\varepsilon(u_i)$ (see the beginning of Section 5 and Table 1), it is easy to check that u_i is $(3\,|\varepsilon(u_i)|)$-receptive, which implies the claim.

Hence, the assumption of Lemma 12 is satisfied. Let $p(u_i, J_j)$ be the numbers whose existence is guaranteed by Lemma 12. We can finally describe Algorithm 2, which consists of the four phases of Algorithm 1, followed by **Phase 5** described below.

Assume a fixed independent set $I = J_j$ was produced by Phase 4 of the algorithm. We construct a sequence of independent sets $I^{(0)}, \dots, I^{(r)}$. At the i-th step of the construction, u_i may or may not be added, and we will ensure that

$$\mathbf{P}(u_i \text{ is added at } i\text{-th step}) = p(u_i, J_j). \tag{4}$$

At the beginning, we set $I^{(0)} = I$. For $1 \leq i \leq r$, we define $I^{(i)}$ as follows. If $u_i \in I$ or I is not favourable for u_i, we set $I^{(i)} = I^{(i-1)}$. Otherwise, by (4) and property (iii) of Lemma 12, the probability that none of u_i's neighbours has been added before is at least

$$1 - \sum_{u_\ell \in \tilde{N}(u_i)} p(u_\ell, J_j) \geq p(u_i, J_j).$$

Thus, by including u_i based on a suitably biased independent coin flip, it is possible to make the probability of inclusion of u_i in Phase 5 (conditioned on $I = J_j$) exactly equal to $p(u_i, J_j)$. The output of Algorithm 2 is the set $I' := I^{(r)}$.

We analyze the probability that a deficient vertex u_i is in I'. By Lemma 8 and Proposition 10,

$$\mathbf{P}(u_i \in I) \geq \frac{88 + \varepsilon(u_i)}{256}.$$

By the above and property (ii) of Lemma 12, the probability that u_i is added to I' during Phase 5 equals

$$\begin{aligned}\mathbf{P}(u_i \text{ is added in Phase 5}) &= \sum_{j=1}^{s} \mathbf{P}(u_i \text{ is added in Phase 5} \mid I = J_j) \cdot \mathbf{P}(I = J_j) \\ &= \sum_{j=1}^{s} p(u_i, J_j) \cdot p(J_j) = \frac{|\varepsilon(u_i)|}{256}.\end{aligned}$$

Since u_i is deficient, $\varepsilon(u_i) < 0$; therefore, we obtain

$$\begin{aligned}\mathbf{P}(u_i \in I') &= \mathbf{P}(u_i \in I) + \mathbf{P}(u_i \text{ is added in Phase 5}) \\ &\geq \frac{88 + \varepsilon(u_i)}{256} - \frac{\varepsilon(u_i)}{256} = \frac{88}{256}.\end{aligned}$$

If w is a vertex of G which is the sponsor of a (necessarily unique) deficient vertex u_i, then the probability of the removal of w in Phase 5 is equal to the probability of the addition of u_i, namely $|\varepsilon(u_i)|/256$. From Lemma 8 and Proposition 10, it follows that $\mathbf{P}(w \in I)$ is high enough for $\mathbf{P}(w \in I')$ to be still greater than or equal to $88/256$.

Finally, if a vertex w is neither deficient nor the sponsor of a deficient vertex, it is not affected by Phase 5, and hence $\mathbf{P}(w \in I') \geq 88/256$ as well. Applying Lemma 1 to Algorithm 2, we infer that $\chi_f(G) \leq 256/88 = 32/11$ as required.

8 Subcubic graphs

The generalisation from triangle-free cubic bridgeless graphs to triangle-free subcubic graphs is perhaps most clear when phrased in terms of the second equivalent definition of the fractional chromatic number as given in Lemma 1.

In Sections 2–7, we showed that for a bridgeless triangle-free cubic graph G', $\chi_f(G') \leq k := 32/11$. Therefore, by Lemma 1, there exists an integer N such that kN is an integer and we can colour the vertices of G' using N-tuples from kN colours in such a way that adjacent vertices receive disjoint lists of colours.

We now show that if G is an arbitrary subcubic graph, then $\chi_f(G) \leq k$. We proceed by induction on the number of vertices of G. The base cases where $|V(G)| \leq 3$ are trivial.

Suppose that G has a bridge and choose a block B_1 incident with only one bridge e. (Recall that a *block* of G is a maximal connected subgraph of G without cutvertices.) Let B_2 be the other component of $G - e$. For $i = 1, 2$, the induction hypothesis implies that B_i $(i = 1, 2)$ admits a colouring by N_i-tuples from a list of $\lfloor kN_i \rfloor$ colours, for a suitable integer N_i. Setting N to be a common multiple of N_1 and N_2 such that kN is an integer, we see that each B_i has an N-tuple colouring by colours $\{1, \ldots, kN\}$. Furthermore, since $k > 2$, we may permute the colours used for B_1 so as to make the endvertices of e coloured by disjoint N-tuples. The result is a valid N-tuple colouring of G by kN colours, showing $\chi_f(G) \leq k$.

We may thus assume that G is bridgeless; in particular, it has minimum degree 2 or 3. We may also assume that it contains a vertex of degree 2 for otherwise we are done by the results of Sections 2–7. If G contains at least two vertices of degree 2, we can form a graph G'' by taking two copies of G and joining the two copies of each vertex of degree 2 by an edge. Since G'' is a cubic bridgeless supergraph of G, we find $\chi_f(G) \leq k$.

It remains to consider the case where G is bridgeless and contains exactly one vertex v_0 of degree 2. Let G_0 be the bridgeless cubic graph obtained by suppressing v_0, and let e_0 denote the edge corresponding to the pair of edges incident with v_0 in G. By Theorem 4, G_0 has a 2-factor F_0 containing e_0, such that $E(F_0)$ intersects every inclusionwise minimal edge-cut of size 3 or 4 in G_0.

Let G_1 be obtained from two copies of G by joining the copies of v_0 by an edge. Thus, G_1 is a cubic graph with precisely one bridge. The 2-factor F_0 of G_0 yields a 2-factor F_1 of G_1 in the obvious way. Moreover, it is not hard to see that every inclusionwise minimal edge-cut of size 3 or 4 in G_1 is intersected by $E(F_1)$. This is all we need to make the argument of Sections 2–7 work even though G_1 is not bridgeless. Consequently, $\chi_f(G_1) \leq k$, and since G is a subgraph of G_1, we infer that $\chi_f(G) \leq k$ as well. This finishes the proof of Theorem 3.

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