

Homomorphism-homogeneous L -colored graphs*

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Abstract

A relational structure is homomorphism-homogeneous (HH -homogeneous for short) if every homomorphism between finite induced substructures of the structure can be extended to a homomorphism over the whole domain of the structure. Similarly, a structure is monomorphism-homogeneous (MH -homogeneous for short) if every monomorphism between finite induced substructures of the structure can be extended to a homomorphism over the whole domain of the structure. In this paper we consider L -colored graphs, that is, undirected graphs without loops where sets of colors selected from L are assigned to vertices and edges. A full classification of finite MH -homogeneous L -colored graphs where L is a chain is provided, and we show that the classes MH and HH coincide. When L is a diamond, that is, a set of pairwise incomparable elements enriched with a greatest and a least element, the situation turns out to be much more involved. We show that in the general case the classes MH and HH do not coincide.

1 Introduction

A *relational structure* \mathcal{A} is a pair $(A, R_{\mathcal{A}})$, where $R_{\mathcal{A}}$ is a tuple $(R_{\mathcal{A}}^i : i \in I)$ of relations such that $R_{\mathcal{A}}^i \subseteq A^{\delta_i}$ (i.e. $R_{\mathcal{A}}^i$ is a δ_i -ary relation on A). The family $\Delta = (\delta_i : i \in I)$ is called the *type* of \mathcal{A} . The type is usually fixed and understood from the context. The underlying set A is called the *domain* of \mathcal{A} .

Relational structures of type (2) can be seen as directed graphs with loops. We will also consider undirected graphs without loops as relational structures of type (2) with one symmetric and irreflexive binary relation.

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For structures $\mathcal{A} = (A, R_{\mathcal{A}})$ and $\mathcal{B} = (B, R_{\mathcal{B}})$ a *homomorphism* $f : \mathcal{A} \rightarrow \mathcal{B}$ is a mapping $f : A \rightarrow B$ such that $(x_1, x_2, \dots, x_{\delta_i}) \in R_{\mathcal{A}}^i$ implies $(f(x_1), f(x_2), \dots, f(x_{\delta_i})) \in R_{\mathcal{B}}^i$ for each $i \in I$. If f is one-to-one then f is called a *monomorphism*. An *isomorphism* $g : \mathcal{A} \rightarrow \mathcal{B}$ is a bijective mapping $g : A \rightarrow B$ such that $(x_1, x_2, \dots, x_{\delta_i}) \in R_{\mathcal{A}}^i \Leftrightarrow (g(x_1), g(x_2), \dots, g(x_{\delta_i})) \in R_{\mathcal{B}}^i$ for each $i \in I$.

An isomorphism from a structure to itself is called an *automorphism*. Similarly, an *endomorphism* is a homomorphism from a structure to itself. Throughout the paper we write

$$f = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \end{pmatrix}$$

for a mapping $f : \{x_1, x_2, \dots, x_n\} \rightarrow \{y_1, y_2, \dots, y_n\}$ such that $f(x_i) = y_i$ for all $i \in \{1, 2, \dots, n\}$.

A structure \mathcal{A} is called *ultrahomogeneous* if every isomorphism between two induced finite substructures of \mathcal{A} can be extended to an automorphism of \mathcal{A} . There is a long-standing effort to classify all ultrahomogeneous relational structures since the work of Fraïssé [1] (see, for example [2, 3]).

In this paper we will use the classification of finite undirected graphs without loops provided by Gardiner in [4]. He has shown that a finite graph is ultrahomogeneous if and only if it is isomorphic to one of the following graphs:

1. a disjoint union of complete graphs all of the same size, $\bigcup_{i=1}^k K_n$,
2. multipartite graphs K_{n_1, n_2, \dots, n_k} with $n_i = n_j = \dots = n_k$,
3. the 5-cycle C_5 ,
4. the line graph $L(K_{3,3})$.

While this class of finite ultrahomogeneous graphs is important in our case we will refer to it as the *Gardiner's class* or simply as *Gardiner graphs*.

Quite recently, Cameron and Nešetřil introduced the following variant of homogeneity [5]. A structure \mathcal{A} is called *homomorphism-homogeneous* (*HH-homogeneous* for short) if every homomorphism between finite induced substructures of \mathcal{A} can be extended to an endomorphism of \mathcal{A} . This notion has motivated a new classification programme. Finite *HH-homogeneous* undirected graphs are classified as complete and null graphs [5]. Other classes of structures where *HH-homogeneous* structures have been fully classified are, for example, partially ordered sets in [6] or in [7] and finite tournaments [8].

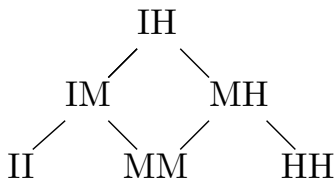
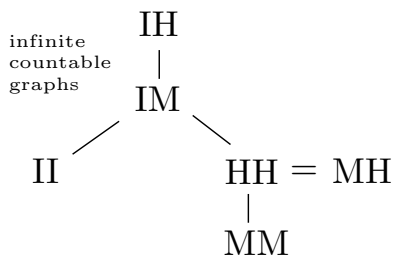


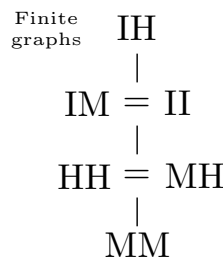
Figure 1: The hierarchy of morphism extension classes for a general relational structure.

Several other variants of homogeneity are also proposed in [5]. For these we follow the notation used in [5, 7]. We say that a structure \mathcal{A} belongs to a class XY if every x -morphism from a finite substructure of \mathcal{A} into \mathcal{A} extends to a y -morphism from \mathcal{A} to \mathcal{A} where pairs (X, x) and (Y, y) can be (I, iso) , $(M, mono)$ and $(H, homo)$.

Many of these classes are related. For example MH is a subclass of IH . The obvious inclusions between the morphism extension classes are depicted in Figure 1. Note that, for simplicity, we omit the inclusions implied by transitivity in all diagrams.



(a) Countably infinite graphs



(b) Finite graphs

Figure 2: The hierarchy of morphism extension classes for graphs [9].

For specific types of relational structures, some classes are known to be equivalent (such as HH and MH for graphs [5, 9]). This leads to simplified inclusion diagrams. Figure 2 depicts the hierarchy for finite and infinite countable graphs [9], and Figure 3 the hierarchy for partially ordered sets [7].

The main question of the classification programme is to give a catalogue of structures belonging to a given class. The full classification of any of the classes is far from complete. The class II is the most extensively studied one, while the class HH and other variants are less explored. In Section 2 we introduce a rather general notion of L -colored graphs where L is a partially ordered set. (We think of L as a poset of admissible combinations of colors ordered by inclusion.) In Sections 3 and 4 we provide classifications of finite



Figure 3: Hierarchy of morphisms extension classes for partially ordered sets [7].

MH -homogeneous L -colored graphs where L is a chain or a diamond. In all the existing classification results, the classes HH and MH coincide. This leads to the question whether there is a structure that is MH but not HH . We give a positive answer to this question in Section 4. A few more types of structures where the classes MH and HH do not coincide are given in Section 5.

2 Multicolored graphs

Let $G = (V, \mathbf{E})$ with $\mathbf{E} = (E_1, E_2, \dots, E_m)$ be a relational structure with a collection \mathbf{E} of symmetric irreflexive binary relations. This structure is called a *multicolored graph*. In case $m = 2$ we say that G is a *bicolored graph*, or shortly a *bigraph*. Finite HH -homogeneous bigraphs have been classified in [10].

In this paper we propose the study of a related but more general notion which yields a clearer, unifying presentation. Let L be a partially ordered set with the ordering relation \preceq , with the least element 0 and the greatest element 1. An L -colored graph is an ordered triple (V, χ', χ'') such that V is a nonempty set, $\chi' : V \rightarrow L$ is an arbitrary function and $\chi'' : V^2 \rightarrow L$ is a function satisfying the following:

1. $\chi''(x, x) = 0$; and
2. $\chi''(x, y) = \chi''(y, x)$ whenever $x \neq y$.

The function χ' provides colors of vertices of G , while χ'' provides colors of edges of G . The two restrictions that we have imposed on χ'' mean that G is without loops and undirected.

A multicolored graph $(V, (E_1, \dots, E_m))$ as introduced in [10] can be thought of as an L -colored graph (V, χ', χ'') where $L = \mathcal{P}(\{1, 2, \dots, m\})$ with set-inclusion as the ordering relation, $\chi'(x) = \emptyset$ (no colors are assigned to vertices), and $\chi''(x, y) = \{j : \{x, y\} \in E_j\}$.

Consequently, the intuition that we have is that $\chi'(x) = 0$ means that there are no colors assigned to x , and $\chi'(x) = 1$ means that the vertex x is colored by all the available colors. Analogously, $\chi''(x, y) = 0$ means that x and y are nonadjacent, while $\chi''(x, y) = 1$ means that the edge $\{x, y\}$ is colored by all the available colors.

A *homomorphism* between two L -colored graphs (V_1, χ'_1, χ''_1) and (V_2, χ'_2, χ''_2) is a mapping $f : V_1 \rightarrow V_2$ such that

$$\chi'_1(x) \preceq \chi'_2(f(x)) \text{ and } \chi''_1(x, y) \preceq \chi''_2(f(x), f(y)),$$

for all x and y in V_1 .

For $W \subseteq V$, a *substructure* of (V, χ', χ'') induced by W is $G[W] = (W, \chi'|_W, \chi''|_W)$, where $\chi'|_W$ and $\chi''|_W$ denote the restrictions of χ' and χ'' to W , respectively.

For an L -colored graph $G = (V, \chi', \chi'')$ and $\alpha \in L$ let $W_\alpha = \{x \in V : \chi'(x) = \alpha\}$ and $G^{(\alpha)} = G[W_\alpha]$.

We say that an L -colored graph $G = (V, \chi', \chi'')$ is *homomorphism-homogeneous* (*HH-homogeneous* for short) if every homomorphism $f : S \rightarrow T$ between finite induced substructures of G extends to an endomorphism of G . We say that an L -colored graph $G = (V, \chi', \chi'')$ is *MH-homogeneous* if every monomorphism $f : S \rightarrow T$ between finite induced substructures of G extends to an endomorphism of G .

Let $G = (V, \chi', \chi'')$ be an L -colored graph, and let $\theta_G \subseteq V^2$ be the reflexive transitive closure of $\theta_G^0 = \{(x, y) \in V^2 : \chi''(x, y) \neq 0\}$. Then θ_G is an equivalence relation on V whose equivalence classes will be referred to as *connected components* of G . An L -colored graph G is *connected* if θ_G has only one equivalence class. Otherwise, it is *disconnected*. We say that G is *complete* if $\chi''(x, y) \neq 0$ for all $x \neq y$.

An L -colored graph $G = (V, \chi', \chi'')$ is *vertex-uniform* if there exists an $\alpha \in L$ such that $\chi'(x) = \alpha$ for all vertices x , and it is *edge-uniform* if there exists a $\beta \in L \setminus \{0\}$ such that $\chi''(x, y) = \beta$ for all vertices x, y such that $x \neq y$. We say that an L -colored graph $G = (V, \chi', \chi'')$ is *uniform* if it is both vertex-uniform and edge-uniform. Up to isomorphism, a finite connected uniform graph is uniquely determined by $n = |V|$, the color of vertices α and the color of edges $\beta \succ 0$, and we denote it by $U(n, \alpha, \beta)$.

If there is no danger of confusion, we shall write simply $\chi(x)$ and $\chi(x, y)$

instead of $\chi'(x)$ and $\chi''(x, y)$, respectively. Also, the set of vertices of G will be denoted by $V(G)$.

Lemma 2.1. *Let G be an MH-homogeneous L -colored graph. Assume that there exist three distinct vertices $a_0, a_1, x \in V(G)$ such that:*

- (i) $\chi(a_0, a_1) \succ 0$ and $\chi(x, a_1) \succ 0$,
- (ii) $\chi(a_0, x) \preceq \chi(a_0, a_1)$ and $\chi(x) \preceq \chi(a_1)$, and
- (iii) $\chi(a_0, x) \prec \chi(a_0, a_1)$ or $\chi(x) \prec \chi(a_1)$.

Then G is not finite.

Proof. Let us construct inductively a sequence of mappings f_2, f_3, \dots , and a sequence of vertices $a_2, a_3, \dots \in V(G)$ with the following properties:

- (1) Let $m(n) = \max\{j \in \{1, \dots, n\} : \chi(x, a_j) \succ 0\}$. (Note that $m(n) \geq 1$ due to (i)). The mapping

$$f_{n+1} = \begin{pmatrix} a_0 & \dots & a_{m(n)-1} & a_{m(n)+1} & \dots & a_n & x \\ a_0 & \dots & a_{m(n)-1} & a_{m(n)+1} & \dots & a_n & a_{m(n)} \end{pmatrix}$$

is a monomorphism from $G[a_0, \dots, a_{m(n)-1}, x, a_{m(n)+1}, \dots, a_n]$ to $G[a_0, \dots, a_n]$.

- (2) G is MH-homogeneous so there is an endomorphism f_{n+1}^* of G which extends f_{n+1} and we let $a_{n+1} = f_{n+1}^*(a_{m(n)})$.
- (3) $a_{n+1} \notin \{x, a_0, \dots, a_n\}$.
- (4) $\chi(a_i, a_j) \succ 0$ for all $i, j \in \{0, \dots, n+1\}$ such that $i \neq j$.
- (5) $\chi(a_0, a_1) \preceq \chi(a_0, a_j)$ and $\chi(a_1) \preceq \chi(a_j)$, for all $1 \leq j \leq n+1$.
- (6) $\chi(a_j, x) \preceq \chi(a_j, a_k)$ and $\chi(x) \preceq \chi(a_k)$ for all $0 \leq j < k \leq n+1$.
- (7) $\chi(a_0, x) \prec \chi(a_0, a_j)$ or $\chi(x) \prec \chi(a_j)$, for all $1 \leq j \leq n+1$.

The inductive construction proceeds in several steps, and the corresponding L -colored subgraphs can be depicted as in Figure 4.

The mapping $f_2 = \begin{pmatrix} a_0 & x \\ a_0 & a_1 \end{pmatrix}$ is a monomorphism from $G[a_0, x]$ to $G[a_0, a_1]$ by (ii), while in case $n > 2$ the requirement (6) for n (inductive hypothesis) and the fact that $\chi(a_j, x) = 0$ for $j > m(n)$ ensure that f_{n+1} is a monomorphism from $G[a_0, \dots, a_{m(n)-1}, x, a_{m(n)+1}, \dots, a_n]$ to $G[a_0, \dots, a_n]$. This shows (1).

Let us show that (3) holds for a_{n+1} assuming (1)–(7) for n .

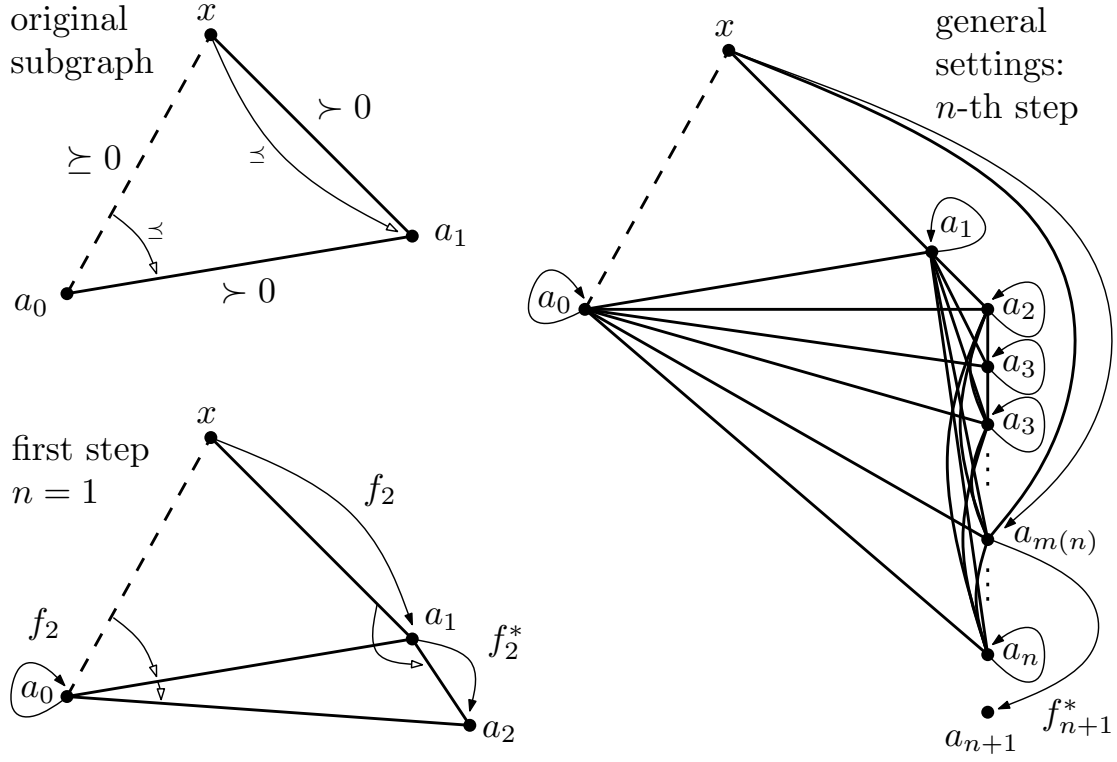


Figure 4: The original subgraph, the first step and general settings for the inductive construction. Bold lines without arrows represent edges—solid lines are those having colors $\succ 0$ and dashed those having colors $\succeq 0$. Thin lines with full arrows represents mappings and thin lines with empty arrows indicate the direction of the succession in colors.

- if $a_{n+1} = x$ then $\chi(a_0, a_{m(n)}) \preceq \chi(f_{n+1}^*(a_0), f_{n+1}^*(a_{m(n)})) = \chi(a_0, x)$ and $\chi(a_{m(n)}) \preceq \chi(f_{n+1}^*(a_{m(n)})) = \chi(x)$, which contradicts (7);
- if $a_{n+1} = a_{m(n)}$ then $0 = \chi(a_{m(n)}, a_{n+1}) = \chi(f_{n+1}^*(x), f_{n+1}^*(a_{m(n)})) \succeq \chi(x, a_{m(n)})$, but $\chi(x, a_{m(n)}) \succ 0$ by definition of $m(n)$ – contradiction;
- if $a_{n+1} = a_j$ for some $j \neq m(n)$ then, by (4), $0 \prec \chi(a_j, a_{m(n)}) \preceq \chi(f_{n+1}^*(a_j), f_{n+1}^*(a_{m(n)})) = \chi(a_j, a_{n+1}) = \chi(a_j, a_j) = 0$ – contradiction.

Let us show that (4) holds for a_{n+1} assuming (1)–(7) for n . Clearly, it suffices to show that $\chi(a_i, a_{n+1}) \succ 0$ for all $0 \leq i \leq n$.

- if $i \neq m(n)$ then $\chi(a_i, a_{n+1}) = \chi(f_{n+1}^*(a_i), f_{n+1}^*(a_{m(n)})) \succeq \chi(a_i, a_{m(n)}) \succ 0$ by the induction hypothesis;
- if $i = m(n)$ then $\chi(a_{m(n)}, a_{n+1}) = \chi(f_{n+1}^*(x), f_{n+1}^*(a_{m(n)})) \succeq \chi(x, a_{m(n)}) \succ 0$ by definition of $m(n)$.

To see that (5) holds for a_{n+1} we use the induction hypothesis and the fact that f_{n+1}^* is a homomorphism:

- $\chi(a_0, a_1) \preceq \chi(a_0, a_{m(n)}) \preceq \chi(f_{n+1}^*(a_0), f_{n+1}^*(a_{m(n)})) = \chi(a_0, a_{n+1})$;
- $\chi(a_1) \preceq \chi(a_{m(n)}) \preceq \chi(f_{n+1}^*(a_{m(n)})) = \chi(a_{n+1})$.

Let us show that (6) holds for a_{n+1} . As above, from (ii) and (5) we immediately get $\chi(x) \preceq \chi(a_1) \preceq \chi(a_{n+1})$. To see that $\chi(a_j, x) \preceq \chi(a_j, a_{n+1})$ for all $j \in \{0, \dots, n\}$ we consider several cases:

- if $j > m(n)$ then $\chi(a_j, x) = 0$ by definition of $m(n)$ so $\chi(a_j, x) \preceq \chi(a_j, a_{n+1})$ holds trivially;
- if $j < m(n)$ then using the induction hypothesis and the fact that f_{n+1}^* is a homomorphism we get the following:
 $\chi(a_j, x) \preceq \chi(a_j, a_{m(n)}) \preceq \chi(f_{n+1}^*(a_j), f_{n+1}^*(a_{m(n)})) = \chi(a_j, a_{n+1})$;
- if $j = m(n)$ then $\chi(a_{m(n)}, x) \preceq \chi(f_{n+1}^*(a_{m(n)}), f_{n+1}^*(x)) = \chi(a_{n+1}, a_{m(n)})$.

Finally, (7) follows from (5) and (iii).

Therefore, G contains an infinite sequence a_0, a_1, a_2, \dots of pairwise distinct vertices, so it cannot be finite. \square

In the rest of the paper we restrict our attention to two types of partially ordered sets L : chains and diamonds.

3 L -colored graphs over chains

In this section we classify finite MH -homogeneous L -colored graphs where L is a bounded chain and show that in this setting the classes MH and HH coincide. So, let L be a chain with the least element 0 and the greatest element 1.

Lemma 3.1. *Let G be a finite L -colored graph which is MH -homogeneous. Assume that x, y, z are three distinct vertices of G satisfying $\chi(x, z) \succ 0$ and $\chi(y, z) \succ 0$. Then:*

- $\chi(x, y) \prec \chi(x, z)$ if and only if $\chi(y) \succ \chi(z)$;
- $\chi(x, y) = \chi(x, z)$ if and only if $\chi(y) = \chi(z)$.

Proof. Clearly, (b) follows immediately from (a) because L is a chain. Let us show (a). Suppose that G is a finite MH -homogeneous L -colored graph,

and let x, y, z be three distinct vertices of G satisfying $\chi(x, z) \succ 0$ and $\chi(y, z) \succ 0$ but not (a). Then either

$$\chi(x, y) \prec \chi(x, z) \quad \text{and} \quad \chi(y) \preceq \chi(z)$$

or

$$\chi(x, y) \succeq \chi(x, z) \quad \text{and} \quad \chi(y) \succ \chi(z).$$

In both cases finiteness of G contradicts Lemma 2.1. \square

Lemma 3.2. *Let G be a finite MH-homogeneous L -colored graph. Then:*

- (a) *for every $\alpha \in L$, every connected component of $G^{(\alpha)}$ is a uniform graph;*
- (b) *for all $x, y \in V(G)$, if $\chi(x, y) \succ 0$ then $\chi(x) = \chi(y)$;*
- (c) *every connected component of G is a uniform graph.*

Proof. (a) Take any $\alpha \in L$ and let S be a connected component of $G^{(\alpha)}$. Then, by the definition of $G^{(\alpha)}$, we have that $\chi(x) = \alpha$ for all $x \in S$. Let us show that $\chi(x, y)$ is constant for all $x, y \in S$ satisfying $x \neq y$. If $|S| = 1$ or $|S| = 2$ the claim is trivial. Assume that $|S| \geq 3$. Since S is a connected component, it suffices to show that whenever $x, y, z \in S$ are three distinct vertices such that $\chi(x, z) \succ 0$ and $\chi(y, z) \succ 0$, then $\chi(x, z) = \chi(y, z) = \chi(x, y)$. So, let $x, y, z \in S$ be three distinct vertices satisfying $\chi(x, z) \succ 0$ and $\chi(y, z) \succ 0$. Since $\chi(y) = \chi(z) = \alpha$, Lemma 3.1 yields that $\chi(x, y) = \chi(x, z)$. Analogously, $\chi(x, y) = \chi(y, z)$.

(b) Assume that there exist $x_1, x_2 \in V(G)$ such that $\chi(x_1, x_2) \succ 0$ and $\chi(x_1) \neq \chi(x_2)$. Without loss of generality we can assume that $\chi(x_1) \prec \chi(x_2)$. Let us now construct inductively a sequence of mappings f_3, f_4, \dots , and a sequence of vertices x_3, x_4, \dots with the following properties:

- (1) the mapping $f_{n+1} : \begin{pmatrix} x_{n-1} \\ x_n \end{pmatrix}$ is a monomorphism from $G[x_{n-1}]$ to $G[x_n]$;
- (2) G is MH-homogeneous so there is an endomorphism f_{n+1}^* of G which extends f_{n+1} and we let $x_{n+1} = f_{n+1}^*(x_n)$;
- (3) $\chi(x_{i-1}) \preceq \chi(x_i)$ for all $i \in \{2, 3, \dots, n\}$.

The mapping $f_3 : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is easily seen to be a monomorphism from $G[x_1]$ to $G[x_2]$ since $\chi(x_1) \prec \chi(x_2)$, while in case $n \geq 3$, the requirement (3) for $i = n$ ensures that f_{n+1} is a monomorphism from $G[x_{n-1}]$ to $G[x_n]$. This

shows (1), and (3) for $i = n + 1$ follows immediately from (2). Note, also, that

$$\chi(x_{i-1}, x_i) = \chi(f_i^*(x_{i-2}), f_i^*(x_{i-1})) \succeq \chi(x_{i-2}, x_{i-1}), \text{ for all } i.$$

Therefore, we have constructed a sequence of vertices x_1, x_2, x_3, \dots such that $\chi(x_1) \preceq \chi(x_2) \preceq \chi(x_3) \preceq \dots$ and $0 \prec \chi(x_1, x_2) \preceq \chi(x_2, x_3) \preceq \chi(x_3, x_4) \preceq \dots$. Since $\chi(x_1) \prec \chi(x_2)$ and since G is finite there exists an n such that $\chi(x_{n-2}) \prec \chi(x_{n-1}) = \chi(x_n)$. Then Lemma 3.1 yields that $\chi(x_{n-2}, x_{n-1}) = \chi(x_{n-2}, x_n) \succ 0$ since $\chi(x_{n-1}) = \chi(x_n)$. By the same lemma we also have $\chi(x_{n-1}, x_n) \prec \chi(x_{n-2}, x_{n-1})$ since $\chi(x_n) \succ \chi(x_{n-2})$. On the other hand, $\chi(x_{n-1}, x_n) \succeq \chi(x_{n-2}, x_{n-1})$ by construction. Contradiction.

(c) It follows from (b) that S is a connected component of G if and only if S is a connected component of $G^{(\alpha)}$ for some $\alpha \in L$. Therefore, every connected component of G is a uniform graph. \square

Theorem 3.3. *Let G be a finite L -colored graph where L is a chain with the least element 0 and the greatest element 1. Then the following are equivalent:*

- (1) G is HH -homogeneous,
- (2) G is MH -homogeneous,
- (3) G has the following structure:
 - every connected component of G is a uniform L -colored graph, and
 - if $U(n_1, \alpha_1, \beta_1)$ and $U(n_2, \alpha_2, \beta_2)$ are connected components of G such that $\alpha_1 \preceq \alpha_2$, then $n_1 \leq n_2$ and $\beta_1 \preceq \beta_2$. Consequently, if $\alpha_1 = \alpha_2$, then $n_1 = n_2$ and $\beta_1 = \beta_2$.

Proof. (3) \Rightarrow (1) is easy.

(1) \Rightarrow (2) is obvious.

(2) \Rightarrow (3). Let G be a finite MH -homogeneous L -colored graph. We already know from Lemma 3.2 that every connected component of G is a uniform graph. So, let S_1 and S_2 be connected components of G such that $G[S_1] \cong U(n_1, \alpha_1, \beta_1)$, $G[S_2] \cong U(n_2, \alpha_2, \beta_2)$ and assume that $\alpha_1 \preceq \alpha_2$. Let x be an arbitrary vertex of S_1 and y an arbitrary vertex of S_2 . Then $f : \begin{pmatrix} x \\ y \end{pmatrix}$ is a monomorphism from $G[x]$ to $G[y]$, since $\chi(x) = \alpha_1 \preceq \alpha_2 = \chi(y)$. So, by the homogeneity requirement, f extends to an endomorphism f^* of G . It is easy to see that an endomorphism maps a connected component of

G into another connected component of G , so $f^*(S_1) \subseteq S_2$, since $f^*(x) = y \in S_2$. Moreover, $f^*|_{S_1}$ is injective (assume that $x, y \in S_1$ are two distinct vertices such that $f^*(x) = f^*(y)$; then $\chi(f^*(x), f^*(y)) = 0$ because G is without loops; on the other hand, $\chi(f^*(x), f^*(y)) \succeq \chi(x, y) = \beta_1 \succ 0$ by the definition of an edge-uniform L -colored graph – contradiction), so $n_1 = |S_1| \leq |S_2| = n_2$. Finally, if $x, y \in S_1$ are two distinct vertices, then $\beta_1 = \chi(x, y) \preceq \chi(f^*(x), f^*(y)) = \beta_2$. \square

4 L -colored graphs over diamonds

In this section we consider L -colored graphs where L is a diamond. We first consider finite vertex-uniform L -colored graphs and show that in this case the classes MH and HH coincide. We then provide an example of an L -colored graph which is MH -homogeneous, but not HH -homogeneous, proving thus that in the general case the classes MH and HH do not coincide for L -colored graphs where L is a diamond. So, let L be a diamond with the least element 0 and the greatest element 1.

Lemma 4.1. *Let G be a finite MH -homogeneous vertex-uniform L -colored graph and assume that there exist $x_0, y_0 \in V(G)$ such that $\chi(x_0, y_0) = 1$. Then the following holds:*

- (1) *For every vertex x there is a vertex y such that $\chi(x, y) = 1$.*
- (2) *Let x, y, z be distinct vertices. If $\chi(x, y) = \chi(y, z) = 1$ then $\chi(x, z) = 1$.*
- (3) *If x and y belong to the same connected component of G then $\chi(x, y) = 1$.*

Proof. (1) Let x be an arbitrary vertex. Then $f = \begin{pmatrix} x_0 \\ x \end{pmatrix}$ extends to an endomorphism f^* of G , so $\chi(x, f^*(y_0)) = \chi(x_0, y_0) = 1$.

(2) Let $\chi(x, y) = \chi(y, z) = 1$. If $\chi(x, z) \prec 1$, Lemma 2.1 yields that G then cannot be finite. Contradiction.

(3) Let S be a maximal set of vertices of G such that $x \in S$ and $\chi(u, v) = 1$ for all $u, v \in S$ with $u \neq v$. Note that $|S| \geq 2$ due to (1). Let us show that S coincides with the connected component W of G that contains x . Suppose to the contrary that this is not the case and take any $z \in W \setminus S$ such that $\chi(z, y) \succ 0$ for some $y \in S$. Without loss of generality we may assume that $y \neq x$ (because $|S| \geq 2$). Note also that $\chi(x, z) \neq 1$ and $\chi(y, z) \neq 1$. Then Lemma 2.1 yields that G is not finite. Contradiction. \square

Proposition 4.2. *Let G be a finite MH -homogeneous vertex-uniform L -colored graph where every vertex has color $\alpha \in L$. Assume that there exist $x_0, y_0 \in V(G)$ such that $\chi(x_0, y_0) = 1$. Then there exists a positive integer n such that every connected component of G is isomorphic to $U(n, \alpha, 1)$.*

Next, we consider finite MH -homogeneous vertex-uniform L -colored graphs satisfying $\chi(x, y) = 1$ for no $x, y \in V(G)$.

Proposition 4.3. *Let G be a finite connected MH -homogeneous vertex-uniform L -colored graph such that $\chi(x, y) = 1$ for no $x, y \in V(G)$. Then G is complete.*

Proof. Assume, to the contrary, that G is not complete. Then there exist $x, y \in V(G)$ such that $x \neq y$ and $\chi(x, y) = 0$. Since G is connected, there exists a sequence v_1, v_2, \dots, v_k of vertices of G such that $x = v_1$, $y = v_k$ and $\chi(v_i, v_{i+1}) \succ 0$ for all $i \in \{1, \dots, k-1\}$. Without loss of generality, we can assume that (v_1, v_2, \dots, v_k) is the shortest such sequence, so that $\chi(v_i, v_j) = 0$ whenever $j - i > 1$. Note that $k \geq 3$ because $\chi(x, y) = 0$.

Now, $f = \begin{pmatrix} v_1 & v_3 \\ v_1 & v_k \end{pmatrix}$ is a partial monomorphism which, by the homogeneity assumption, extends to an endomorphism f^* of G . Let $z = f^*(v_2)$. Note that $\chi(x, z) \succ 0$ and $\chi(y, z) \succ 0$. Therefore, x , y and z provide a configuration which, by Lemma 2.1, ensures that G is not finite. Contradiction. \square

If G is a finite vertex-uniform L -colored graph which is connected and complete, all endomorphisms are automorphisms, and it is easy to see that G is HH -homogeneous if and only if G is MH -homogeneous if and only if G is ultrahomogeneous. On the other hand, if G is a finite vertex-uniform L -colored graph which is not connected and has the property that $\chi(x, y) \prec 1$ for all $x, y \in V(G)$, then by Proposition 4.3 every connected component of G is complete and all components have to be isomorphic. So, we have the following partial classification result which depends on the classification of all finite ultrahomogeneous edge colored graphs (and this is a long-standing open problem):

Theorem 4.4. *Let L be a diamond with the least element 0 and the greatest element 1. The following are equivalent for a finite vertex-uniform L -colored graph G where every vertex is colored by $\alpha \in L$:*

- (1) G is HH -homogeneous,
- (2) G is MH -homogeneous,

(3) G is a disjoint union of $k \geq 1$ copies of H , where

- H is $U(n, \alpha, 1)$ for some positive integer n ; or
- H is an ultrahomogeneous L -colored graph such that $0 \prec \chi(x, y) \prec 1$ for all $x, y \in V(G)$ such that $x \neq y$, and $\chi(x) = \alpha$ for all $x \in V(G)$.

However, if $L = M_2$ is the diamond on four elements $0, b, r, 1$ where $0 \prec b \prec 1, 0 \prec r \prec 1$ and where b and r are incomparable (b and r stand for *blue* and *red*, respectively), we can provide the full classification as follows. For an $\alpha \in M_2$ let $G_{(\alpha)} = (V, E_\alpha)$ be the (ordinary undirected) graph where $E_\alpha = \{\{x, y\} : \chi(x, y) = \alpha\}$.

Theorem 4.5. *The following are equivalent for a finite vertex-uniform M_2 -colored graph G where every vertex is colored by $\alpha \in M_2$:*

(1) G is HH -homogeneous,

(2) G is MH -homogeneous,

(3) G is a disjoint union of $k \geq 1$ copies of H , where

- H is $U(n, \alpha, 1)$ for some positive integer n ; or
- H is vertex uniform, $H_{(r)}$ is one of the Gardiner graphs and $H_{(b)}$ is its complement.

As the example below shows, Theorems 4.4 and 4.5 cannot be extended to finite L -colored graphs where L is a diamond and graphs are not required to be vertex-uniform.

Example 1 Let G be an M_2 -colored graph on four vertices a, b, c, d where the vertices and the edges are colored as follows: $\chi(a) = \chi(b) = r, \chi(c) = \chi(d) = b, \chi(a, c) = \chi(c, d) = \chi(b, d) = r, \chi(a, d) = \chi(b, c) = b$ and $\chi(a, b) = 0$ (see Figure 5).

Then G is clearly an MH -homogeneous graph. To see that G is not an HH -homogeneous graph it suffices to note that the partial homomorphism $f = \begin{pmatrix} a & b \\ a & a \end{pmatrix}$ cannot be extended to an endomorphism of G .

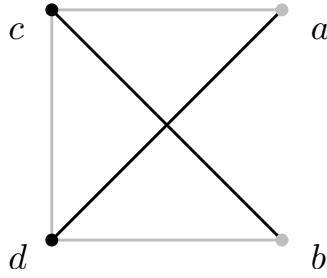


Figure 5: An example of a finite L -colored graph that is MH -homogeneous but not HH -homogeneous.

5 Concluding remarks

A simple relational structure presented in Example 1 can easily be generalised to provide a whole class of structures that are all MH -homogeneous but not HH -homogeneous.

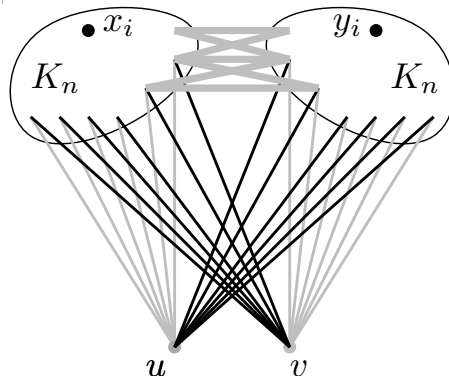


Figure 6: A class of finite L -colored graphs which are all MH - but not HH -homogeneous.

The construction is depicted in Figure 6. Fix $n \in \{1, 2, \dots, \omega\}$. (Note that in case $n = \omega$ we get an example of a countably infinite structure that is MH -homogeneous but not HH -homogeneous.) Take two cliques both of size n whose vertices and edges are colored black. Join the vertices of these two cliques by fat gray edges. Finally, add two new nonadjacent vertices u and v colored gray, and join the two vertices and the vertices of the two cliques by black and gray edges as in Figure 6. Then, as in Example 1, we can show that this graph is MH -homogeneous but not HH -homogeneous.

A question that arises immediately is whether one can avoid the need for colored vertices at the expense of introducing loops.

Consider the finite edge-colored graph depicted in Figure 7 with no colors

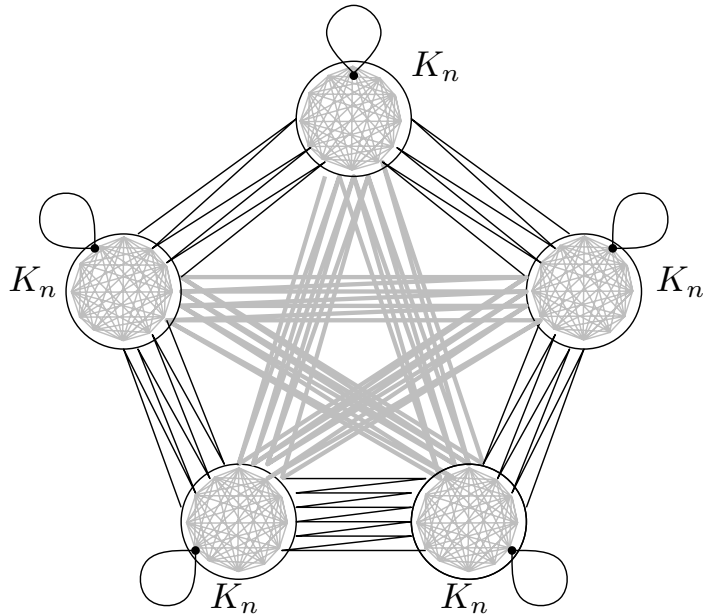


Figure 7: An edge-colored graph with loops that is MH - but not HH -homogeneous.

assigned to vertices that we construct as follows. Given $n > 1$, take five copies of K_n and color their edges gray. Now join these cliques by complete bipartite graphs using two mutually disjoint 5-cycles where the edges of one 5-cycle are black, while the edges of the other 5-cycle are gray. Furthermore, add a black loop to each vertex.

This graph is easily seen to be MH -homogeneous. To see that it is not HH -homogeneous, consider a partial homomorphism unifying two neighboring cliques (this is possible due to black-colored loops). Then every endomorphism that extends such a partial homomorphism would enforce the existence of an edge colored both black and gray.

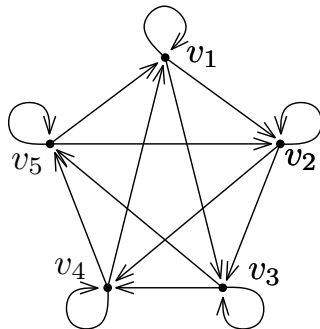


Figure 8: A digraph with loops that is MH -homogeneous but not HH -homogeneous.

Finally in Figure 8 we present a directed graph with loops that is MH -homogeneous but not HH -homogeneous. To see that this digraph is not HH -homogeneous consider a partial homomorphism $f = \begin{pmatrix} v_1 & v_4 & v_5 \\ v_1 & v_5 & v_5 \end{pmatrix}$. Then every endomorphism that extends f would enforce the existence of a bidirectional edge.

We close the paper with several open problems whose solutions would be helpful in understanding the structure of homogeneous L -colored graphs with respect to various types of homogeneity discussed in this paper. Let L be an arbitrary partially ordered set.

Problem 1 Classify all finite HH -homogeneous and MH -homogeneous L -colored graphs.

Problem 2 Do classes MH and HH coincide for finite vertex-uniform L -colored graphs?

Problem 3 Do classes MH and HH coincide for countable vertex-uniform L -colored graphs?

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